Mathematics of Business, Accounting, and Finance

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Preface

In recent years teachers in collegiate schools of business and other professional schools have become increasingly aware that most students enrolled in elementary business courses lack both an adequate knowledge of the basic principles of mathematics and a facility in fundamental arithmetic operations. In many colleges, courses in business mathematics or commercial algebra have been inaugurated to help students attain a high degree of facility in fundamental arithmetic operations and a clear understanding of algebraic principles. Such a course may provide the only college training in mathematics that many students receive. In other colleges, an additional course in mathematics of finance, or mathematics of investment, is also included in the curriculum.

In this revision of the earlier edition of *Mathematics of Business and Accounting*, the first nine chapters are planned to meet the objectives usually outlined for a course in business mathematics by providing a comprehensive review of the fundamental arithmetic operations, as well as a greatly enlarged review of the algebraic principles usually taught in secondary schools. The number of problems has been increased sufficiently to permit the book to be used repeatedly without unnecessary duplication of assignments, and the authors have stressed the types of problems which arise in business operations.

The remainder of the book covers all the materials usually found in courses in mathematics of finance, and also includes a treatment of installment credit not usually found in such texts. The section on depreciation has been enlarged to include methods the use of which is permitted under recent amendments to the revenue code.

A chapter on life insurance has been added. The problems are based on rates of interest most frequently used, and on mortality tables currently used in all states. In the discussion of life annuities, additional tables have not been introduced, first, because the tables in current use will soon be replaced, and, second, because the basic principles can be illustrated just as well by the use of the CSO Tables.

Mathematics of Business, Accounting, and Finance

1

Addition and Subtraction of Integers

Introduction

Students, clerks, businessmen, and accountants all must be able to deal with numbers accurately and quickly. This skill can be acquired only through practice and through an understanding of the basic principles of mathematics. There are only four fundamental operations: addition, subtraction, multiplication, and division. These are performed by the use of *numbers*, either whole or decimal. A whole number, such as 8, 43, 327, or 1,268, is called an *integer*.

Addition combinations

To add numbers together is to find their sum, a task which every one should be able to accomplish with speed and accuracy. The numbers which are added are called *addends*. Thus:

Addend	3
Addend	5
Addend	8
Sum (or Total)	16

If we take the ten digits, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0, in pairs, we can see that there are only 100 possible combinations. Skill in addition can be gained only by drilling on these combinations until the response is automatic.

EXERCISE 1.1

Give the sums of the following integers, emphasizing speed and accuracy.

1.	8	1	2	6	2	0	3	9	0	2	4	7	1	8	1	8
	2	5	7	1	5	9	8	2	8	8	4	4	3	8	7	6

2.	4	1	8	4	8	9	1	6	2	8	4	5	3	3	3	4
	3	4	3	5	2	9	6	2	1	4	3	6	1	4	2	8
3.	9	5	6	2	3	2	4	7	7	5	4	6		6	7	7
	3	4	4	9	9	4	7	3	7	3	2	3	6	5	2	8
4.	3	7	5	9	8	5	6	9	3	4	5	7	7	6	9	4
	7	9	7	6	7	8	9	5	6	6	3	5	6	7	8	9

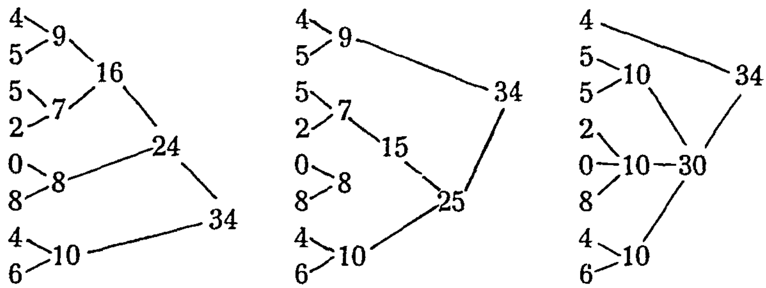
Those whose work involves the use of numbers soon gain facility in addition A student, however, may not have used addition frequently enough to be accurate and efficient To improve your speed, drill on the combinations which cause you to hesitate Although beginners consider each combination as a series of numbers, those who are more practiced immediately see that 3, 5, and 8 total 16 instead of going through the mental process of combining 3 and 5 to make 8, then adding 8 to make 16 With a little practice you will begin to think not that the sum of 3 and 5 is 8, and that 8 and 8 is 16, but that 3 and 5 and 8 is 16

EXERCISE 1.2

Give the sums of the following integers Emphasize accuracy and speed

1.	1	6	5	1	1	0	5	3	8	1	1	6	5	8	4
	4	2	1	7	2	5	3	2	9	9	5	1	9	9	7
	7	2	7	0	6	0	3	1	1	8	1	5	0	4	2
2.	4	1	1	8	4	3	2	7	2	7	2	9	4	2	2
	3	2	4	6	3	1	5	2	4	7	4	2	5	8	7
	1	7	1	1	2	3	2	6	7	4	9	5	8	7	5
3.	3	9	4	8	4	4	9	6	8	5	3	2	5	5	7
	8	2	7	8	3	5	9	2	4	6	4	6	6	3	3
	2	8	8	4	8	9	2	4	6	4	6	5	3	6	5
4.	6	9	3	9	3	3	8	6	4	7	4	9	8	5	5
	4	3	7	6	6	5	6	6	8	5	6	5	9	8	9
	3	6	9	5	9	6	7	7	8	6	7	9	5	9	8
5.	5	7	8	8	9	7	4	6	5	5	4	7	6	5	7
	9	9	7	9	7	5	9	8	9	9	6	6	7	9	9
	7	8	5	5	8	8	5	9	9	8	9	5	4	3	6

In solving the following problems strive for accuracy as well as speed. If you are seeking to improve your speed, do not think of the sum of 4, 5, 5, 2, 0, 8, 4, 6 as 4 plus 5 plus 5 plus 2 plus 0 plus 8 plus 4 plus 6. Think in terms of the combination of two or more numbers, such as 4 and 5 makes 9, and 2 and 5 makes 7. Note the three possible combinations that follow to give us 34.



It is immaterial what combinations you use. It is important, however, that you begin to think in terms of the combinations faster than you can say or write them. If you practice these drills for fifteen minutes a day for two or three weeks, you will save much time on future examinations and on other occasions when you may be required to add numbers quickly and accurately.

EXERCISE 1.3

Give the sums of the following.

1.	4	8	1	7	3	1	4	7	4	1	9	2	4	8	5	4
	5	5	7	2	4	1	1	1	0	2	8	5	6	7	6	8
	5	5	8	0	8	7	4	8	9	2	7	9	8	9	9	7
	2	5	3	3	5	9	9	6	8	8	6	6	5	3	4	8
	0	7	9	7	1	4	3	4	5	0	0	2	6	7	8	7
	8	3	8	4	5	5	8	1	3	7	6	3	0	9	9	6
	4	8	2	6	6	7	4	6	3	6	5	8	5	7	2	8
	6	7	5	5	1	8	8	9	9	4	0	5	5	3	9	9

2.	7	9	6	9	2	6	1	2	5	6	5	6	5	6	9	9
	3	4	9	8	2	9	3	2	3	0	3	0	3	4	8	8
	5	4	6	3	9	5	7	6	9	5	9	5	6	4	8	7
	9	0	3	7	1	8	8	8	8	6	8	6	6	6	6	2
	4	3	7	8	5	0	0	7	8	7	8	7	7	4	5	7
	8	8	9	8	8	7	4	6	6	8	6	5	4	6	6	5
	3	1	5	9	7	6	5	5	6	7	7	0	6	5	9	5
	8	4	0	7	9	9	5	8	6	7	6	4	4	1	6	7

3.	6	8	9	7	7	7	4	4	8	2	3	5	6	7	4	7
	5	1	3	1	2	0	1	2	1	7	4	6	5	4	3	7
	8	3	2	3	5	9	2	7	9	2	5	4	7	6	4	4
	4	1	5	5	2	2	2	8	2	3	6	4	4	7	5	6
	7	2	3	2	4	1	5	6	6	4	6	4	5	7	7	5
	6	8	2	3	9	5	7	7	3	5	6	7	8	6	1	9
	3	4	6	9	1	2	6	5	2	6	7	4	4	4	9	3
	2	7	7	4	3	1	3	7	5	1	4	9	3	5	3	2

4.	3	9	3	5	3	7	2	8	7	9	8	7	4	3	2	4
	6	7	3	3	2	6	7	8	7	8	7	6	5	4	3	5
	6	8	6	6	9	5	4	9	6	5	4	3	8	7	6	3
	4	2	6	7	2	2	2	5	9	4	3	2	9	8	7	6
	7	8	4	9	5	4	5	6	5	2	3	5	6	7	8	2
	8	9	6	2	9	4	8	8	4	3	4	6	7	8	9	9
	8	8	8	8	7	7	9	3	6	9	7	5	4	5	2	9
	4	6	3	5	4	4	2	9	9	7	7	4	6	3	1	6

Column addition

Our number system was probably based on the number of fingers on a person's hand Indeed our word for digit is derived from the Latin word for finger, *digitus* At the number ten, primitive man ran out of fingers on which to count A natural practice under such circumstances would be to use ten as the highest number known, adding the additional numbers to it, such as ten-and-one, ten-and-two and so forth Though the name eleven has been substituted for ten-and-one, and twelve for ten-and-two, it is easy to see that thirteen is derived from three-and-ten, fourteen from four-and-ten, and so on

Our number system uses the *principle of position* or *place* value, that is, the value of a number depends on two factors first, the digit used, and second, the position of the digit Thus the digit to the right in a whole number indicates units only, but the second digit from the right indicates tens Thus the number 25 is made up of two digits, 2 and 5 The 2 is the ten's digit, it indicates 2 tens The 5 is the unit's digit The number 25 is thus equivalent to 2 tens plus 5 units (i e, $25 = 2 \times 10 + 5 \times 1$) The number 52, on the other hand, is equivalent to 5 tens plus 2 units (i e, $52 = 5 \times 10 + 2 \times 1$)

Each position has been assigned a value ten times greater than the one preceding it We thus say that the *base* of our number system is 10

The names of only the positions of numbers which you will probably ever need to know are as follows:

quadrillions	hundred trillions	ten trillions	trillions*	hundred billions	ten billions	billions	hundred millions	ten millions	millions	hundred thousands	ten thousands	thousands	hundreds	tens	units
5,	3	5	6,	2	5	7,	7	8	6,	9	1	2,	7	2	5

In two-column addition, the usual procedure is to add the unit's column together first, then the ten's column. Some people feel that they save time by adding two columns at a time. There are several methods of making such additions quickly but probably the easiest is to modify the unit's digits and ten's digits until the combinations are fairly simple. Thus the addition of 37 and 49 can be considered the addition of 37 and 40 and 9 (86 is the answer). By this method the problem

$$\begin{array}{r} 30 \\ 38 \\ 46 \\ 13 \\ 21 \\ \underline{22} \end{array}$$

might be considered as:

$$\begin{aligned} 30 + 38 &= 68 \\ 68 + 40 &= 108; \quad 108 + 6 = 114 \\ 114 + 13 &= 127 \\ 127 + 21 &= 148 \\ 148 + 20 &= 168; \quad 168 + 2 = 170 \end{aligned}$$

The wisdom of adopting such short cuts has to be determined by each person for himself. If you need to add only short columns of two figures each, undoubtedly you can learn to do it with rapidity. Not many of us are called on to make such calculations frequently. Since some practice in these combinations appears desirable, the following exercise is included.

*In the English and German systems, this column is read as billion; in the United States and France a billion is the equivalent of a thousand millions, and a trillion is the equivalent of a thousand billions. According to the English and German systems, a billion is a million of millions, a trillion is a million of billions, and each higher denomination is a million times the one preceding.

EXERCISE 1.4

Test your speed and accuracy in the following two-column additions

1.	2.	3.	4.	5.	6.	7.	8.
27	31	16	22	39	48	10	18
16	14	27	43	47	15	17	26
30	43	31	20	14	16	25	34
23	37	40	18	15	24	33	42

9.	10.	11.	12.	13.	14.	15.	16.
19	28	58	81	68	39	53	14
27	29	68	58	46	49	68	37
35	37	78	45	33	59	26	64
46	45	34	23	11	69	41	93
44	16	27	69	79	79	56	21
14	20	28	47	26	26	14	10
41	43	38	34	19	19	29	44
49	32	48	12	29	29	47	98

Horizontal addition

Once the addition combinations are known, it is possible to add numbers horizontally with just as much dispatch as it is to add them vertically In many instances it is necessary to add numbers which are written side by side If you do the following exercise you should be able to increase your speed in horizontal addition This skill should serve you in good stead if you are called upon to make such additions on payroll records, inventory records, or working papers in accounting Try to use combinations as much as possible

EXERCISE 1.5

Add the following horizontally

1. $13 + 17 + 21 + 16 + 14 = ?$
2. $32 + 11 + 14 + 29 + 12 = ?$
3. $54 + 19 + 26 + 61 + 33 = ?$
4. $42 + 49 + 56 + 27 + 39 = ?$
5. $82 + 69 + 26 + 76 + 95 = ?$
6. $11 + 26 + 69 + 53 + 78 = ?$
7. $62 + 27 + 57 + 75 + 69 = ?$
8. $55 + 47 + 39 + 23 + 31 = ?$
9. $19 + 36 + 47 + 24 + 15 = ?$
10. $52 + 43 + 98 + 67 + 25 = ?$

A check of the results is obtained when the numbers are added both horizontally and vertically and then the answers are added both horizontally and vertically For example

$$\begin{array}{rrrrrrr}
3+ & 5+ & 9+ & 1+ & 7+ & 2= & 27 \\
5+ & 4+ & 8+ & 3+ & 2+ & 6= & 28 \\
8+ & 1+ & 2+ & 7+ & 4+ & 3= & 25 \\
6+ & 2+ & 6+ & 7+ & 3+ & 5= & 29 \\
8+ & 2+ & 7+ & 4+ & 5+ & 1= & 27 \\
\underline{3}+ & \underline{3}+ & \underline{5}+ & \underline{8}+ & \underline{2}+ & \underline{7}= & \underline{28} \\
33+ & 17+ & 37+ & 30+ & 23+ & 24= & 164
\end{array}$$

EXERCISE 1.6

Add the following:

1.
$$\begin{array}{l}
4+8+1+7+3=? \\
5+5+7+2+4=? \\
5+6+8+0+8=? \\
2+5+3+3+5=? \\
0+7+9+7+1=? \\
\underline{8}+\underline{3}+\underline{8}+\underline{4}+\underline{5}=\underline{?} \\
?+?+?+?+?+?
\end{array}$$

2.
$$\begin{array}{l}
1+4+7+4+1=? \\
1+1+1+0+2=? \\
7+4+8+9+2=? \\
9+9+6+8+8=? \\
4+3+4+5+0=? \\
\underline{5}+\underline{8}+\underline{1}+\underline{3}+\underline{7}=\underline{?} \\
?+?+?+?+?+?
\end{array}$$

3.
$$\begin{array}{l}
39+48+10+18=? \\
47+15+17+26=? \\
14+16+25+34=? \\
15+24+33+42=? \\
23+32+41+43=? \\
\underline{34}+\underline{40}+\underline{49}+\underline{10}=\underline{?} \\
?+?+?+?=?
\end{array}$$

4.
$$\begin{array}{l}
19+28+70+64=? \\
27+29+38+77=? \\
35+37+22+81=? \\
46+45+93+72=? \\
44+16+81+29=? \\
\underline{14}+\underline{20}+\underline{57}+\underline{40}=\underline{?} \\
?+?+?+?=?
\end{array}$$

5.
$$\begin{array}{l}
19+28+55+71=? \\
27+29+69+38=? \\
35+37+52+63=? \\
46+45+32+78=? \\
44+16+59+72=? \\
\underline{14}+\underline{20}+\underline{83}+\underline{57}=\underline{?} \\
?+?+?+?=?
\end{array}$$

6.
$$\begin{array}{l}
49+54+23+35=? \\
65+39+16+88=? \\
37+95+33+22=? \\
51+77+41+66=? \\
72+26+19+51=? \\
\underline{76}+\underline{81}+\underline{23}+\underline{75}=\underline{?} \\
?+?+?+?=?
\end{array}$$

Addition of long columns

Work habits should be developed which make it as easy as possible to solve problems which entail the addition of several columns. The following work habits are suggested: first add each column by combinations; then record the total of each column separately. To do the latter the following four ways are shown. The first two ways are standard, the third way is

sometimes called the *accountant's method*, and the fourth way is sometimes called the *banker's method*

I	II	III	IV
2 22	4,328	4,328	4,328
4,328	3,782	3,752	3,752
3,752	4,278	4,278	4,278
4,278	6,143	6,143	6,143
6,143	2,274	2,274	2,274
2,274	3,822	3,822	3,822
3,822	2 22	27	27
24,597	24,597	27	29
		23	25
		22	24 or
		24,597	24,597

In I and II, put the ten's digit of each column sum as a little number at the top or the bottom of the next column to the left

In III, put down each column sum and then add these column sums Add either from left to right or from right to left

In IV, the 27 is obtained by adding the unit's column The 29 is obtained by adding to the sum of the ten's column the 2 of the 27 from the sum of the unit s column, and so on The answer desired is made up of the figures lowest in position in each column These figures are underlined in example IV

No matter what method is used, one advantage of recording the total of each column separately is that if the worker is interrupted after he has added one or more columns he can resume his work where he left off

EXERCISE 1.7

Add the following Do not always use the same method

1.	2.	3.	4.
4,314	1,489	5,059	3,469
1,018	8,908	3,094	4,558
215	7,794	9,032	8,561
2,562	1,547	7,815	1,492

5.	6.	7.	8.
14,314	67,136	86,288	12,436
10,024	21,662	64,911	70,513
97,654	79,237	74,328	14,927
32,100	36,846	84,719	56,451

9.	10.	11.	12.
891,730	655,845	477,975	243,871
581,611	297,799	739,889	333,475
912,377	278,959	256,520	946,380
598,125	179,865	253,644	217,605
731,541	297,589	799,395	567,845
468,832	783,798	585,588	921,832

13.	14.	15.	16.	17.
1,623	2,633	9,617	3,278	5,191
5,937	8,248	4,547	1,419	5,956
2,884	6,552	2,523	3,024	3,752
2,360	8,695	5,812	7,777	7,249
1,514	4,013	4,745	8,965	2,141
9,830	2,271	7,791	8,689	4,713
1,927	5,927	9,388	3,551	6,503
4,478	7,853	7,548	1,913	3,961
1,507	3,076	6,827	3,620	7,495
2,928	2,985	2,121	9,181	6,831
3,514	8,506	6,960	3,268	2,029
9,408	1,022	2,149	6,087	1,807
5,503	9,741	6,784	1,032	1,519
1,728	3,819	2,971	1,957	7,267

Verification of addition

In any work which requires many additions, you will no doubt use an adding machine or some type of calculator. Such devices tend to assure accuracy and should be used when conditions warrant.

If not much time is spent in adding figures, however, such mechanical devices may be uneconomical. Consequently it is often necessary to develop accuracy in addition. If mistakes in addition are not found quickly, they result in a considerable loss of time and money. An error on a deposit ticket or a check stub can result in disproportionate embarrassment. Bookkeepers and accountants particularly must guard against errors in addition. By constant checking and verification, addition errors in business can be kept at a minimum.

There are three methods for verifying (or checking) answers in addition: (1) adding in reverse order; (2) casting out 9's; and (3) casting out 11's.

Adding in reverse order

If originally each column was added from top to bottom, or each row from left to right, the answers can be checked by adding from bottom to top or from right to left. By adding in the reverse order an entirely

different set of addition combinations is used If the same answer is obtained both times, this fact is usually sufficient verification of accuracy When the accountant's method of adding each column and recording the subtotals is adopted, verification by reverse order addition seems to be by far the most practical and accurate method of verification

Casting out 9's

Addition may be checked by the use of check numbers, the most common of which are 9 and 11 The method of proof by the use of check numbers consists essentially of dividing each addend by the check number, ignoring the quotients, and adding the remainders if any The sum of the remainders from the addends should be equal to the remainder of the answer, or total, after all the check numbers have been cast out of it

Nine is one of the easiest check numbers to use because the 9's in any number may be cast out by adding the digits in the number and deducting 9 or any multiple of 9 from the sum The balance is called the remainder, or *excess of nines* This term, excess of nines, is used frequently in the first two chapters of this text It is wise to go through the following examples carefully to assure an understanding of the meaning of the term

Number	*Sum of the Digits*	*Excess of 9's (Remainder after All 9 s Have Been Cast Out)*
10	$1 + 0 = 1$	1
21	$2 + 1 = 3$	3
72	$7 + 2 = 9,\ 9 - 9 = 0$	0
110	$1 + 1 + 0 = 2$	2
453	$4 + 5 + 3 = 12,\ 12 - 9 = 3$	3
3,539	$3 + 5 + 3 + 9 = 20,\ 20 - 2 \times 9 = 2$	2

The excess of 9's in any number can be found fairly rapidly by the mental process of dropping or canceling all 9's or digits totaling 9 For example, in the number 4,539 it can be seen quickly that $4 + 5$ is 9 and hence may be dropped or canceled, and that the last digit 9 may be canceled also It is readily apparent that the remainder is 3

Go through the next example and see how many numbers have been dropped or canceled

Number	*Excess of 9 s*	*Number*	*Excess of 9 s*
63	0	273	3
199	1	4,653	0
451	1	9,372	3
819	0	37,264	4

Verify the addition in the following example:

Addends		*Excess of 9's*
3,539		2
4,357		1
9,641		2
5,821		7
4,763		2
28,121	5	$14 - 9 = 5$

Cast out all the 9's in each of the addends. Total the remainders or excess of 9's in the addends. The sum of these numbers is 14. Casting the 9's out leaves an excess of 9's of 5. (If you desire, the $7 + 2$ can be cast out as you make the sum.) The next step is to cast the 9's out of the sum found by adding the addends, that is 28,121. The remainder after the 9's have been cast out is 5. Since these two figures are the same, we say the answer has been verified. If the sum of the remainders is not the same there has been an error either in the sum or in the verification.

EXERCISE 1.8

Add and check by casting out 9's.

1.	2.	3.	4.
3,645	1,101	1,914	8,689
5,364	4,989	4,874	9,667
4,646	2,937	3,892	7,246
2,407	4,587	2,908	2,968

5	6.	7.	8.
64,479	12,891	23,321	48,621
57,862	22,598	12,726	79,181
15,249	34,756	99,624	34,930
26,346	56,564	89,667	63,514
21,467	67,774	78,689	59,477

9.	10.	11.	12.
35,469	79,198	59,495	49,486
57,758	88,126	69,895	98,173
68,356	90,432	71,875	67,883
24,459	14,438	97,379	75,133
13,313	25,369	14,478	78,443
98,050	36,978	11,141	72,776

Casting out 11's

A common error in computation is the interchange of adjacent digits. It is said that the digits are transposed if \$9.30 is written as \$3.90 or as

\$9 03 In checking addition by casting out 9's, such an error would not be disclosed, but if 11 is used as a check number, a transposition in the sum will be apparent

Basically, proof by casting out 11's is similar to proof by casting out 9's since it depends on comparing the sum of the remainders in the addends after division by 11, with the remainders after the sum of the addends has been divided by 11

It is more difficult to cast out 11's than it is to cast out 9's Consider the remainders when the following numbers are divided by 11

Number		*Remainder*
23	$(3 - 2 = 1)$	1
47	$(7 - 4 = 3)$	3
89	$(9 - 8 = 1)$	1
12	$(2 - 1 = 1)$	1
58	$(8 - 5 = 3)$	3
79	$(9 - 7 = 2)$	2
91	$(12 - 9 = 3)$	3

In two digit numbers the remainder is found by subtracting the ten's digit from the unit's digit If the ten's digit is larger than the unit's digit 11 may be added to the unit's digit Thus when 91 is divided by 11 the remainder is found by adding 11 and 1, and subtracting the 9 from this sum

To cast 11's out of larger numbers find the sum of the digits in the *odd* places beginning at the right of the number the unit's place, the hundred's place, the ten thousand's place, and so on Then find the sum of the digits in the *even* places starting with the ten's place From the sum of digits in the odd places deduct the sum of the digits in the even places Add 11 if necessary The difference is the remainder, or excess of 11's Consider the following examples

Number	*Sum of Digits in the Odd Places*	*Sum of Digits in the Even Places*	*Difference*
2,868	16	8	8
477	11	7	4
48,732	13	11	2
2,558	13	7	6
52,583	13	10	3
107,218	10	9	1
213,783	11 + 11	13	9
325,172	5 + 11	15	1

In the last two examples, since the sum of digits in the odd places is less than the sum of digits in the even places, either add 11 to first sum or subtract 11, if possible, from the second sum before the difference is determined. Thus in the number 213,783, rather than adding 11 to 11 and deducting 13 to get a difference of 9, just deduct 11 from 13 and subtract the difference 2 from the 11 to get the remainder of 9.

EXERCISE 1.9

Cast the 11's out of the following numbers and write the remainders. Do not copy the numbers.

1. 89; 68; 45; 37; 13; 67; 35; 29; 12; 46; 97; 72; 65; 41
2. 5,359; 4,506; 7,249; 6,998; 3,158; 2,784; 9,258; 8,837
3. 13,727; 95,261; 83,573; 97,117; 94,183; 52,578; 118,372
4. 107,218; 765,339; 411,268; 672,543; 111,987; 58,234,772

Verify the addition in the following example:

Addends	*Sum of the Digits in the Odd Places*	*Sum of the Digits in the Even Places*	*Remainder*
3,297	9	12 (deduct 11)	8
5,283	5	13 (deduct 11)	3
5,356	9 (add 11)	10	10
7,512	7 (add 11)	8	10
4,387	10	12 (deduct 11)	9
68,375	14	15 (deduct 11)	10
94,210			

The sum of the remainders of the addends after casting out all 11's should be the same as the remainder after casting out all 11's from the sum. The sum is 94,210. Casting the 11's out leaves a remainder of 6. When the remainders of the addends are added together and the 11's are cast out, the remainder is 6. Since the two remainders are the same, the addition is verified.

EXERCISE 1.10

Add and check by casting out 11's.

1.	2.	3.	4.
40,951	52,323	74,843	68,755
23,015	93,688	14,936	93,688
54,330	5,948	47,943	94,531
44,444	2,481	71,294	21,736
89,530	77,898	84,384	66,550

5.	6.	7.	8.
541,498	609,755	589,741	122,411
67,508	578,621	721,315	179,613
249,005	423	4,587	900,615
593,441	4,312	650,593	696,040
312,650	60,975	898,491	624,485
483,298	60,891	723,408	123,001

Subtraction combinations

The inverse of addition is subtraction If $8 + 12 = 20$, then $20 - 8 = 12$ The number from which another number is to be subtracted (here 20) is called the *minuend* The number to be subtracted (here 8) is called the *subtrahend* The answer (here 12) is called the *difference* The difference is the *excess* of the minuend over the subtrahend

Minuend	20
Subtrahend	8
Difference	12

Those whose work requires them to make frequent subtractions become proficient at it, those who make subtractions infrequently are likely to be slow and uncertain Most students beginning the study of business need some drill on the subtraction combinations The student should run through these simple drills until he has gained a satisfactory speed

EXERCISE 1.11

Subtract the following

1.	3	7	4	9	10	5	2	6	10	3	4	2	8	7	5	9	12	8	11	8
	3	0	4	1	1	5	2	6	5	1	1	0	8	6	4	9	4	1	6	7
2.	4	11	12	9	2	4	14	10	8	6	3	6	6	9	8	12	16	10	8	5
	3	5	6	8	1	2	7	9	4	2	2	1	5	3	6	5	8	2	5	3
3.	5	12	8	10	11	9	13	10	18	10	12	10	11	9	9	11	11	7	8	6
	2	7	2	6	3	2	7	3	9	7	3	8	7	4	5	9	8	5	3	4
4.	9	11	13	9	17	11	14	10	14	7	16	12	14	12	13	7	16	13	7	15
	6	2	5	7	9	4	9	4	5	3	7	8	6	9	9	4	9	6	2	7

Standard and Austrian methods of subtraction

Either of two methods of subtraction can be used: standard or Austrian *(also called complementary) In the standard method, the process of*

taking away is emphasized. Thus 7 − 4 is thought of as leaving 3. In the Austrian method, the subtraction is carried on more in the nature of addition. Using this method one seeks to find not what 4 taken away from 7 leaves, but rather the number which must be added to 4 to give 7. Indeed we constantly face problems of this nature. If A has $345 and wants to buy a used car which costs $600, he wants to know how much *more* he must save before he can buy the car.

A cashier in a market is to be paid $7.24 out of a $10 bill. She does not deduct the $7.24 from $10 and pay the difference. Instead she determines the amount of change by saying, "And 1 is 25, and 25 is 50, and 50 is $8, and 1 is $9, and 1 is $10," and she gives you $2.76.

The advantage of one method over the other for ordinary subtraction is not sufficiently great to justify a student's giving up the method with which he is already familiar. Great facility can be gained in either method, both of which are illustrated.

Subtraction by standard method

637 − 254 = ?

	4 from 7 leaves 3.
637	5 cannot be deducted from 3, so 1 is borrowed from 6, reducing 6 to 5 and increasing the 3 to 13.
254	5 from 13 leaves 8.
383	2 from 5 leaves 3. So 637 − 254 = 383.

Subtraction by Austrian method

637 − 254 = ?

	4 plus 3 equals 7. Write the 3.
637	5 plus 8 equals 13. Write the 8.
254	The 1 from the 13 is added to the next number in the subtrahend. Thus the 2 in the number 254 is raised to 3.
383	3 must be added to 3 to make 6. Write the 3. Therefore 637 − 254 = 383.

EXERCISE 1.12

Subtract the following:

1.	2.	3.	4.	5.
10,905	98,936	83,523	26,574	130,793
7,256	75,489	24,396	24,396	65,997

6.	7.	8.	9.	10.
494,393	61,612	83,123	86,660	13,224
29,378	38,855	47,357	80,483	7,085

11.	78,824	12.	50,291	13.	46,164	14.	90,804	15.	17,583
	59,394		26,657		27,357		3,029		8,285
16.	51,963	17.	15,251	18.	69,922	19.	29,505	20.	27,367
	29,386		7,869		28,403		7,469		9,179
21.	37,063	22.	117,623	23.	34,949	24.	83,000	25.	164,624
	8,405		23,659		3,689		77,599		98,397
26.	94,500	27.	153,115	28.	97,378	29.	22,523	30.	16,813
	35,906		73,786		80,469		18,948		7,405

Verification of subtraction

It is difficult to think of a complicated problem in subtraction Usually since only two numbers are involved, the reasonableness of the answer can be appraised quickly However, an answer which appears to be reasonable may not be accurate Because inaccurate financial records are virtually worthless, the accuracy of every problem in subtraction should be checked

Subtraction can be verified in three ways (1) by addition and subtraction, (2) by casting out 9 s, and (3) by casting out 11's

Verification by addition and subtraction

In the addition and subtraction method, the subtraction is checked either by addition or by further subtraction If it is found that $11 - 9 = 2$, and that the difference 2, plus the subtrahend 9, is equal to the minuend 11, it is reasonably certain that the original subtraction is correct The answer can also be verified by subtracting the difference 2 from the minuend 11 to see if this difference 9 equals the original subtrahend A more complex example follows

Subtraction	*Addition check*	*Subtraction check*
537	283	537
254	254	283
283	537	254

Since only two numbers are involved in most subtraction problems, verification by addition can be accomplished quickly In most, if not all cases, it is the most expeditious method to use

EXERCISE 1.13

Subtract the following. Verify your answer by addition or by subtraction.

1. 434 157	2. 857 376	3. 800 769	4. 928 666	5. 980 574	6. 955 209	7. 675 456
8. 952 133	9. 967 672	10. 816 567	11. 448 259	12. 917 524	13. 8,000 6,507	14. 9,001 3,099
15. 8,007 4,118	16. 9,998 8,779	17. 5,874 3,875	18. 560,724 387,256	19. 382,274 158,746	20. 2,800,745 734,856	

Verification by casting out 9's

The same method of casting out 9's developed to verify addition can be used to check the accuracy of subtraction. The procedure is to find the difference between the excess of 9's in the minuend and the subtrahend. This difference should be the same as the excess of 9's in the answer to the problem being checked.

	Excess of 9's
537	6
254	2
283	4

The difference between 6 and 2 is the same as the excess of 9's in the answer, 283.

If the excess of 9's in the subtrahend is greater than the excess of 9's in the minuend, add 9 to the excess of 9's of the minuend before taking the difference.

EXERCISE 1.14

Subtract and prove by casting out 9's.

1. 95,493 78,824	2. 62,765 52,921	3. 47,461 35,277	4. 17,385 8,258	5. 27,736 19,079
6. 35,396 29,386	7. 15,251 7,869	8. 69,922 67,841	9. 71,211 62,322	10. 25,909 17,964
11. 37,380 28,547	12. 71,762 63,857	13. 53,786 49,497	14. 97,354 64,530	15. 77,202 69,325

Verification by casting out 11's

Subtraction as well as addition can be verified by casting out 11's The use of this method, however, has little practical application To prove subtraction by this method, proceed to find the excess of 11's in the minuend and subtrahend as explained on page 11 The difference between them should equal the excess of 11's found in the answer The following examples illustrate the method

	Excess of 11's		*Excess of 11's*
2507	10	238,741	8
856	9	127,885	10
1651	1	110,856	9

Note $8 + 11 - 10 = 9$

In each example it can be seen that the difference between the excess of 11's in the minuend and the subtrahend is equal to the excess of 11's in the answer This fact may be accepted as verification of the answer

EXERCISE 1.15

Subtract and check the differences in the following problems by casting out 11's

1. 10,442 6,874	2. 85,679 38,827	3. 728,507 86,672	4. 293,105 187,411
5. 135,004 97,837	6. 748,336 527,947	7. 278,556 149,773	8. 821,777 697,005

Horizontal subtraction

Many records, such as accounting and inventory records, are kept in columnar form A number to be subtracted appears beside the minuend rather than under it, and after the difference is computed, it is written on the same horizontal line Although most persons are slowed down disproportionately when first confronted with horizontal subtraction after a little practice they find that it is just as easy to subtract horizontally as vertically. Either the standard or the Austrian method can be used horizontally

EXERCISE 1.16

Solve the following:

1. $98{,}936 - 24{,}598 = ?$
2. $29{,}505 - 27{,}367 = ?$
3. $164{,}624 - 94{,}500 = ?$
4. $27{,}357 - 23{,}569 = ?$
5. $83{,}000 - 28{,}403 = ?$
6. $153{,}115 - 97{,}378 = ?$
7. $33{,}523 - 16{,}813 = ?$
8. $117{,}623 - 34{,}949 = ?$
9. $80{,}469 - 73{,}786 = ?$
10. $86{,}660 - 59{,}394 = ?$

Balancing an account

A combination of vertical and horizontal addition and subtraction is used to determine and verify balances in inventory records, bank accounts, and many accounting records. In the following example only deductions are made from the beginning balance. The answer is verified when it is seen that the total of the beginning balance less the total of the deductions is equal to the remaining balance.

Given the number of items of inventory at the beginning of the day and the total number of each item withdrawn during the day, if there are no additions to inventory, find the number of each item that should be on hand at the end of the day.

Item	*Inventory at the Beginning of the Day*	*Withdrawals During the Day*	*Balance at the End of the Day*
A	5,827	2,863	2,964
B	4,713	3,558	1,155
C	2,773	1,587	1,186
D	6,004	4,778	1,226
	19,317	12,786	6,531

Frequently accounts are kept on columnar paper with column headings similar to the following:

		Balance	
Debits	*Credits*	*Debit*	*Credit*

In most accounts one type of entry tends to dominate. Thus an account payable tends to have a credit balance, and an account receivable tends to have a debit balance. Here you are not concerned with the accounting practices but rather with the arithmetical techniques which are used.

To find the balance in an account when both figures are in the same column—that is, when both are credits or both are debits—add the figures

together and write the sum in the column with the same title under the heading *Balance* If the figures are in different columns on the same horizontal line—that is, if debits are in one and credits in the other, or vice versa—the smaller is subtracted from the larger and the balance written in the same column as the larger

Illustration Find the resulting balances in each of the following accounts Verify the accuracy of your work

Beginning Balance		*Changes During Period*		*Resulting Balance*	
Debit	*Credit*	*Debit*	*Credit*	*Debit*	*Credit*
$3 827 00			$2,543 00	$1,284 00	
$5,127 00			$8,432 00		$3,305 00
	$4,338 00	$6,114 00		$1,776 00	
$1 557 00		$2,086 00		$3,643 00	
	$1,326 00		$4,446 00		$5 772 00

To verify the accuracy of the results obtained, compare the totals of the balances at the beginning adjusted by the changes made during the period to the sum of the *Resulting Balances*

EXERCISE 1.17

Find the balance of each of the following accounts and verify the results

	Beginning Balance		*Changes*		*Resulting Balance*	
	Debits	*Credits*	*Debits*	*Credits*	*Debits*	*Credits*
1.	$2,125 43			$1,075 19		
2.		$3,856 30	$2,175 40			
3.		$1,764 28		$2,834 90		
4.		$2,764 19	$5,289 34			
5.	$234 83		$1,452 31			
6.	$5,917 23			$2,117 40		

Combined addition and subtraction

It is often necessary in business both to add and to subtract in one set of operations Although there are some short cuts, experience in the classroom seems to indicate that when such operations are carried on without mechanical aids it is wiser to find the sum of the items to be added and deduct from this sum the sum of the items which are to be

deducted. For example, to find the value of 387 − 238 + 467, first add 387 and 467 together (387 + 467 = 854), then find the difference between 854 and 238, namely, 616. Therefore 387 − 238 + 467 = 616.

When there are several increase items and several decrease items, time may be saved by finding: (1) the sum of the figures to be added; (2) the sum of the figures to be deducted; (3) the difference between the two.

Illustration: During the first quarter of this year, Departments A, B, C, D, E, and F showed the following changes. Find the net increase or decrease.

Department	*Increase or Decrease (—)*
A	$ 124.45
B	− 409.58
C	357.92
D	− 95.80
E	245.20
F	437.80

	$124.45	
	357.92	
	245.20	
	437.80	
The sum of the increases		$1,165.37
	$409.58	
	95.80	
The sum of the decreases		505.38
Net increase		$ 659.99

EXERCISE 1.18

Find the net balance in the following:

1.	2.	3.	4.	5.
487	2,572	87,911	$ 33,812	$ 12,452.82
− 228	− 3,714	− 55,772	− 12,427	8,332.64
372	− 1,886	27,338	− 5,116	− 9,109.47
− 529	2,237	− 13,082	22,601	− 10,098.08
113	3,117	− 8,780	3,512	1,790.20

The arithmetic problems which arise when columnar records are kept usually involve a combination of addition and subtraction. The solution to such problems involves an application of the principles developed here.

EXERCISE 1.19

Solve the following problems

The following data were taken from the inventory records of an aircraft company Calculate the inventory of stock on hand at the end of the month

	Parts Number	*Number of Items Beginning of Month*	*Number of Items Received*	*Number of Items Disbursed*	*Inventory at End of Month*
1.	A-1694	11,475	45,679	28,176	
2.	A-1695	85,464	90,248	124,404	
3.	A-1169	979,409	744 000	821,419	
4.	B-4401	24,159	12,340	25,079	
5.	B-4457	9,000	1,840	2,880	
6.	Totals				

The following data are from a bank ledger Determine the totals and the closing balance

	Depositor	*Opening Balance*	*Deposits Made*	*Checks Drawn*	*Closing Balance*
7.	Edwards	\$287 35	\$108 34	\$172 35	
8.	Jones	438 67	58 26	71 58	
9.	Knight	53 82	272 45	138 27	
10.	Richards	1,172 87	558 37	427 27	
11.	Thomas	856 88	337 29	582 11	
12.	Totals				

2

Multiplication and Division of Integers

Introduction

Most calculations in business are made by the use of machines. Indeed familiarity with the operations of a calculating machine is often an aid in obtaining an initial position in a business firm. A person with a sound understanding of the fundamental arithmetical operations and a high degree of skill in their performance is more likely to be selected for promotion to positions carrying greater responsibility.

Although a person in a position of responsibility may delegate to others the actual tasks of multiplication or division, he must be so adept with numbers that he can see a miscalculation even in a cursory study of a report. Those in top management positions must be able quickly to estimate and determine the reasonableness of the figures supplied to them. Both able managers and beginners in business must have that facility in computation which comes only from familiarity with the fundamental operations.

Multiplication of integers

Multiplication is the process of repeating or adding any given number or quantity a certain number of times, or of finding the result of such repeated additions by means of a brief computation. It is really a short way to add. For example, 4×3 is the same as $4 + 4 + 4$, and 3×4 is the same as $3 + 3 + 3 + 3$.

In multiplication, the number to be multiplied is called the *multiplicand;* the number by which the multiplicand is multiplied is called the *multiplier;* and the result is called the *product.* The distinction between the multiplicand and the multiplier is not of great importance. Observe that regardless of the order in which numbers are multiplied together the product is the same:

Multiplicand	9	3
Multiplier	3	9
Product	27	27

Each number when multiplied by one or more numbers to give a product is a *factor* of that product When only two numbers are multiplied, there are two factors, the multiplicand and the multiplier Since the product is the same regardless of the order of multiplication, it is customary to consider the smaller of the two numbers as the multiplier

Multiplication combinations

In multiplication as in addition and subtraction there are a limited number of possible combinations, commonly called the *multiplication facts*, or *multiplication tables* In order to carry out the easiest type of multiplication it is necessary to know these facts through 9×9 Review is necessary to develop your speed

EXERCISE 2.1

Complete the multiplication tables by multiplying the number at the head of each column by the number at the left, then noting the product in the proper square

	1	2	3	4	5	6	7	8	9
1.	2	4	6	?	?	?	?	?	?
2.	3	?	?	?	?	?	?	?	?
3	4	?	?	?	?	?	?	?	?
4.	5	?	?	?	?	?	?	?	?
5.	6	?	?	?	?	?	?	?	?
6.	7	?	?	?	?	?	?	?	?
7.	8	?	?	?	?	?	?	?	?
8.	9	?	?	?	?	?	?	?	?

Although any problem in multiplication can be carried out by a person who knows the multiplication facts up through 9×9, a knowledge of

other combinations is also valuable. Ordinarily the multiplication tables are memorized through 12×12. The following drill goes through 20×20.

EXERCISE 2.2

Multiply the number at the head of each column by the number at the left-hand side of the table, and write the product in the proper square.

	10	11	12	13	14	15	16	17	18	19	20
9.	11	121	132	?	?	?	?	?	?	?	?
10.	12	?	?	?	?	?	?	?	?	?	?
11.	13	?	?	?	?	?	?	?	?	?	?
12.	14	?	?	?	?	?	?	?	?	?	?
13.	15	?	?	?	?	?	?	?	?	?	?
14.	16	?	?	?	?	?	?	?	?	?	?
15.	17	?	?	?	?	?	?	?	?	?	?
16.	18	?	?	?	?	?	?	?	?	?	?
17.	19	?	?	?	?	?	?	?	?	?	?
18.	20	?	?	?	?	?	?	?	?	?	?

While you may not choose to learn all the combinations up to 20×20, it is worth while to memorize the following:

$$
\begin{array}{ll}
11 \times 11 = 121 & 16 \times 16 = 256 \\
12 \times 12 = 144 & 17 \times 17 = 289 \\
13 \times 13 = 169 & 18 \times 18 = 324 \\
14 \times 14 = 196 & 19 \times 19 = 361 \\
15 \times 15 = 225 & 20 \times 20 = 400
\end{array}
$$

Long or written multiplication

If the smaller of two numbers being multiplied exceeds 12 or some other relatively low number, the process of multiplication is usually written out. Each digit is put down as the steps of multiplication are taken. The result of multiplying the multiplier by a single digit is called a partial product.

Multiplicand	487	
Multiplier	384	
	1948	partial product
	3896	partial product
	1461	partial product
	187,008	

The partial product from the ten's digit is set over one place to the left of the product of the unit's digit In effect, multiplication by the ten's digit implies that a zero is omitted Placing the partial product of the ten's digit one place to the left, and the partial product of the hundred's digit two places to the left of the partial product of the unit's digit, compensates for the zeros which are understood but which are never written X (B28) J6 87236

EXERCISE 2.3

Find the product of the following factors

1. 243 × 127	6. 320 × 771	11. 666 × 707	16. 8,334 × 617
2. 118 × 67	7. 446 × 931	12. 1,384 × 728	17. 9,010 × 208
3. 306 × 58	8. 364 × 188	13. 8,337 × 517	18. 5,397 × 663
4. 445 × 139	9. 982 × 374	14. 2,556 × 983	19. 4,339 × 872
5. 782 × 440	10. 518 × 931	15. 2,704 × 740	20. 9,653 × 816

Multiplication by inspection

When multiplication is done infrequently, the wisest procedure usually is to follow the rules of long or written multiplication and to write down each partial product, adding them together to get the product Under certain instances, however, the product can be obtained by what may be called multiplication by inspection without writing down each step in the multiplication process These so-called short-cut methods are not of uniform value to all In the final analysis, each person must decide which, if any, of the methods illustrated he cares to use

Several simple short cuts can be used in multiplication if either factor fits a particular description

1 If one factor ends in *ciphers* (or zeros), multiply by the number exclusive of the zeros and annex as many ciphers as there are on the end of the multiplier and multiplicand

$$258 \times 300 = 258 \times 3 \times 100 = 77{,}400$$

That is, multiply 258 by 3, and annex 2 ciphers Consider such a problem as one in short rather than long multiplication

2. If one factor is a number slightly less than 100, multiply by 100, and deduct from the result the product of the multiplicand and the difference between 100 and the multiplier. For example:

$$327 \times 99 = 327 \times 100 - 327 \times 1 = 32{,}700 - 327 = 32{,}373$$

3. If one factor is a number slightly more than 100, multiply by 100 and add to the result the product of the multiplicand and the difference between the multiplier and 100. For example:

$$327 \times 102 = 327 \times 100 + 327 \times 2 = 32{,}700 + 654 = 33{,}354$$

4. If one factor is 5, the product is obtained by multiplying by 10 and dividing by 2, since $10 \div 2 = 5$. To multiply by 10, simply add a zero. For example,

$$158 \times 5 = 1{,}580 \div 2 = 790$$

5. If one factor is 25, multiply by 100 and divide by 4, since $100 \div 4 = 25$. To multiply by 100, simply add two zeros. For example,

$$438 \times 25 = 43{,}800 \div 4 = 10{,}950$$

6. If both factors are only two-digit numbers, multiplication may be carried out separately for each digit of the multiplier but only the final product be written. The following examples illustrate how some are able to use this method to save time.

Illustrations:

a. Find the product of 39×37.

$7 \times 9 = 63$.	Write 3 and carry 6.	
$7 \times 3 = 21$.	$21 + 6 = 27$.	Carry 27.
$3 \times 9 = 27$.	$27 + 27 = 54$.	Write 4 and carry 5.
$3 \times 3 = 9$.	$9 + 5 = 14$.	Write 14.
	The product is 1,443.	

b. Find the product of 56×47.

$7 \times 6 = 42$.	Write 2 and carry 4.	
$7 \times 5 = 35$.	$35 + 4 = 39$.	Carry 39.
$4 \times 6 = 24$.	$24 + 39 = 63$.	Write 3 and carry 6.
$4 \times 5 = 20$.	$20 + 6 = 26$.	Write 26.
	The product is 2,632.	

7. With a little practice you can multiply any number by an integer of 20 or less without writing down each step in the multiplication process. The following illustrations show how the steps are taken.

Illustrations

a Find the product of 327×6

$6 \times 7 = 42$	Write 2 and carry 4	
$6 \times 2 = 12$	$12 + 4 = 16$	Write 6 and carry 1
$6 \times 3 = 18$	$18 + 1 = 19$	The product is 1,962

b Find the product of 482×7

$7 \times 2 = 14$	Write 4 and carry 1	
$7 \times 8 = 56$	$56 + 1 = 57$	Write 7 and carry 5
$7 \times 4 = 28$	$28 + 5 = 33$	The product is 3 371

c Find the product of 327×14

$14 \times 7 = 98$	Write 8 and carry 9	
$14 \times 2 = 28$	$28 + 9 = 37$	Write 7 and carry 3
$14 \times 3 = 42$	$42 + 3 = 45$	The product is 4,578

The following problems are intended to give you drill in the various short-cut methods described Work them by using the appropriate method This minimum application of the methods may prove valuable to you

EXERCISE 2.4

Find the following products

1. $384 \times 100 = ?$
2. $507 \times 400 = ?$
3. $638 \times 300 = ?$
4. $427 \times 500 = ?$
5. $2,412 \times 200 = ?$
6. $6,510 \times 60 = ?$
7. $4,800 \times 3,000 = ?$
8. $6,787 \times 2,000 = ?$
9. $451 \times 900 = ?$
10. $750 \times 800 = ?$
11. $221 \times 102 = ?$
12. $871 \times 101 = ?$
13. $415 \times 104 = ?$
14. $275 \times 103 = ?$
15. $874 \times 101 = ?$
16. $1,003 \times 99 = ?$
17. $2,500 \times 98 = ?$
18. $185 \times 97 = ?$
19. $240 \times 96 = ?$
20. $3,050 \times 99 = ?$
21. $84 \times 5 = ?$
22. $290 \times 50 = ?$
23. $778 \times 5,000 = ?$
24. $754 \times 500 = ?$
25. $1,881 \times 5 = ?$
26. $946 \times 5 = ?$
27. $7,742 \times 50 = ?$
28. $870 \times 500 = ?$
29. $642 \times 5,000 = ?$
30. $5,000 \times 820 = ?$
31. $36 \times 25 = ?$
32. $154 \times 25 = ?$
33. $1,274 \times 25 = ?$
34. $6,184 \times 25 = ?$
35. $1,340 \times 25 = ?$
36. $440 \times 25 = ?$

37. $25 \times 684 = ?$
38. $250 \times 1{,}600 = ?$
39. $7{,}448 \times 250 = ?$
40. $5{,}672 \times 2500 = ?$
41. $87 \times 77 = ?$
42. $43 \times 14 = ?$
43. $27 \times 15 = ?$
44. $82 \times 13 = ?$
45. $53 \times 92 = ?$
46. $87 \times 56 = ?$
47. $84 \times 24 = ?$
48. $121 \times 6 = ?$
49. $38 \times 12 = ?$
50. $272 \times 15 = ?$

Estimated products

It is important to be able to estimate the answer to a multiplication problem—to determine approximately what the answer should be. By serving as a quick check of the exact answer, an approximation can forestall any serious error resulting from a mistake in multiplication.

In fact, one of the primary objectives to be gained from a course in business mathematics is the ability to estimate the reasonableness of an answer to a mathematical problem. Until one is capable of performing the fundamental operations quickly and accurately, he is incapable of estimating the reasonableness of an answer. Since numbers are one of the principal means of recording and communicating facts in business, anyone in a position of responsibility must be able to appraise quickly the reasonableness of any product, sum, difference, or quotient.

Estimating a product is not a substitute for actual calculation. An estimation merely furnishes some criteria on which to judge the reasonableness of the product. In dealing with large numbers, it is difficult to estimate a product unless the numbers are first modified to contain only one or two digits other than zeros. When zeros are substituted for other digits, the number is said to be *rounded.* The following rules are commonly observed in the process of rounding:

1. If the number dropped is less than 5, a zero is substituted for the number and the remaining digits are unchanged.

2. If the number dropped is more than 5, the last digit retained is increased by one unit.

For example, 4,294 rounded to the nearest 10 is 4,290;
4,294 rounded to the nearest 100 is 4,300;
4,294 rounded to the nearest 1,000 is 4,000.

3. In order to avoid cumulative errors resulting from rounding all numbers to a higher number or to a lower number, the following procedure is used if the number dropped is 5: the last digit retained, if an odd number, is raised to an even number; the last digit retained, if an even number, is not raised.

For example, 465 rounded to the nearest 10 is 460,
475 rounded to the nearest 10 is 480,
305 rounded to the nearest 10 is 300,
295 rounded to the nearest 10 is also 300

44,465 rounded to the nearest 10 is 44,460,
44,465 rounded to the nearest 100 is 44 500,
44,465 rounded to the nearest 1,000 is 44,000,
44,465 rounded to the nearest 10,000 is 40,000

45,575 rounded to the nearest 10 is 45,580,
45,575 rounded to the nearest 100 is 45,600,
45,575 rounded to the nearest 1,000 is 46,000,
45,575 rounded to the nearest 10,000 is 50,000

EXERCISE 2.5

Round the following numbers as directed

1. Round each of the following to the nearest 10 9, 127, 125, 155, 8,511
2. Round each of the following to the nearest 10 5, 285, 954, 1,966, 1,251
3. Round each of the following to the nearest 100 49, 210, 515, 949, 954, 58, 150, 12, 444, 2,500
4. Round each of the following to the nearest 1,000 389, 210, 515, 1,224, 501, 500, 35,000, 6,500
5. Round each of the following to the nearest 1,000,000 500,000, 4,500,000, 15,500,000, 1,244,923, 985,492

To estimate a product, the multiplier and the multiplicand are both rounded until each contains only one digit other than one or more zeros, then the rounded numbers are multiplied, giving the estimated product For example, find the estimated product of

Product	*Rounded Numbers*	*Estimated Product*
1,824 × 687	2,000 × 700	1,400,000
583 × 72	600 × 70	42,000
238 × 747	200 × 700	140,000

The first estimated product is obtained quickly by noting that 2 × 7 = 14 followed by 5 zeros (3 from 2,000 and 2 from 700) Thus the problem becomes one in multiplication by inspection

EXERCISE 2.6

Estimate the product in each of the following:

1. 432×845
2. $1,932 \times 5,449$
3. $7,360 \times 2,456$
4. $7,654 \times 3,678$
5. $9,825 \times 1,653$
6. $25,165 \times 876$
7. $14,632 \times 1,769$
8. $55,000 \times 54,469$
9. $44,978 \times 53,543$
10. $89,987 \times 21,789$

Verification of multiplication

In addition to understanding the basic principles of multiplication, you should gain sufficient skill to assure absolute accuracy in your work. In setting up an accounting, inventory, or any other type of control system, an effort is made to interrelate the work in such a way that a mistake made at one point is apparent when accounts or records fail to balance.

You should be confident that your work is correct. One method of assuring accuracy is to develop the practice of going over your multiplication a second time. If this method is followed consistently, other methods may not be necessary. Any one of at least three methods can be used to verify products: (1) interchanging the order of the factors; (2) casting out 9's; and (3) casting out 11's.

1. Interchanging the order of the factors is based on the *commutative law of mathematics*, which states that the multiplier and the multiplicand may be interchanged and still give the same product. This method is convenient to use only when the multiplier and the multiplicand contain approximately the same number of digits and when neither is large. If two large numbers, such as 4,975 and 3,621, are being multiplied, the interchange of the two makes the problem of verification time consuming.

2. Casting out 9's to verify multiplication is based on the same principle used to verify sums and differences.

In checking multiplication by casting out 9's, just as in checking addition, it is necessary to find the excess of 9's in each factor. In the number 632 the excess of 9's is 2. In multiplication and division the excess of 9's is usually called the *residue*. Thus in the number 632 the residue is 2.

To verify multiplication by casting out 9's, find the residue of each factor. The *product* of these residues should equal the residue of the product of the multiplier and the multiplicand.

Illustration Find the product of 488 × 384, and check by casting out 9's

		Residue
	488	2
	× 384	× 6
	187,392	12
Residue	3	3

The residues of the two factors are 2 and 6. The product of 2 and 6 is 12, the residue of 12 is 3. The residue of the product is 3. The multiplication is verified.

3. Casting out 11's to verify multiplication is also similar to casting out 11's to verify sums and differences. To check by casting out 11's, proceed as follows:

Beginning from the right of the multiplicand, first add the figures in the odd places, and then the figures in the even places. From the sum of the digits in the odd places, deduct the sum of the digits in the even places. If the first sum is smaller than the second sum, add 11 or any multiple of 11 before subtracting the sum of the figures in the even places. Call this *difference a*. Next, find the sum of the digits in the odd places and the sum of the digits in the even places of the multiplier. Deduct the sum of the digits in the even places from the sum of the digits in the odd places (if necessary, 11 or any multiple of 11 may be added before subtracting). Call this latter *difference b*. Then take the original product which is being checked, cast out all 11's by following the procedures used above.

The final difference found for the product should be equal to the product of *difference a* multiplied by *difference b* less, if necessary, some multiple of 11.

Illustrations Find the products of 487 × 384 and 5,783 × 48, and check by casting out 11's

487	$11 - 8 = 3$	
384	$7 + 11 - 8 = 10$	
187,008	$16 - 8$	$\overline{30} - 22 = 8$
	$= 8$	
5,783	$10 - 2 = 8 = 8$	
48	$8 - 4 = 4$	
277,584	$16 + 11 - 17$	$\overline{32} - 22 = 10$
	$= 10$	

EXERCISE 2.7

Find the product and verify by interchanging the multiplier and the multiplicand:

1. 386×438
2. 478×569
3. 768×942
4. 387×484
5. 548×716
6. $3{,}786 \times 4{,}578$
7. 782×603
8. 694×459
9. $9{,}436 \times 6{,}574$
10. $5{,}891 \times 3{,}760$

Multiply and verify by casting out 9's:

11. 421×299
12. $1{,}251 \times 3{,}702$
13. $1{,}694 \times 7{,}218$
14. $6{,}836 \times 1{,}742$
15. $9{,}045 \times 4{,}731$
16. $7{,}658 \times 4{,}437$
17. $8{,}456 \times 1{,}282$
18. $3{,}209 \times 7{,}321$
19. $1{,}652 \times 4{,}724$
20. $3{,}733 \times 9{,}042$

Multiply and verify by casting out 11's.

21. $7{,}639 \times 4{,}117$
22. $6{,}650 \times 7{,}209$
23. $9{,}354 \times 9{,}238$
24. $6{,}398 \times 5{,}247$
25. $7{,}410 \times 6{,}219$
26. $5{,}119 \times 6{,}991$
27. $7{,}653 \times 4{,}987$
28. $4{,}596 \times 5{,}321$
29. $5{,}891 \times 4{,}001$
30. $3{,}055 \times 4{,}777$

Division of integers

Division is the process of finding how many times one number is contained in another. It is the inverse of multiplication. For example, since $8 \times 3 = 24$, then $24 \div 8 = 3$. The number which is being divided, here 24, is called the *dividend;* the number by which the dividend is divided, here 8, is called the *divisor;* and the answer, here 3, is called the *quotient.*

$$\begin{array}{r|l} & \text{Quotient (3)} \\ \hline \text{Divisor (8)} & \text{Dividend (24)} \end{array}$$

If the product of the quotient and the divisor is less than the dividend, the difference is called the *remainder.* If the product is equal to the dividend it is said that the divisor goes into the dividend an *exact* number of times.

Since division is the inverse of multiplication, the division combinations are the inverse of the multiplication combinations. The divisor and quotient are factors of the dividend, just as the multiplier and the multiplicand are factors of the product.

Factors

An understanding of factors and the ability to recognize them is a help in many arithmetic problems, particularly those involving fractions. Numbers are classified in many ways, but from the standpoint of factoring, they are of two kinds, *prime* and *composite*. A *prime number* is one which has no whole number divisors except one and itself. Thus 1, 2, 3, 5, 7, 11, 13, 17, 19, are the prime numbers less than 20. These numbers are prime numbers since there is no whole number which can be divided into any of them without leaving a remainder.

A *composite number* is the product of two or more factors other than 1 and itself. Thus 18 is a composite number with 2 and 9 as factors, or 6 and 3 as factors. Since 9 is itself the product of 3 and 3, the *prime factors* of 18 are 2, 3, and 3 (i.e., $2 \times 3 \times 3 = 18$). Factors are generally stated in pairs or groups. Thus the factors of 24 are 12 and 2, or 3, 4, and 2, or 3, 2, 2, and 2.

Frequently it is desirable to find the factors of a number. The following rules, used to determine whether one number is exactly divisible by another, may prove helpful in finding the factors of a number.

1. Two is an exact divisor of any even integer, such as 24, 56, 124, and 326.

2. Five is an exact divisor of any integer ending in 0 or 5, such as 35, 60, 165, and 340.

3. Three is an exact divisor of any integer the sum of whose digits is exactly divisible by 3, such as 15, 18, 27, 36, and 111.

4. Any integer whose last digit is the same as the divisor and whose other digits from left to right are divisible singly or in pairs by the divisor, is divisible by the divisor. For example, the following integers are divisible by 7: 287, 3,577, 56,707, 63,217, and 3,507.

If called upon to find the factors of a number, one should determine first whether it is exactly divisible. If it is exactly divisible, the number by which it is exactly divisible is a factor.

When one factor has been determined, divide the number by that factor and proceed to find all the other factors possible in the quotient.

Illustration. Determine all pairs of factors of 48.

Dividing by 2, we have 24 and 2
Dividing by 3, we have 16 and 3
Dividing by 4, we have 12 and 4
Dividing by 6, we have 8 and 6

EXERCISE 2.8

Determine all pairs of factors (except 1) which apply to the following:

1. 27, 36, 45, 54, 72
2. 42, 52, 56, 64, 24
3. 81, 75, 32, 39, 51
4. 80, 96, 156, 182, 210
5. 289, 304, 321, 98, 144

Division by inspection

Short division, or division by inspection, is the process of finding the quotient without writing down the various steps in the division process. It amounts primarily to an application of the division combinations to relatively small numbers. A thorough knowledge of the division combinations and some drill will increase your speed in solving such problems.

EXERCISE 2.9

Solve the following by inspection.

1. $274 \div 2$
2. $275 \div 5$
3. $2{,}335 \div 5$
4. $3{,}612 \div 6$
5. $320 \div 5$
6. $2{,}793 \div 3$
7. $5{,}688 \div 8$
8. $279 \div 3$
9. $2{,}736 \div 9$
10. $2{,}135 \div 7$
11. $824 \div 8$
12. $246 \div 6$
13. $6{,}448 \div 8$
14. $2{,}824 \div 2$
15. $5{,}670 \div 7$
16. $24 \div 6$
17. $112 \div 7$
18. $1{,}208 \div 8$
19. $1{,}535 \div 5$
20. $42 \div 7$
21. $648 \div 8$
22. $620 \div 4$
23. $4{,}266 \div 6$
24. $120 \div 8$
25. $273 \div 3$
26. $728 \div 8$
27. $72 \div 9$
28. $729 \div 9$
29. $243 \div 9$
30. $658 \div 7$
31. $272 \div 8$
32. $255 \div 15$
33. $252 \div 9$
34. $610 \div 5$
35. $1{,}332 \div 12$
36. $276 \div 4$
37. $247 \div 13$
38. $4{,}788 \div 14$
39. $336 \div 7$
40. $252 \div 18$
41. $234 \div 13$
42. $171 \div 9$
43. $270 \div 15$
44. $1{,}216 \div 8$
45. $408 \div 17$

Using the rules of divisibility given, carry out the division in only the problems which are exactly divisible.

46. $2{,}100 \div 5$
47. $570 \div 3$
48. $47{,}146 \div 11$
49. $7{,}701 \div 3$
50. $9{,}369 \div 9$
51. $47{,}136 \div 8$
52. $4{,}213 \div 2$
53. $51{,}824 \div 3$
54. $5{,}467 \div 11$
55. $6{,}476 \div 6$

Long, or written, division

There are two methods of long division, standard and continental. Although both methods are demonstrated in the problems which follow, use the method with which you are already familiar. One method is not sufficiently better than the other to justify learning a new method.

1. The standard method is usually taught and used in the United States. By this method the problem 34,952 ÷ 136 usually takes the following form.

Since the dividend is 34,952 and the divisor is 136, we have

```
         257
  136 ) 34952
        272
        ---
         775
         680
         ---
          952
          952
          ---
            0
```

Steps of solution

a. 13 (of 136) goes into 34 of the dividend 2 times. Put down 2 in the quotient.

b. 2 × 136 equals 272. Subtract 272 from 349, giving 77. Bring down 5.

c. 13 (of 136) goes into 77 (of 775) 5 times. Put down 5 in the quotient.

d. 5 × 136 equals 680. Subtract 680 from 775, giving 95. Bring down 2.

e. 13 (of 136) goes into 95 (of 952) 7 times. Put down 7 in the quotient.

f. 7 × 136 gives 952. Subtract 952 from 952, giving 0. Thus the quotient is 257 and there is no remainder. That is,

$$34{,}952 \div 136 = 257, \quad \text{or} \quad 34{,}952 = 136 \times 257$$

2. The continental method of division, which has been taught from time to time in various sections of the United States, utilizes the Austrian method of subtraction. Using this method, the problem 34,952 ÷ 136 takes the following form.

```
         257
  136 ) 34952
        0775
         0952
          000
```

Steps of solution

a. 13 (of 136) times 2 is less than 34 of the dividend and 13 times 3 is greater than 34. Put down 2 in the quotient.

b. Multiply 136 by 2, but do not write down the 272. Instead, write down only the difference between 272 and 349, using the Austrian method of subtraction. This process is carried out by taking each digit separately; thus 2×6 is 12. Since 7 must be added to 12 to obtain the 9 (of 349) or really 19, write 7 under the 9.

c. 2×3 is 6, plus 1 from the 19 above, is 7. To 7 it is necessary to add 7 to get the 4 (of 349) or really 14. Write 7 under the 4.

d. 2×1 is 2, plus the 1 from the 14 above, is 3. Then to 3 add 0 to get the 3 (of 349). Write 0 under the 3.

e. Bring down the 5 from the dividend and write it after the 077, giving 0775.

f. 13 (of 136) times 5 is less than 77 (of 0775), and 13 times 6 is greater than 77. Put down 5 in the quotient. In multiplying $136 \times 5 = 680$, do not write the 680. Again using the Austrian method of subtraction, write only the difference between 680 and 775. This is done a step at a time as was done earlier.

g. 5×6 is 30, plus 5 is 35. Write the 5 under the 5.

h. 5×3 is 15, plus the 3 of 35, plus 9 is 27. Write 9 under the 7.

i. 5×1 is 5, plus the 2 of 27, plus 0 is 7. Write 0 under the 7.

j. Bring down 2, making 952.

k. 13 (of 136) times 7 is less than 95 (of 952), and 13 times 8 is greater than 95. Put down 7 in the quotient.

l. 7×6 is 42; 2 plus 0 is 2. Write 0 under the 2.

m. 7×3 is 21, plus the 4 of 42, plus 0 is 25. Write the 0 under the 5.

n. 7×1 is 7, plus the 2 of 25, plus 0 is 9. Write 0 under the 9.

EXERCISE 2.10

Find the quotient, using either the standard or the continental method of division.

1. $8,366 \div 47$
2. $5,830 \div 55$
3. $2,250 \div 75$
4. $2,106 \div 27$
5. $7,560 \div 36$
6. $1,078 \div 22$
7. $4,680 \div 32$
8. $9,301 \div 131$
9. $5,830 \div 110$
10. $9,675 \div 225$
11. $78,912 \div 18$
12. $769,045 \div 185$
13. $387,068 \div 926$
14. $392,413 \div 919$
15. $384,794 \div 457$
16. $863,010 \div 2,007$
17. $498,774 \div 2,571$
18. $984,340 \div 2,765$
19. $895,860 \div 2,765$
20. $673,266 \div 2,222$

The remainder

When the divisor is not a factor of the dividend there will be a remainder. The remainder may be expressed as a common fraction with the remainder as the numerator and the divisor as the denominator or as a decimal fraction. In the following problem solved by the standard method the remainder is shown as a common fraction, then as a decimal fraction.

1. Remainder as a common fraction

```
       83
32 ) 2660
     256
      100
       96
        4 (the remainder)
```

That is $2{,}660 \div 32 = 83\frac{4}{32} = 83\frac{1}{8}$

2. Remainder as a decimal fraction. Put a decimal point after the last digit to the right in both the dividend and the quotient and continue the division process.

```
       83.125
32 ) 2660.000
     256
      100
       96
        40
        32
         80
         64
         160
         160
```

That is $2{,}660 \div 32 = 83.125$

When these two remainders ($\frac{1}{8}$ and 0.125) are compared, it is seen that $\frac{1}{8} = 0.125$. This kind of equality is discussed in the chapter on decimal *fractions*.

EXERCISE 2.11

Divide, finding the integral quotient and stating the remainder, if any, as a common fraction and as a decimal fraction.

1. $55{,}832 \div 64$
2. $19{,}224 \div 224$
3. $21{,}818 \div 128$
4. $32{,}110 \div 520$
5. $59{,}675 \div 110$
6. $769{,}119 \div 185$
7. $387{,}529 \div 926$
8. $674{,}377 \div 2{,}222$
9. $987{,}844 \div 3{,}872$
10. $424{,}011 \div 17{,}421$

Verification of division

Two methods commonly used to check division are multiplication and casting out 9's. In checking by multiplication, the quotient is multiplied by the divisor, and the remainder, if any, is added to this product.

Illustration: Divide 2,687 by 32 and check by multiplication.

$$2{,}687 \div 32 = 83\tfrac{31}{32}$$

Check: $83 \times 32 =$ 2,656
Add the remainder 31
2,687. Since this equals the original dividend, the quotient is verified.

To check division by casting out 9's, find the residues of the dividend, divisor, quotient, and remainder. Find the product of the residues of the quotient and the divisor. The residue of this product, plus the residue of the remainder, should equal the residue of the dividend.

Illustration: Divide 4,597 by 32, and check by casting out 9's.

$$\begin{array}{r|l} & \;\;143 \\ \hline 32 & 4597 \\ & 32 \\ \hline & 139 \\ & 128 \\ \hline & \;\;117 \\ & \;\;\;\;96 \\ \hline & \;\;\;\;21 \end{array}$$

The residues are:

Dividend	7
Divisor	5
Quotient	8
Remainder	3

The product of the residues of the divisor and the quotient is (5×8) 40.

The residue of 40 is	4
Add the residue of the remainder	3
The sum is	7

The residue of the dividend is 7. Therefore the answer checks.

By a similar process not illustrated here, division can be checked by casting out 11's.

EXERCISE 2.12

Divide and verify by multiplication.

1. $6{,}381 \div 9$
2. $9{,}301 \div 131$
3. $8{,}729 \div 203$
4. $9{,}540 \div 36$
5. $7{,}839 \div 13$
6. $1{,}320 \div 22$
7. $13{,}041 \div 27$
8. $19{,}040 \div 224$
9. $11{,}376 \div 237$
10. $20{,}064 \div 132$

Divide and verify by casting out 9's

11. 1,765 − 42	16. 12,800 − 360
12. 1,564 − 47	17. 16,960 − 320
13. 1,280 − 32	18. 28,514 − 218
14. 7,416 − 24	19. 176,866 − 526
15. 1,559 − 11	20. 159,427 − 391

Finding an average

Often in business and accounting, it is necessary to determine the arithmetic average of a series of numbers (each number is called a *term* of the series). An average is found by adding all the terms together and dividing by the number of terms.

Illustration. Find the average of 28, 34, 43, 82, 55, 27, and 32.

Since $28 + 34 + 43 + 82 + 55 + 27 + 32 = 301$, and $\frac{301}{7} = 43$, the average of these seven terms is 43.

Averages are used often in business in making estimations. For example, a company which has a fleet of five cars finds that during the last quarter they were driven 1,200, 1,300, 800, 2 000, and 1,700 miles, respectively. The total distance driven was 7,000 miles, so the average distance was 1,400 miles. There was no car which was driven exactly the average number of miles. On the basis of total expenses the average cost per mile can be computed. When sufficient past data are known, the task of estimating or controlling future expenditures may be simplified.

EXERCISE 2.13

Find the average of the following:

1. 98, 76, 45, 78, 48
2. 2,874, 5,732, 4,116, 1,804, 7,279
3. 7,382, 1,127, 1,123, 1,075
4. 972, 948, 882, 867, 968
5. 1,122, 1,234, 1,308, 1,269, 1,568

Extracting a square root

A challenging type of division entails finding a divisor for a number which is equal to the quotient. This process, known as extracting the square root, can be done relatively easily by the use of logarithms, or by the use of a slide rule. It can also be done by arithmetic.

When a number is multiplied by itself it is said to be squared. If the number is a whole number with no decimals the product is said to be a perfect square. The following multiplication shows the relationship of the square of a two-place number to the product.

39	
39	
81	This partial product is the square "of the last digit 9."
27	This is the product of 9 and 3.
27	This is the product of 3 and 9.
9	This is the square of the first digit 3.
1,521	This is the square of 39.

The product 1,521 contains the square of the first digit (3) in the number, twice the product of the two digits in the number, and the square of the second digit (9).

If this process of squaring a number is reversed, an understanding of the process of extracting the square root can be more easily understood. If the number whose square root is wanted is separated into groups of two figures each, beginning at the decimal point, the number of groups will determine the number of digits in the square root. For example, if a number less than 10 is squared there will be only two digits to the left of the decimal point. If any number more than 10, but less than 100, is squared there will be two sets of digits to the left of the decimal point. To extract a square root, proceed as follows:

Step 1. Separate the number into groups of two figures each beginning at the decimal point.

15 21. 00

Step 2. Find the largest number whose square is contained in the first set of digits.

```
      3
   |‾‾‾‾‾‾‾‾‾‾‾
   | 15  21. 00
 3 |  9
   |___________
```

Step 3. Subtract the square of the number and bring down the next pair of digits.

```
      3
   |‾‾‾‾‾‾‾‾‾‾‾
   | 15  21. 00
 3 |  9
    ___________
      6  21
```

Step 4. From the relationship previously examined it is known that the remainder here consists of the sum of the square of the second digit,

and twice the product of the first and second digits For this reason double the root already found, 3, and annex a zero This given 60 as a trial divisor

Step 5 Add to this trial divisor the number selected as the estimated quotient For example, if 8 is selected as the estimated quotient, the trial divisor is increased from 60 to 68 and when multiplied by 8 gives 514 Since 514 is much less than 621, try 9 as the estimated quotient When 9 is added to the trial divisor of 60 the result is 69, which when multiplied by 9 gives the desired product 621

	3	9	
	15	21	00
3	9		
	6	21	
69	6	21	

Thus the square root of 1,521 is 39

The symbol $\sqrt{}$, called the *radical sign*, is used to indicate the extraction of a root We have just seen that the square root of 1,521 is 39 It can be written as 39, or it can be indicated as $\sqrt{1{,}521}$ The number which appears under the radical sign (here 1,521) is called the *radicand* If only the radical symbol is used, it indicates the square root To indicate the cube root of a number, such as 27, the same symbol is used, but a small number, called the *index* or *order of the root* is written in the *v* of the radical sign $\sqrt[3]{27}$

The method used in the preceding illustration may be used to find the square root of any number The following illustration is given in more detail

Illustration Find $\sqrt{82{,}656.25}$

Step 1 Separate the number into groups of two figures each, beginning at the decimal point and moving in both directions In this illustration the number would be separated 8 26 56 25

Step 2 Find the largest integer whose square is less than or equal to the first group of digits on the left In this problem the first group of digits contains only one number (8) The largest integer whose square is contained in 8 is 2 since the square of 2 is 4 and the square of 3 is 9 Subtract the square from the first group of digits

	2			
	8	26	56	25
2	4			
	4			

Step 3. Bring down the next pair of digits (here 26) and write them after the difference just calculated: 4 26.

$$\begin{array}{r|l} & 2 \\ \hline & 8\ 26\ 56.\ 25 \\ 2 & 4 \\ \hline & 4\ 26 \end{array}$$

Step 4. Double the first number obtained (here 2) and write it to the left of the group of digits (here 4 26). Then

$$\begin{array}{r|l} & 2 \\ \hline & 8\ 26\ 56.\ 25 \\ 2 & 4 \\ \hline 4 & 4\ 26 \end{array}$$

Step 5. It is necessary to select a digit which, when it is written to the right of the doubled digit ($2 \times 2 = 4$) and multiplied by itself, does not exceed the group of digits (here 426). If 7 is tried, it can be seen that $47 \times 7 = 329$. Try 9: $49 \times 9 = 441$. This is too large, so try 8. $48 \times 8 = 384$. Subtract this product.

$$\begin{array}{r|l} & 2\ 8 \\ \hline & 8\ 26\ 56.\ 25 \\ 2 & 4 \\ \hline & 4\ 26 \\ 48 & 3\ 84 \\ \hline & \quad 42\ 56. \end{array}$$

Bring down the next pair of digits.

Step 6. Double the answer so far obtained (here 28): $28 \times 2 = 56$.

Step 7. To the right of the product or sum (here 56), write a digit so that the three-place number so obtained will, when multiplied by the last digit added, give either the number (here 42 56) or a number slightly lower. By trial it can be seen that 6 is too small and 8 too large. Here select 7. $567 \times 7 = 3{,}969$. Subtract this product.

$$\begin{array}{r|l} & 2\ 8\ \ 7 \\ \hline & 8\ 26\ 56.\ 25 \\ 2 & 4 \\ \hline & 4\ 26 \\ 48 & 3\ 84 \\ \hline & \quad 42\ 56. \\ 567 & \quad 39\ 69. \\ \hline & \quad\ \ 2\ 87.\ 25 \end{array}$$

Step 8 Bring down the next group of digits—25. Since the last group of digits brought down is to the right of the decimal point in this illustration, a decimal point is now placed in the answer. (Since it is the last group, it may contain only one digit.)

Step 9 Double the amount found so far (here 287): 287 × 2 = 574.

Step 10 Select a digit which when written after the product or sum (here 574) and is used as a multiplier of the number which results, does not exceed the number (here 287 25). By trial, 5 is selected: 5745 × 5 = 287 25. Subtract this product.

$$\begin{array}{r} 28725 \\ \underline{28725} \\ 0 \end{array}$$

There is no remainder. Since there is no remainder, $\sqrt{82{,}656.25}$ is 287.5.

The problem need not be written step by step, but can appear as follows:

$$\sqrt{82{,}656.25} \qquad \begin{array}{r|l} & 8\,26\,56.25 \quad 287.5 \\ 2 & 4 \\ \hline & 4\,26 \\ 48 & 3\,84 \\ \hline & 42\,56 \\ 567 & 39\,69 \\ \hline & 2\,87\,25 \\ 5745 & 2\,87\,25 \\ \hline \end{array}$$

This answer may be verified by showing that 287.5 times itself equals 82,656.25.

EXERCISE 2.13

Find the following:

1. $\sqrt{163}$
2. $\sqrt{1.815}$
3. $\sqrt{16.129}$
4. $\sqrt{117{,}136}$
5. $\sqrt{50.176}$
6. $\sqrt{67.21}$
7. $\sqrt{1{,}189.96}$
8. $\sqrt{52.9984}$
9. $\sqrt{83.9856}$
10. $\sqrt{715.6644}$

REVIEW PROBLEMS

Chapters 1 and 2

Add and check.

1.	2.	3.	4.	5.
3,782	4,782	8,187	4,273	3,324
816	5,127	5,332	3,082	5,082
332	3,837	7,873	5,903	7,172
1,827	8,371	2,297	7,272	6,083
583	5,282	836	8,387	792
2,407	478	4,128	5,112	4,206

6.	7.	8.	9.	10.
48,287	82,723	14,209	428,331	113,882
33,229	51,312	83,057	587,082	416,508
8,453	82,771	41,229	48,816	37,822
7,112	53,227	3,778	382,009	616,743
82,036	8,406	40,039	528,317	82,387
40,037	31,228	8,337	402,807	6,293

11.	12.	13.	14.	15.
37,832	45,803	52,117	78,930	6,783
14,772	70,056	18,883	7,003	23,651
9,082	21,713	6,115	32,988	18,673
23,776	9,991	33,827	458	3,227
918	43,886	2,227	3,802	10,809
5,907	4,988	23,762	13,756	9,067
28,224	25,556	4,556	4,228	6,005
21,212	43,009	12,668	15,030	11,227
56,337	1,104	5,786	3,109	8,256
3,334	8,786	2,117	4,873	19,569

Subtract the following, and check.

16.	17.	18.	19.	20.
3,872	8,273	7,106	4,336	2,348
2,695	5,437	4,382	3,879	1,596

21.	22.	23.	24.	25.
43,327	81,116	41,239	21,443	438,336
18,632	63,827	38,837	16,827	382,574

26.	27.	28.	29.	30.
17,923	41,223	11,004	238,942	498,452
8,096	40,875	9,872	67,568	392,987

Find the estimated and exact products of the following, and check.

31.	32.	33.	34.	35.
3,827	18,127	3,229	12,326	4,827
372	4,007	487	812	3,206

36. 2,782	37. 1,327	38. 4,307	39. 27,822	40. 43,517
538	422	593	483	83
41. 18,422	42 56,932	43. 28,403	44. 132,882	45. 774 912
3,086	1,245	10,706	21,831	5,002

Find the estimated and exact quotients of the following and check

46. 28,782 — 1,562
47. 31,753 — 872
48. 16 993 — 224
49. 26,268 — 332
50. 83 812 — 4,475
51. 238,917 — 45,037
52. 386,881 — 27,238
53. 616,542 — 3,896
54. 314,816 — 25 600
55. 296,782 — 18,545

56. $\frac{78,912}{872}$ 57. $\frac{33,558}{4,325}$ 58. $\frac{46,005}{2,225}$ 59. $\frac{2,652}{345}$ 60. $\frac{328,556}{8,762}$

Find the balance of each of the following accounts and verify the results

61.

Debits	*Credits*	*Debits*	*Credits*
\$3 872 28	\$2,548 36		
2,638 41	4,662 78		
5,882 06	8,116 42		
7,338 20	5,816 68		
1,556 82	3,114 48		

62.

Debits	*Credits*	*Debits*	*Credits*
\$18,482 32	\$21,693 67		
9,822 90	11,568 21		
12,338 27	9,092 08		
6,552 43	4,916 75		
13,784 92	13,628 44		
5 443 89	5,452 65		

63.

Debits	*Credits*	*Debits*	*Credits*
\$8,927 36	\$7,552 48		
5,723 06	9,213 27		
2,032 93	4,507 48		
7,337 80	7,828 72		
5,213 27	3,726 40		
8,338 65	6,827 58		
3,029 63	3,458 77		
2,906 04	6,763 09		

Find the net increase or decrease of the following.

64.	65.	66.	67.	68.	69.
3,627	2,782	5,827	\$ 4.87	\$ 38.28	\$ 18.06
−2,563	1,807	−4,117	−5.38	−67.48	−12.28
4,228	−3,508	−2,776	−9.27	−45.43	8.33
−3,117	−1,848	4,808	11.35	15.74	−27.36
1,001	2,193	3,698	14.28	9.07	5.55
			−6.83	−5.84	−6.24

70. $4.87 − 5.86 + 13.72 − 8.45 + 6.34 − 4.33 = ?

71. Last year the Alpha Corporation earned $1,631,975. This was equivalent to $1.45 for each common share. Find the number of common shares.

72. During the last six months Best and Company earned $48,030. This was equivalent to $1.25 for each common share. Find the number of common shares.

73. An acre is a measure of area containing 4,840 square yards. How many square feet in an acre?

74. There are 8 salesmen in Department B of the Outlet Company. If their total sales for a 5-day week were $5,080, what were the average sales per day per salesman?

75. In a previous problem it was found that an acre contains 43,560 square feet. For comparing land costs the value is often stated on a square foot basis. Find the cost per square foot for land quoted at $8,712 per acre; $15,246.00 per acre.

76. On a used-car lot there are 243 motor cars with a total value of $308,124. What is the average price of each car?

77. The monthly sales in a certain store were as follows: January, $17,148; February, $18,219; March, $20,483; April, $22,382; May, $21,052; and June, $21,114. Find the total sales for the first six months of the year, and find the average monthly sales.

78. Find the average of the following amounts: $827.37; $553.23; $82.36; $197.48; $333.56; $67.24; and $411.16.

79. Find the average of the following amounts: $12,782.06; $15,997.24; $8,728.44; $11,338.86; and $7,083.33.

80. In one year the mortgages on 13,571 farms were foreclosed. The estimated value of these mortgages was $41,994,962.79. Find the average amount of each mortgage foreclosed.

81. In one year *Lloyd's Register of Ships* reports that there were 29,763 ships with a gross tonnage of 68,509,430. Find the average gross tonnage per ship.

82. How many square feet are there in a rectangular area that is 156 inches long and 132 inches wide?

83. The balance in A's account at the end of March was $816 78 During the month of April he made deposits totaling $408 65 and wrote checks amounting to $946 13 If the balance of his account shown on the bank statement at the end of April was $497 24, find the amount of the checks he had written which were still outstanding

84. Richard Cook's account showed a balance at the end of May of $382 27 During the month of June he made deposits totaling $243 18 and wrote checks amounting to $416 13 If his balance shown on the bank statement at the end of June was $316 45, find the amount of the checks he had written which were still outstanding

85. Tuition at a university is $21 per unit The 9,875 students take on the average 15 units each semester During the academic year of two semesters, how much in tuition should the business office collect?

86. Tuition at a university is $13 50 per unit The 8,215 students take on the average 15 units each quarter During the academic year of three quarters, how much in tuition should the business office collect?

87. A stamping machine makes 40 contacts a minute Each time it makes a contact it stamps a machine part In calculating the time to produce 225,000 units, how many stamping machines must be used if the job is to be finished in one 40-hour week?

88. Each of a group of employees of an artificial flower making shop can make a flower in 30 seconds An order comes in for 1,200 dozen flowers How many employees must be assigned to the job to finish the order in three 8-hour working days?

89. The newsstand sales of a metropolitan newspaper for 5 days last year totaled 324,815 papers For the corresponding 5 days this year, the newsstand sales totaled 337,450 papers Find the gain in average daily newsstand sales

90. A merchant s inventory at the beginning of the year was $13,151 60 and at the end of the year it was $17,827 25 Purchases during the year were $105,551 81 His administrative and selling expenses for the year were $8,740 75, and all other expenses totalled $3,794 21 If his total sales were $113,847 65, how much did he gain or lose?

91. A truck which empty weighs 8,250 pounds weighed 20,206 pounds when loaded with steel Z beams 5 by 3 inches Such beams weigh 14 pounds per lineal foot How many feet of beams were on the truck?

92. Daily sales in the men's clothing department of Barbee's department store averaged $1,625 If 5 salesmen are employed, what are the average daily sales per person?

93. Robinson's Department store had total sales last year of \$29,080,680. Find the average monthly sale.

94. A timber 12 × 24 inches weighs 80 pounds per foot. How much should a timber 6 × 12 inches weigh per foot?

95. There are 24 grains in 1 pennyweight, and 20 pennyweights in 1 ounce. How many grains in an ounce?

96. If gold is worth \$35 an ounce and there are 12 ounces in a pound, how much is gold per pound?

97. If gold is worth \$35 an ounce and a gold brick weighs 400 ounces, how much is a gold brick worth?

98. An independent druggist rents 1,600 square feet of floor space in a downtown location. His minimum annual rental is \$10,200. How much is his monthly rental per square foot?

99. The price of quicksilver is \$295 for a 76-pound flask. Find the equivalent price per pound.

100. At the beginning of the year a merchant had an inventory of goods which had cost him \$21,782.00. During the year his purchases amounted to \$86,751.00. At the end of the year he had merchandise on hand valued at \$23,781.00. What was the cost of the goods he had sold?

101. A buyer in a department of the May Company expects March sales to be \$24,000. The retail value of the inventory in the department at the beginning of the month is \$4,200. The buyer wants to decrease the inventory to \$3,600 by the end of the month. If the monthly sales reach the expected volume, how much merchandise must be bought at retail value by the department during the month?

102. The Acme Department Store had a retail stock worth \$21,600 on January 1. It wants to reduce its stock to \$14,000 by the end of June. Sales during the first quarter totaled \$31,000 while purchases at retail valuation were \$34,000. What is the value of the present inventory? How much must sales exceed purchases if the inventory is to stand at the desired level by the end of June?

103. How many 1-inch cubes can be placed in a box with inside dimensions of 4 inches wide, 5 inches deep, and 6 inches long?

104. Add:

	18 hours	28 minutes	36 seconds
	46 hours	48 minutes	54 seconds
	27 hours	14 minutes	17 seconds

105. Add:

	5 yards	4 feet	11 inches
	7 yards	2 feet	9 inches
	5 yards	2 feet	7 inches

3

Common Fractions

Introduction

Since commodities sold at retail are usually sold in relatively small quantities, prices are generally stated only in dollars and cents; fractional parts of cents are not used. Even though a price for several units is stated so that a fractional part of a cent may be involved, prices of single units are not calculated with such precision.

When large numbers of units are involved, however, the price per unit is stated with greater precision. For example, in the grain market where sales are customarily made in units of 5,000 bushels, the price per bushel is quoted in dollars, cents, and fractional parts of a cent. Price changes are customarily recorded in "points," each equivalent to $\frac{1}{8}$ of a cent.

Common fractions

When precision is needed in business and industry, it is often necessary to use not only whole numbers but also values of less than a single unit. In a sense, these values which are less than a unit can be thought of as parts of numbers, called *common fractions*. If a quantity is divided into three equal parts, for instance, these parts are called thirds. The fraction $\frac{1}{3}$ represents one of these parts, $\frac{2}{3}$ represents two of them, and $\frac{3}{3}$ represents all three parts, or the whole quantity.

A common fraction is an indicated quotient of two whole numbers, such as $\frac{1}{2}$ or $\frac{2}{3}$ or $\frac{4}{9}$. The term in a fraction which indicates the number of fractional units taken is called the *numerator*; it is the number written above the line. The number which shows into how many equal parts the unit is divided is called the *denominator*; it is the number written below the line. A *proper fraction* is one whose numerator is less than its denominator, such as $\frac{1}{3}$ or $\frac{3}{8}$.

Improper fractions

Any fraction with a numerator greater than the denominator is called an *improper fraction.* Thus such fractions as $\frac{5}{3}$, $\frac{11}{7}$, and $\frac{23}{8}$ are improper fractions. Since an improper fraction really represents a whole number plus a part of a number, it may be changed to a proper fraction plus an integer. A combination of a proper fraction and an integer is called a *mixed number;* for example, $2\frac{1}{2}$ and $5\frac{1}{4}$ are mixed numbers.

To change an improper fraction into a mixed number, divide the numerator by the denominator and write the result as a whole or mixed number.

Illustration: Change $\frac{21}{8}$, $\frac{5}{3}$, and $\frac{12}{6}$ to mixed numbers.
$\frac{21}{8} = 2\frac{5}{8}$; $\frac{5}{3} = 1\frac{2}{3}$; and $\frac{12}{6} = 2$, a whole number.

To change a mixed number into an improper fraction, multiply the whole number by the denominator of the fraction, add the numerator of the fraction, and write the result over the denominator of the fraction.

Illustration: Change the following mixed numbers to improper fractions: $2\frac{3}{8}$; $3\frac{7}{12}$; and $4\frac{2}{3}$.

$$2\frac{3}{8} = \frac{2 \times 8 + 3}{8} = \frac{19}{8}; \quad 3\frac{7}{12} = \frac{3 \times 12 + 7}{12} = \frac{43}{12}; \quad \text{and}$$

$$4\frac{2}{3} = \frac{4 \times 3 + 2}{3} = \frac{14}{3}.$$

EXERCISE 3.1

Change the following improper fractions into mixed numbers:

1. $\frac{8}{5}$	2. $\frac{15}{7}$	3. $\frac{11}{4}$	4. $\frac{16}{5}$	5. $\frac{7}{3}$
6. $\frac{27}{4}$	7. $\frac{35}{8}$	8. $\frac{42}{9}$	9. $\frac{48}{6}$	10. $\frac{64}{16}$

Change the following mixed numbers into improper fractions:

11. $2\frac{1}{3}$	12. $3\frac{2}{5}$	13. $3\frac{5}{8}$	14. $2\frac{3}{7}$	15. $5\frac{2}{3}$
16. $2\frac{5}{12}$	17. $1\frac{9}{16}$	18. $4\frac{2}{9}$	19. $6\frac{1}{4}$	20. $4\frac{4}{15}$

Reduction of fractions

In the arithmetic processes of addition and subtraction only like numbers can be used. Fractions with unlike denominators cannot be added or subtracted until they are converted to fractions with a common denominator. A fraction such as $\frac{5}{10}$ has the same value as the fraction $\frac{1}{2}$. The process of changing a fraction from one denominator to another is referred to as "reducing" or "changing" the fraction although there is no actual change in its value.

To reduce a fraction to a lower term, divide both the numerator and the denominator by a common factor Thus $\frac{7}{21}$ can be reduced to $\frac{1}{3}$ by dividing both the numerator and the denominator by 7 The fraction $\frac{144}{256}$ can be reduced to $\frac{18}{32}$ by dividing both the numerator and the denominator by 8, and the resulting fraction can be reduced further to $\frac{9}{16}$ by dividing both 18 and 32 by 2

To change a fraction to a higher denominator, multiply the numerator and the denominator by the same number Thus to change $\frac{3}{4}$ to 16ths, multiply both the numerator and the denominator by 4, to change $\frac{11}{16}$ to 32nds, multiply both the numerator and the denominator by 2

EXERCISE 3.2

Reduce the following fractions to lowest terms

1. $\frac{2}{4}$, $\frac{6}{8}$, $\frac{5}{10}$, $\frac{15}{20}$, $\frac{27}{33}$
2. $\frac{48}{64}$, $\frac{25}{75}$, $\frac{35}{60}$, $\frac{128}{168}$, $\frac{36}{360}$
3. $\frac{68}{72}$, $\frac{34}{144}$, $\frac{42}{156}$, $\frac{65}{169}$, $\frac{36}{96}$
4. $\frac{5}{25}$, $\frac{20}{48}$, $\frac{65}{90}$, $\frac{72}{116}$, $\frac{36}{72}$
5. $\frac{18}{32}$, $\frac{25}{30}$, $\frac{80}{100}$, $\frac{12}{16}$, $\frac{169}{234}$
6. Change $\frac{1}{8}$ to 32nds, $\frac{1}{4}$ to 12ths
7. Change $\frac{1}{5}$ to 15ths, $\frac{1}{10}$ to 40ths
8. Change $\frac{2}{7}$ to 14ths, $\frac{2}{3}$ to 42nds
9. Change $\frac{3}{8}$ to 40ths, $\frac{5}{9}$ to 45ths
10. Change $\frac{5}{12}$ to 72nds, $\frac{9}{16}$ to 64ths
11. Change $\frac{3}{4}$ into a fraction of the same value whose denominator is 8, 12, 20, 28, 64, 100
12. Change $\frac{2}{3}$ into a fraction of the same value whose denominator is 6, 9, 15, 18, 24, 27, 30, 36
13. Change $\frac{5}{6}$ into a fraction of the same value whose denominator is 12, 18, 24, 36, 42, 54, 66, 78
14. Change $\frac{3}{5}$ into a fraction of the same value whose denominator is 10, 20, 30, 40, 75, 80, 90, 100
15. Change $\frac{1}{2}$ into a fraction of the same value whose denominator is 14, 32, 78, 144, 236, 582

Multiples

A *multiple* of a number is the product of that number and any integer. The multiples of 2 are 4, 6, 8, 10, 12, etc The multiples of 3 are 6, 9, 12, 15, 18, etc The multiples of 4 are 8, 12, 16, 20, 24, etc

From these examples it can be seen that 12 is a multiple of 2, 3, and 4, and that 12 and 24 are multiples of both 3 and 4 A factor common to two or more numbers is called a *common multiple*

The *lowest common multiple* of two or more numbers is the smallest number which contains each of the given set of numbers as factors Thus

the lowest common multiple of 3 and 4 is 12; the lowest common multiple of 2 and 4 is 4; the lowest common multiple of 2, 3, 6, and 9 is 18.

Frequently it is desirable to change fractions to a common denominator. To keep the amount of work at a minimum it is desirable to use the smallest possible denominator, or as it is usually referred to, the *lowest common denominator* (L.C.D.). This is the smallest number into which each denominator can be divided exactly and consequently is the same as the lowest common multiple of the given denominators.

Finding the lowest common denominator

One method of finding the lowest common denominator is to express each denominator in terms of its prime factors or numbers. To find the L.C.D. of the fractions $\frac{3}{8}$, $\frac{7}{12}$, $\frac{1}{6}$, and $\frac{2}{3}$, first state the prime factors of the denominators.

The prime factors of 8 are 2, 2, and 2 $(2 \times 2 \times 2 = 8)$.
The prime factors of 12 are 2, 2, and 3 $(2 \times 2 \times 3 = 12)$.
The prime factors of 6 are 2 and 3 $(2 \times 3 = 6)$.
The fourth denominator, 3, is already a prime factor, 3.

Although the product of 4×2 is 8, the number 4 is not a prime factor of 8 since 4 is the product of two factors, 2 and 2. On the other hand, such numbers as 5, 7, 11, 13, 19, 23, and 37 are prime numbers.

Once each denominator has been expressed in terms of its prime factors, the lowest common denominator (or the lowest common multiple of the denominators) is readily found as the product of each combination of prime factors. That is, the lowest common denominator must contain each prime factor the greatest number of times it is contained in any one denominator. Thus the lowest common denominator of 8, 6, 12, and 3 must contain three 2's, since $2 \times 2 \times 2$ contains all possible combinations of 2's found in the prime factors of the four denominators. The lowest common denominator must contain only one 3 as a prime factor since in none of the denominators is the factor 3 found more than once. Thus the L.C.D. must be $2 \times 2 \times 2 \times 3$, or 24.

A somewhat different procedure for finding a L.C.D., but one employing the same principles, is to write all the denominators in a line as separate numbers. Then divide through the denominators with a prime number. If the prime number used as a divisor is not a factor of any one or more of the denominators, copy the denominator, or denominators, on the next line along with the quotients. (In the following illustration note the 3 in the first division). Repeat this process until no other prime number except 1 can be used as a divisor of at least two denomina-

tors Then find the product of all divisors and all remaining quotients For example

$$\begin{array}{r|rrrr} 2 & 8 & 12 & 6 & 3 \\ \hline 2 & 4 & 6 & 3 & 3 \\ \hline 3 & 2 & 3 & 3 & 3 \\ \hline & 2 & 1 & 1 & 1 \end{array}$$

The L C D is therefore

$$2 \times 2 \times 3 \times 2 \times 1 \times 1 \times 1 = 24$$

EXERCISE 3 3

Find the lowest common multiple of the following

1. 4, 6, 9, 12
2. 4, 9, 12, 15
3. 6, 10, 15, 18
4. 2, 5, 6, 9
5. 8, 12, 18, 24
6. 7, 14, 21, 28
7. 12, 16, 20, 25
8. 3, 4, 9, 12, 18
9. 5, 12, 15, 18, 24
10. 4, 8, 10, 12, 15

Comparing fractions

To facilitate the comparison of fractions with one another, change them to fractions with a common denominator It is difficult to deter mine whether $\frac{9}{16}$ is greater than $\frac{7}{12}$ until they are both changed to 48ths Then it is seen that $\frac{9}{16}$ is equal to $\frac{27}{48}$ while $\frac{7}{12}$ is equal to $\frac{28}{48}$ When two fractions have the same denominator, the numerator determines which is of the greater magnitude, hence it is seen that $\frac{7}{12}$ is greater than $\frac{9}{16}$

EXERCISE 3.4

Arrange the following fractions in the order of their magnitude, beginning with the smallest

1. $\frac{5}{8}, \frac{3}{4}, \frac{7}{12}, \frac{2}{3}, \frac{7}{10}$
2. $\frac{3}{8}, \frac{5}{12}, \frac{7}{16}, \frac{1}{3}, \frac{3}{10}$
3. $\frac{9}{16}, \frac{7}{12}, \frac{5}{8}, \frac{2}{3}, \frac{3}{5}$
4. $\frac{3}{20}, \frac{1}{12}, \frac{5}{36}, \frac{3}{16}, \frac{1}{5}$
5. $\frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}$

Arrange the following fractions in the order of their magnitude, beginning with the largest

6. $\frac{7}{8}, \frac{11}{12}, \frac{13}{16}, \frac{17}{24}$
7. $\frac{7}{12}, \frac{9}{16}, \frac{11}{20}, \frac{13}{24}$
8. $\frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{7}{9}$
9. $\frac{5}{12}, \frac{2}{5}, \frac{3}{8}, \frac{1}{3}, \frac{4}{9}$
10. $\frac{1}{8}, \frac{2}{15}, \frac{3}{20}, \frac{4}{25}, \frac{5}{32}$

Addition of common fractions

It is only infrequently that many fractions need to be added or subtracted in business and accounting. Usually the problems encountered are relatively simple. If a person understands the real meaning of fractions he will experience little difficulty in using them. The simplest sort of addition problem is of the type $\frac{3}{5} + \frac{1}{5}$. If one remembers that $\frac{3}{5}$ represents 3 parts of something which has been divided into 5 parts, and the $\frac{1}{5}$ represents one of these parts, he will have no difficulty in seeing that the sum of the two is $\frac{4}{5}$.

When the denominators of two or more fractions to be combined are not the same, it is necessary first to convert them to fractions with a common denominator. To find the sum of fractions having the same denominator, add the numerators and use the sum as the numerator of a new fraction whose denominator is the common denominator.

Illustration: Find the sum of $\frac{3}{8} + \frac{7}{12} + \frac{1}{6} + \frac{2}{3}$.

The lowest common denominator is found to be 24. Changing each fraction to 24ths.

$$\frac{3}{8} + \frac{7}{12} + \frac{1}{6} + \frac{2}{3} = \frac{9}{24} + \frac{14}{24} + \frac{4}{24} + \frac{16}{24} = \frac{43}{24} = 1\frac{19}{24}$$

In many problems dealing with fractions, the answer found may have a denominator larger than necessary. In such a case the answer should ordinarily be stated in its simplest form—that is, reduced to lowest terms.

When only two fractions are to be added, the basic principle followed is exactly the same as the one described for adding any number of fractions. When only two fractions are to be added, however, the rules outlined can be followed easily by the following procedure:

1. Multiply the numerator of each fraction by the denominator of the other and add the products. This sum is the numerator of the sum of the fractions.

2. The denominator of the two fractions is the product of the two denominators.

3. Reduce the results to lowest terms.

Illustrations:

a. Find the sum of $\frac{2}{5} + \frac{3}{7}$.

Under the ordinary procedure.

$$\frac{2}{5} = \frac{14}{35}; \quad \frac{3}{7} = \frac{15}{35}; \quad \frac{14 + 15}{35} = \frac{29}{35}$$

Using the method just outlined, the numerator of the sum is equal to

$2 \times 7 + 3 \times 5 = 29$, and the denominator of the sum is $5 \times 7 = 35$ The answer is therefore $\frac{29}{35}$ The answer cannot be reduced

b Find the sum of $\frac{3}{8} + \frac{5}{12}$

$$\frac{3}{8} + \frac{5}{12} = \frac{3 \times 12 + 5 \times 8}{8 \times 12} = \frac{76}{96} = \frac{76 - 4}{96 - 4} = \frac{19}{24}$$

c Find the sum of $\frac{3}{8} + \frac{11}{16}$

$$\frac{3}{8} + \frac{11}{16} = \frac{3 \times 16 + 11 \times 8}{8 \times 16} = \frac{136}{128} = 1\frac{8}{128} = 1\frac{1}{16}$$

Using the ordinary procedure,

$$\frac{3}{8} = \frac{6}{16}, \quad \frac{6}{16} + \frac{11}{16} = \frac{17}{16} = 1\frac{1}{16}$$

In this third illustration it can be seen that the ordinary procedure is shorter

EXERCISE 3.5

Add and reduce to lowest terms

1. $\frac{1}{3} + \frac{3}{4}$
2. $\frac{3}{8} + \frac{7}{12}$
3. $\frac{3}{5} + \frac{2}{9}$
4. $\frac{5}{9} + \frac{1}{12}$
5. $\frac{1}{6} + \frac{1}{4}$
6. $\frac{7}{12} + \frac{3}{8} + \frac{1}{3}$
7. $\frac{7}{8} + \frac{3}{5} + \frac{1}{10}$
8. $\frac{5}{8} + \frac{2}{3} + \frac{1}{12}$
9. $\frac{13}{60} + \frac{7}{12} + \frac{9}{20}$
10. $\frac{5}{8} + \frac{7}{12} + \frac{7}{18}$
11. $\frac{2}{3} + \frac{1}{9} + \frac{1}{6} + \frac{1}{4}$
12. $\frac{3}{16} + \frac{1}{32} + \frac{3}{8} + \frac{1}{4}$
13. $\frac{1}{5} + \frac{12}{30} + \frac{3}{15} + \frac{1}{3}$
14. $\frac{7}{20} + \frac{11}{25} + \frac{7}{10} + \frac{3}{5}$
15. $\frac{7}{8} + \frac{11}{32} + \frac{3}{4} + \frac{1}{16}$

Addition of mixed numbers

To add mixed numbers it is necessary to apply the rules for the addition of integers as well as the rules for the addition of fractions, since a mixed number such as $2\frac{3}{5}$ indicates the addition of a whole number (2) plus a fraction ($\frac{3}{5}$) (i e, $2\frac{3}{5} = 2 + \frac{3}{5}$) The addition of mixed numbers may be carried out by first adding the whole numbers, then adding the fractions, and finally combining the sums obtained

Illustration Find the sum of $2\frac{3}{5} + 3\frac{5}{6} + 1\frac{5}{8} + 7\frac{4}{15}$

Adding the integers

$2 + 3 + 1 + 7 = 13$

Adding the fractions

$$\frac{3}{5} + \frac{5}{6} + \frac{5}{8} + \frac{4}{15} = \frac{72 + 100 + 75 + 32}{120} = \frac{279}{120} = 2\frac{39}{120} = 2\frac{13}{40}$$

Combining the two sums

$13 + 2\frac{13}{40} = 15\frac{13}{40}$

EXERCISE 3.6

Add and reduce to lowest terms:

1. $3\frac{2}{3} + 2\frac{1}{6} + 2\frac{4}{9}$
2. $4\frac{1}{4} + 2\frac{5}{6} + 1\frac{1}{3}$
3. $7\frac{3}{8} + 4\frac{7}{12} + 2\frac{5}{24}$
4. $14\frac{9}{16} + 8\frac{5}{12}$
5. $2\frac{5}{8} + 3\frac{1}{2} + 5\frac{1}{6}$
6. $4\frac{3}{8} + 1\frac{1}{12} + \frac{9}{16}$
7. $3\frac{5}{12} + \frac{5}{16} + \frac{3}{8}$
8. $2\frac{2}{7} + \frac{9}{14} + 3\frac{1}{2}$
9. $5\frac{1}{3} + \frac{4}{9} + \frac{11}{6}$
10. $4\frac{1}{12} + \frac{5}{8} + 3$
11. $1\frac{7}{40} + 3\frac{9}{16} + \frac{7}{10}$
12. $\frac{7}{12} + 1\frac{5}{8} + 3$
13. $\frac{8}{13} + 2\frac{8}{39} + 9\frac{1}{3}$
14. $5\frac{1}{4} + 4 + 7\frac{2}{7}$
15. $\frac{21}{8} + 2\frac{5}{6} + 5$
16. $\frac{17}{6} + \frac{13}{8} + 2\frac{3}{4}$
17. $\frac{7}{12} + \frac{21}{16} + 5$
18. $3\frac{2}{7} + 4 + \frac{17}{14}$
19. $5\frac{3}{10} + 8\frac{7}{20} + 41\frac{4}{15}$
20. $8\frac{4}{9} + 5\frac{11}{18} + \frac{1}{6}$

Subtraction of common fractions

When one fraction is subtracted from another, first the L.C.D. must be determined. Then each denominator must be changed to the L.C.D., and the numerator of each fraction changed accordingly. The subtraction can then be made, either horizontally or vertically, by deducting the numerator of the subtrahend from the numerator of the minuend and writing the difference over the lowest possible denominator.

Illustration: Find the difference $\frac{3}{10} - \frac{2}{7}$.

$$\frac{3}{10} - \frac{2}{7} = \frac{21 - 20}{70} = \frac{1}{70}$$

EXERCISE 3.7

Subtract the following:

1. $\frac{7}{12} - \frac{3}{8}$
2. $\frac{11}{16} - \frac{7}{20}$
3. $\frac{13}{24} - \frac{7}{16}$
4. $\frac{5}{8} - \frac{1}{6}$
5. $\frac{7}{18} - \frac{4}{15}$
6. $\frac{1}{2} - \frac{1}{4}$
7. $\frac{7}{8} - \frac{5}{16}$
8. $\frac{3}{4} - \frac{3}{5}$
9. $\frac{5}{8} - \frac{1}{3}$
10. $\frac{7}{10} - \frac{3}{8}$
11. $\frac{1}{2} - \frac{1}{3}$
12. $\frac{3}{4} - \frac{3}{5}$
13. $\frac{9}{10} - \frac{4}{5}$
14. $\frac{7}{8} - \frac{3}{16}$
15. $\frac{5}{6} - \frac{3}{4}$
16. $\frac{3}{5} - \frac{1}{6}$
17. $\frac{1}{2} - \frac{3}{7}$
18. $\frac{7}{12} - \frac{1}{2}$
19. $\frac{5}{9} - \frac{1}{3}$
20. $\frac{2}{3} - \frac{1}{4}$

Subtraction of mixed numbers

To subtract one mixed number from another, find the difference of the fractional parts of the minuend and the subtrahend, and find the difference of the integer parts of the minuend and the subtrahend. The sum of these two differences is the answer desired.

$$4\frac{7}{12} - 2\frac{3}{8} = 4 - 2 + \frac{7}{12} - \frac{3}{8} = 2 + \frac{14 - 9}{24} = 2\frac{5}{24}$$

Frequently the fractional part of the minued is less than the fractional part of the subtrahend, for example, in the problem $287\frac{1}{3} - 37\frac{5}{8}$, the $\frac{1}{3}$ is smaller than the $\frac{5}{8}$ In such a case take 1 from the integer part of the minuend and add it to the fractional part of the minuend, making an improper fraction Then subtract these new mixed numbers

$$287\tfrac{1}{3} - 37\tfrac{5}{8} = 287 - 37 + \tfrac{1}{3} - \tfrac{5}{8}$$

Since $\frac{1}{3}$ is smaller than $\frac{5}{8}$, take 1 (or $\frac{3}{3}$) from 287 and add it to $\frac{1}{3}$, making $\frac{4}{3}$ Then

$$287\frac{1}{3} - 37\frac{5}{8} = 286 - 37 + \frac{4}{3} - \frac{5}{8} = 249 + \frac{32 - 15}{24} = 249\frac{17}{24}$$

EXERCISE 3.8

Subtract the following

1. $2\frac{5}{8} - 1\frac{1}{4}$	6. $83\frac{11}{14} - 72\frac{6}{7}$	11. $33 - 18\frac{3}{8}$
2. $5\frac{5}{12} - 2\frac{3}{16}$	7. $127\frac{1}{4} - 38\frac{2}{3}$	12. $47\frac{5}{8} - 32$
3. $9\frac{7}{12} - 5\frac{3}{8}$	8. $18\frac{7}{24} - 15\frac{9}{16}$	13. $27\frac{7}{16} - 25$
4. $5\frac{7}{8} - 3\frac{2}{3}$	9. $16 - 3\frac{7}{12}$	14. $18 - 17\frac{9}{20}$
5. $4\frac{7}{15} - 2\frac{3}{5}$	10. $27 - 22\frac{11}{16}$	15. $81\frac{3}{8} - 80\frac{7}{12}$

Multiplication with common fractions

Multiplication with common fractions may be multiplication of a fraction by a whole number ($\frac{1}{2} \times 5$), multiplication of an integer by a fraction ($6 \times \frac{1}{4}$), or a fraction multiplied by a fraction ($\frac{1}{2} \times \frac{1}{4}$)

In the discussion of multiplication it was suggested that multiplication was a simple substitute for addition, thus $\frac{1}{2} \times 5$ means the same as adding $\frac{1}{2}$ five times, which we see is equal to $2\frac{1}{2}$ Writing both 5 and $\frac{1}{2}$ as fractions, we have $\frac{1}{2} \times \frac{5}{1}$ Multiplying both numerators together and both denominators together, we have $\frac{5}{2}$ or $2\frac{1}{2}$

Under the commutative law of mathematics, the multiplier and the multiplicand can be interchanged and still give the same product Thus we know that $5 \times \frac{1}{2}$ is equal to $2\frac{1}{2}$ In effect, $5 \times \frac{1}{2}$ means that 5 is to be added not once, but a fractional part of a whole—in this case only $\frac{1}{2}$ a time

The multiplication of fractions is not always indicated clearly as multiplication For example, "Find $\frac{1}{2}$ of 10" implies multiplication, just as $\frac{1}{2}$ of $\frac{2}{3}$ implies multiplication Multiplying a fraction by a fraction, such as $\frac{1}{2} \times \frac{2}{3}$, can perhaps be understood better if it is thought of as $\frac{1}{2}$ of $\frac{2}{3}$ Then it is not difficult to understand that $\frac{1}{2}$ of $\frac{2}{3}$ is equal to $\frac{1}{3}$, or $\frac{1}{2}$ of the original quantity ($\frac{2}{3}$)

You should understand the meaning of the multiplication of fractions as well as know that the general rule is to multiply the numerators together to find the numerator of the product, and to multiply the denominators together to find the denominator of the product.

Illustrations:

a. Find $\frac{1}{3}$ of $\frac{4}{5}$.

$$\frac{1}{3} \times \frac{4}{5} = \frac{4}{15}$$

b. Find the product of $\frac{1}{3} \times \frac{2}{7} \times \frac{5}{9}$.

$$\frac{1}{3} \times \frac{2}{7} \times \frac{5}{9} = \frac{1 \times 2 \times 5}{3 \times 7 \times 9} = \frac{10}{189}$$

EXERCISE 3.9

Find the following products:

1. $5 \times \frac{3}{4}$	4. $8 \times \frac{5}{12}$	7. $\frac{2}{3} \times 21$	10. $\frac{1}{7} \times 6$	13. $\frac{8}{9} \times \frac{3}{4} \times \frac{3}{8}$
2. $4 \times \frac{2}{3}$	5. $8 \times \frac{3}{16}$	8. $\frac{5}{16} \times 12$	11. $\frac{1}{2} \times \frac{1}{4}$	14. $\frac{3}{5} \times \frac{1}{6} \times \frac{4}{9}$
3. $8 \times \frac{5}{6}$	6. $\frac{1}{2} \times 9$	9. $\frac{1}{9} \times 81$	12. $\frac{3}{16} \times \frac{4}{5}$	15. $\frac{7}{8} \times \frac{5}{6} \times \frac{2}{3}$

Cancellation

Since the numerator and the denominator of a fraction may both be divided by the same number without changing the value of a fraction, any factor of the numerator and any factor of the denominator may also be divided by the same number without changing the value of the fraction. Multiplication can often be simplified by carrying out such division before multiplying.

For example, it is possible to find $\frac{2}{3}$ of $\frac{9}{10}$ by multiplying $\frac{2 \times 9}{3 \times 10}$ and reducing the product to lowest terms. If, however, the numerator and the denominator are written as prime factors, the problem becomes $\frac{2 \times 3 \times 3}{3 \times 2 \times 5}$. Since there are common factors in the numerator and the denominator, both the numerator and the denominator may be reduced by dividing 2 into one factor of the numerator and one factor of the denominator, and 3 into one factor of the numerator and one factor of the denominator. Leaving the quotients obtained from the division as factors of the numerator and the denominator, we reduce the problem to $\frac{1 \times 1 \times 3}{1 \times 1 \times 5} = \frac{3}{5}$.

This process of dividing a factor of the numerator and a factor of the denominator by a common divisor, or factor, is called *cancellation.* It usually is written in the following form:

$$\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}} \times \frac{\overset{3}{\cancel{9}}}{\underset{5}{\cancel{10}}} = \frac{3}{5}$$

That is, 2 is divided into one factor of the numerator, and the quotient (1, in this case) is substituted for the factor When 10 in the denominator is divided by the 2, the quotient of 5 is written in place of the 10 in the denominator Since 3 is a factor of 9 in the numerator and 3 in the denominator, the quotients 3 and 1, respectively, are substituted for the original factors Once all cancellation has been carried out, the remaining quotients in the numerator are multiplied by any uncanceled factors in the numerator to form the numerator of the product, and the remaining quotients in the denominator and any uncanceled factors in the denominator are multiplied together to form the denominator of the product

Illustrations Find the product

a $\frac{5}{6} \times \frac{9}{35} = \frac{\overset{1}{\cancel{5}}}{\underset{2}{\cancel{6}}} \times \frac{\overset{3}{\cancel{9}}}{\underset{7}{\cancel{35}}} = \frac{3}{14}$

b $\frac{5}{8} \times \frac{4}{5} \times \frac{3}{4} = \frac{\overset{1}{\cancel{5}}}{8} \times \frac{\overset{1}{\cancel{4}}}{\underset{1}{\cancel{5}}} \times \frac{3}{\underset{1}{\cancel{4}}} = \frac{3}{8}$

EXERCISE 3.10

Multiply the following, using cancellation whenever possible

1. $\frac{9}{16} \times \frac{7}{12} \times \frac{8}{21}$	6 $\frac{3}{7} \times \frac{14}{27} \times \frac{1}{3}$
2. $\frac{5}{8} \times \frac{4}{9} \times 3$	7. $10 \times \frac{2}{3} \times \frac{1}{30}$
3. $18 \times \frac{5}{6} \times \frac{2}{3}$	8 $\frac{7}{8} \times \frac{5}{16} \times \frac{4}{14}$
4. $\frac{7}{26} \times \frac{5}{21} \times \frac{3}{4}$	9. $\frac{3}{4} \times \frac{8}{9} \times \frac{5}{12}$
5. $\frac{8}{5} \times \frac{15}{32} \times \frac{4}{3}$	10. $\frac{7}{12} \times \frac{9}{16} \times \frac{4}{21}$

Multiplication of mixed numbers

When the multiplier or the multiplicand, or both, are mixed numbers, the product may be found either (a) by changing the mixed numbers into improper fractions and multiplying, or (b) by observing the rules for the multiplication of whole numbers and the rules for the multiplication of fractions In other words, it is necessary to multiply as follows if the mixed numbers are not put into improper fraction form first

1. Find the product of the fractions in the multiplier and the multiplicand.

2. Find the product of the integer in the multiplicand and the fraction in the multiplier, if there is one.

3. Find the product of the fraction in the multiplicand, if there is one, and the integer in the multiplier.

4. Find the product of the integers in the multiplier and the multiplicand.

5. Find the sum of the partial products.

Illustrations: Find the following products:

a. $2\frac{1}{4} \times 1\frac{3}{5}$

First method

$$2\frac{1}{4} \times 1\frac{3}{5} = \frac{9}{\underset{1}{\cancel{4}}} \times \frac{\overset{2}{\cancel{8}}}{5}$$

$$= \frac{18}{5} = 3\frac{3}{5}$$

Second method

$$\begin{array}{ll} 2\frac{1}{4} & \\ 1\frac{3}{5} & \\ \hline \frac{3}{20} & (\frac{3}{5} \times \frac{1}{4} = \frac{3}{20}) \\ 1\frac{1}{5} & (\frac{3}{5} \times 2 = 1\frac{1}{5}) \\ \frac{1}{4} & (\frac{1}{4} \times 1 = \frac{1}{4}) \\ 2 & (1 \times 2 = 2) \\ \hline \end{array}$$

$3\frac{3}{5}$, since $\frac{1}{4} + \frac{1}{5} + \frac{3}{20} = \frac{3}{5}$

b. $24\frac{7}{8} \times 15\frac{3}{4}$

First method

$$24\frac{7}{8} \times 15\frac{3}{4} = \frac{199}{8} \times \frac{63}{4}$$

$$= \frac{12{,}537}{32}$$

$$= 391\frac{25}{32}$$

Second method

$$\begin{array}{ll} 24\frac{7}{8} & \\ 15\frac{3}{4} & \\ \hline \frac{21}{32} & (\frac{3}{4} \times \frac{7}{8} = \frac{21}{32}) \\ 18 & (\frac{3}{4} \times 24 = 18) \\ 13\frac{1}{8} & (\frac{7}{8} \times 15 = 13\frac{1}{8}) \\ 360 & (15 \times 24 = 360) \\ \hline \end{array}$$

$391\frac{25}{32}$, since $\frac{1}{8} + \frac{21}{32} = \frac{25}{32}$

c. Often cancellation can be used.
Find the product of $\frac{9}{4} \times \frac{8}{5} \times \frac{5}{12}$

$$\frac{9}{4} \times \frac{8}{5} \times \frac{5}{12} = \frac{\overset{3}{\cancel{9}}}{\underset{1}{\cancel{4}}} \times \frac{\overset{\overset{1}{\cancel{2}}}{\cancel{8}}}{\underset{1}{\cancel{5}}} \times \frac{\overset{1}{\cancel{5}}}{\underset{\underset{2}{\cancel{6}}}{\cancel{12}}} = \frac{3}{2} = 1\frac{1}{2}$$

EXERCISE 3.11

Find the following products

1. $2\frac{3}{8} \times 3\frac{1}{5}$	6. $35 \times 3\frac{2}{7}$	11. $5\frac{1}{3} \times 3\frac{3}{8}$
2. $125 \times 1\frac{9}{25}$	7. $30 \times 3\frac{2}{5}$	12. $16\frac{2}{3} \times 39\frac{1}{2}$
3 $72 \times 2\frac{3}{8}$	8. $27\frac{1}{4} \times 32\frac{1}{3}$	13. $17\frac{1}{5} \times 18\frac{1}{43}$
4. $2\frac{3}{11} \times 3\frac{6}{25}$	9. $8\frac{7}{12} \times 5\frac{1}{3}$	14. $24\frac{7}{8} \times \frac{95}{97}$
5. $2\frac{5}{6} \times 3\frac{3}{7}$	10. $48\frac{1}{2} \times 3\frac{2}{3}$	15. $331\frac{1}{2} \times 12\frac{148}{221}$

Division of common fractions

It is helpful in understanding division by fractions to review the relationship between the multiplication of fractions and the division of whole numbers To find $\frac{1}{2}$ of 6 implies multiplication of $6 \times \frac{1}{2}$ The answer is 3 We know that if 6 is divided by 2 the quotient is 3 Upon examination it is seen that the two problems are basically the same

$$6 \times \tfrac{1}{2} \text{ is the same as } 6 \times 1 \div 2, \text{ or } 6 \div 2 = 3$$
$$6 \div 2 = 3$$

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Indeed, one number written above another in fractional form is often used to indicate division Thus $6 \div 2$ might be written $\frac{6}{2}$ In mathematics the fraction $\frac{1}{2}$ is said to be the reciprocal of 2 *A reciprocal of a number is 1 divided by the number* Thus the reciprocal of 2 is $\frac{1}{2}$, the reciprocal of 3 is $\frac{1}{3}$, the reciprocal of 4 is $\frac{1}{4}$ The product of a number and its reciprocal is always 1 For example

$$4 \times \tfrac{1}{4} = 1$$
$$3 \times \tfrac{1}{3} = 1$$
$$2 \times \tfrac{1}{2} = 1$$

We have already shown that the quotient obtained by dividing one number by a second number $(6 \div 2 = 3)$ is the same as the product obtained by multiplying the first number by the reciprocal of the second $(6 \times \frac{1}{2} = 3)$

Does the same relationship hold for fractions? The relationship is the same *The reciprocal of a fraction is the fraction formed by interchanging the numerator and the denominator—that is, by inverting the fraction* Thus the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, of $\frac{3}{4}$ is $\frac{4}{3}$, of $\frac{2}{5}$ is $\frac{5}{2}$

The product obtained by multiplying a fraction by its reciprocal is 1 For example, $\frac{2}{3} \times \frac{3}{2} = 1$, $\frac{3}{4} \times \frac{4}{3} = 1$

In the division of common fractions, use is made of this knowledge of reciprocals since the division of common fractions is carried out by multiplying the dividend by the reciprocal of the divisor This fundamental rule has three general applications since the division of common

fractions includes: (1) dividing a common fraction by an integer; (2) dividing an integer by a common fraction; and (3) dividing one common fraction by another.

1. To divide a common fraction by an integer, multiply the fraction by the reciprocal of the integer, and simplify the resulting fraction, if possible.

Illustration: $\frac{5}{8} \div 4 = \frac{5}{8} \times \frac{1}{4} = \frac{5}{32}$.

Such problems are not difficult to understand. Do not accept the rule you have learned as an explanation of the solution. You can reason out the solution as follows. From our understanding of division we know that in this problem we are asked to take $\frac{5}{8}$ of a whole (1) and divide it into four equal portions. We know that $\frac{5}{8}$ is equal to $\frac{20}{32}$. If $\frac{20}{32}$ are broken into four equal parts, each part will be $\frac{5}{32}$.

2. To divide an integer by a common fraction, multiply the integer by the reciprocal of the common fraction and simplify the product.

Illustration: $4 \div \frac{5}{8} = \frac{4}{1} \times \frac{8}{5} = \frac{32}{5} = 6\frac{2}{5}$.

The quotients of the problems of division worked up to this point have always been smaller than the dividend. Here the quotient of $6\frac{2}{5}$ is larger than the dividend (4). In this problem one is asked how many portions of $\frac{5}{8}$ of a unit each are to be found in four units. We know that in four units there is a total of $\frac{32}{8}$. If these $\frac{32}{8}$ are divided into groups of 5 each, there are 6 complete groups and a remainder of $\frac{2}{5}$.

3. To divide one common fraction by another, invert the divisor and multiply, or, in other words, multiply the dividend by the reciprocal of the divisor.

Illustration: $\frac{7}{8} \div \frac{3}{4} = \frac{7}{\underset{2}{\cancel{8}}} \times \frac{\overset{1}{\cancel{4}}}{3} = \frac{7}{6} = 1\frac{1}{6}$.

The basic question in this problem is how many $\frac{3}{4}$ are there in $\frac{7}{8}$. When the problem is stated as how many $\frac{6}{8}$ are there in $\frac{7}{8}$ it is not difficult to see that there are 1 and $\frac{1}{6}$ in the dividend. By multiplying the dividend by the reciprocal of the divisor, $\frac{7}{8} \times \frac{4}{3}$, the same answer is more readily obtained.

Division of mixed numbers

To divide one mixed number by another, change the mixed numbers to improper fractions and proceed as in the division of common fractions.

Illustration Find the quotient of $12\frac{1}{2} \div 3\frac{1}{3}$

$$12\tfrac{1}{2} = \tfrac{25}{2}, \quad 3\tfrac{1}{3} = \tfrac{10}{3}$$

$$\frac{25}{2} \div \frac{10}{3} = \frac{\overset{5}{\cancel{25}}}{2} \times \frac{3}{\underset{2}{\cancel{10}}} = \frac{15}{4} = 3\tfrac{3}{4}$$

EXERCISE 3.12

Find the following quotients

1. $\frac{1}{4} \div \frac{1}{3}$	11. $12 \div \frac{3}{4}$	21. $11\frac{2}{3} \div 7$
2. $\frac{5}{8} \div \frac{3}{4}$	12. $6 \div \frac{1}{3}$	22. $12\frac{3}{5} \div 9$
3. $\frac{7}{12} \div \frac{1}{3}$	13. $\frac{1}{4} \div 5$	23. $48\frac{1}{3} \div 15$
4. $\frac{4}{9} \div \frac{7}{12}$	14. $\frac{3}{8} \div 16$	24. $21\frac{3}{5} \div 27$
5. $\frac{9}{16} \div \frac{7}{24}$	15. $\frac{1}{2} \div 20$	25. $135 \div 5\frac{5}{8}$
6. $\frac{3}{8} \div \frac{3}{4}$	16. $2 \div 4\frac{1}{2}$	26. $35 \div 2\frac{2}{3}$
7. $\frac{3}{4} \div \frac{13}{40}$	17. $3\frac{1}{3} \div 2\frac{2}{9}$	27. $56 \div 1\frac{3}{7}$
8. $\frac{5}{6} \div \frac{4}{5}$	18. $5\frac{1}{8} \div 1\frac{3}{5}$	28. $26 \div 2\frac{2}{3}$
9. $4 \div \frac{1}{2}$	19. $7\frac{1}{2} \div 5$	29. $129 \div 5\frac{3}{8}$
10. $6 \div \frac{2}{3}$	20. $5 \div 5\frac{5}{8}$	30. $49\frac{1}{2} \div 3\frac{2}{3}$

Complex fractions

When division is indicated by writing the dividend over the divisor, and when both dividend and divisor are common fractions, the indicated division gives rise to what are known as *complex fractions*—that is, a fraction which has a fraction for the numerator and a fraction for the denominator, such as $\frac{\frac{2}{5}}{\frac{8}{15}}$

Before further computations can be carried out, it is usually necessary to eliminate or to simplify complex fractions Inasmuch as one number written above another number indicates division, a complex fraction is usually simplified by carrying out the indicated division, observing the rules for the division of fractions Thus to simplify the fraction $\frac{\frac{2}{5}}{\frac{8}{15}}$, multiply the numerator by the reciprocal of the denominator

$$\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{5}}} \times \frac{\overset{3}{\cancel{15}}}{\underset{4}{\cancel{8}}} = \frac{3}{4}$$

Illustration: Simplify $\dfrac{\frac{7}{24}}{\frac{21}{36}}$

$$\frac{\frac{7}{24}}{\frac{21}{36}} = \frac{\cancel{7}}{\underset{2}{\cancel{24}}} \times \frac{\overset{\overset{1}{\cancel{3}}}{\cancel{36}}}{\underset{\cancel{7}}{\cancel{21}}} = \tfrac{1}{2}$$

Complex fractions are said to be compound if the numerator and denominator themselves are made up of expressions capable of solution. Thus $\dfrac{\frac{1}{2} \times \frac{1}{4}}{\frac{1}{3} \div \frac{3}{8}}$ is considered a compound complex fraction. Such an expression is simplified by performing the indicated operations in the numerator and in the denominator before carrying out the division.

Illustration: $\dfrac{\frac{1}{2} \times \frac{1}{4}}{\frac{1}{3} \div \frac{3}{8}} = \dfrac{\frac{1}{8}}{\frac{1}{3} \times \frac{8}{3}} = \dfrac{\frac{1}{8}}{\frac{8}{9}} = \frac{1}{8} \times \frac{9}{8} = \frac{9}{64}$.

EXERCISE 3.13

Simplify the following:

1. $\dfrac{\frac{9}{16}}{\frac{13}{20}}$
2. $\dfrac{\frac{3}{8}}{\frac{7}{12}}$
3. $\dfrac{\frac{7}{16}}{\frac{11}{24}}$
4. $\dfrac{\frac{9}{16}}{\frac{5}{8}}$
5. $\dfrac{2\frac{5}{8}}{3\frac{1}{2}}$
6. $\dfrac{5\frac{5}{12}}{6\frac{1}{2}}$
7. $\dfrac{\frac{5}{6}}{\frac{15}{36}}$
8. $\dfrac{2\frac{3}{16}}{2\frac{5}{8}}$
9. $\dfrac{\frac{1}{2} + \frac{3}{4}}{\frac{3}{8} - \frac{1}{4}}$
10. $\dfrac{\frac{7}{9} \times \frac{1}{2}}{\frac{1}{3}}$
11. $\dfrac{\frac{1}{4} \times \frac{1}{5}}{\frac{1}{3} \div \frac{1}{4}}$
12. $\dfrac{\frac{4}{5} \times \frac{10}{11}}{\frac{1}{5} \div \frac{5}{3}}$
13. $\dfrac{\frac{5}{8}}{\frac{7}{12} + \frac{3}{4}}$
14. $\dfrac{\frac{1}{2}}{\frac{3}{4} - \frac{1}{2}}$
15. $\dfrac{\frac{2}{5} \div \frac{3}{7}}{\frac{5}{7} \times \frac{3}{8}}$

4

Decimal Fractions

Introduction

In order to help management in the control of expenditures accountants—particularly cost accountants—determine what a standard or reasonable expenditure should be for each process and material Then the total cost of each operation or material—it might be in a manufacturing plant, a laundry, a hospital, etc —is divided by the number of times the operation has been performed, or the number of units into which the material has been divided By comparing the actual cost with the predetermined standard management can readily determine what departments are not giving a satisfactory performance, can appraise the desirability of changing processes or can measure the amount of savings which may result if certain changes are made Like most other calculations which must be made in financial commercial, and industrial enterprises these calculations must be made with great precision

Though common fractions give greater precision than whole numbers a fraction with a large numerator or denominator is unwieldy unless the denominator is a power of ten If the denominator is a power of ten such as 10, 100 1 000 etc, the fraction can be written as a *decimal fraction*—commonly called a decimal—by writing the numerator after a dot called the *decimal point* The denominator is not written, but its value is indicated by the number of digits appearing in the decimal Thus $\frac{7}{10}$ is written 0 7, and $\frac{23}{100}$ as 0 23, but in writing $\frac{7}{100}$ as a decimal it is necessary to insert a zero to the right of the decimal point since *a decimal fraction must contain as many digits to the right of the decimal point as there would be zeros in the denominator if the decimal were written as a common fraction* Thus the common fraction $\frac{15}{1000}$ in decimal form is written 0 015

Kinds of decimals

A common fraction written as a decimal fraction is called a *pure decimal.* Thus $\frac{9}{10}$ when written as 0.9 is a pure decimal.* If the number were $97\frac{9}{10}$ or any other whole number and a fraction, a decimal could still be used to represent the fractional part by writing the whole number, followed by a decimal point and then the decimal fraction; thus $97\frac{9}{10}$ may be written 97.9 and referred to as a *mixed decimal.*

Fundamental operations with decimals

When a common fraction is written as a decimal fraction, the four fundamental operations—addition, subtraction, multiplication, and division—can be carried on just as if it were a whole number. Care must be taken to locate the decimal point properly.

Addition of decimals

Decimals and mixed decimal numbers are added in the same way as integers are added, except that the decimal points must be in a vertical column when the vertical addition method is used.

Illustrations:

$$\text{a.}\quad \begin{array}{r} 0.528 \\ 0.318 \\ \underline{0.287} \\ 1.133 \end{array} \qquad\qquad \text{b.}\quad \begin{array}{r} 2.584 \\ 3.121 \\ \underline{5.108} \\ 10.813 \end{array}$$

Often the decimal part of one number is not so long as the decimal parts of other numbers being added. In this case, if they are exact numbers, either add zeros (0) sufficient to make all decimal parts the same, or assume that the zeros are there.

$$\begin{array}{l} 2.5837 \\ 3.2500 \\ \underline{4.1080} \\ 9.9417 \end{array} \qquad\qquad \begin{array}{l} 2.5837 \\ 3.25 \\ \underline{4.108} \\ 9.9417 \end{array}$$

Though many find it more difficult to add decimal numbers horizontally than vertically, practice makes it possible to gain facility in horizontal addition.

$$2.5837 + 3.25 + 4.108 = 9.9417$$

*Note. The form used in this text is to insert a 0 *before* the decimal point in writing all pure decimals. Actually the 0 preceding the decimal point in the decimal 0.9 has no meaning; it has been inserted only in the interest of clarity. The decimal could just as well be written .9.

Most addition problems in business and accounting involve only two places beyond the decimal point This fact simplifies horizontal addition

Illustration

\$43 82	+	\$18 46	+	\$8 27	=	\$70 55
27 93	+	15 08	+	12 80	=	55 81
18 27	+	41 29	+	37 43	=	96 99
\$90 02	+	\$74 83	+	\$58 50	=	\$223 35

Subtraction of decimals

The rules for subtracting decimal numbers are the same as the rules for subtracting integers The decimal points must be in the correct place and there must be the same number of digits to the right of the decimal point in both numbers Use either the standard or the Austrian method

Illustrations

a		b	
	38 274		15 2500
	− 22 856		− 8 3724
	15 418		6 8776

Frequently in subtraction the numbers are written horizontally

Illustration

\$37 74	−	\$28 27	=	\$ 9 47
43 84	−	17 39	=	26 45
28 29	−	16 82	=	11 47
\$109 87	−	\$62 48	=	\$47 39

EXERCISE 4.1

Add the following

1.	2	3.	4.	5.
38 270	5 840	22 3740	27 296	1 1152
25 820	8 272	58 1162	9 310	0 8270
43 207	12 300	31 3700	14 007	2 0306
12 316	18 000	14 1006	6 200	3 8100
3 420	6 082	8 2700	11 026	1 7230

6. 37 7820 + 8 2230 + 12 5682 + 5 3198 + 6 2800
7. 18 3600 + 27 8103 + 6 7120 + 5 3000 + 8 9926
8. 9 260 + 6 338 + 8 927 + 10 082 + 5 881
9. 62 423 + 37 111 + 5 882 + 12 007 + 22 870
10. 5 55277 + 4 88260 + 1 78782 + 0 38200 + 6 11682

Complete the following.

11. \$18.47 + \$16.28 + \$37.12 + \$56.69 = ?
12. 31.07 + 22.19 + 48.46 + 63.16 = ?
13. 71.33 + 42.26 + 53.58 + 112.14 = ?
14. 52.48 + 107.11 + 82.62 + 88.74 = ?
15. ? + ? + ? + ? = ?

Subtract the following.

16.	17.	18.	19.	20.
51.270	262.532	15.3060	32.827	64.532
38.272	127.800	8.0834	27.910	64.478

21. 61.26 − 28.74
22. 83.36 − 54.87
23. 18.300 − 13.348
24. 127.374 − 83.226
25. 36.820 − 27.553
26. 34.421 − 16.070

Complete the following.

27. \$56.68 − \$41.27 = ?
28. 73.22 − 61.18 = ?
29. 32.62 − 24.84 = ?
30. 126.13 − 107.44 = ?
31. ? − ? = ?

Multiplication of decimals

When two decimal numbers are multiplied, the operation is the same as when two integers are multiplied. The position of the decimal point in the product is determined by counting off from the right as many places as there are places in the multiplicand plus the places in the multiplier.

Illustration: Find the product of 38.274 and 5.43.

$$\begin{array}{r} 38.274 \\ \underline{5.43} \\ 114822 \\ 153096 \\ \underline{191370} \\ 207.82782 \end{array} \qquad \text{or} \qquad \begin{array}{r} 5.43 \\ \underline{38.274} \\ 2172 \\ 3801 \\ 1086 \\ 4344 \\ \underline{1629} \\ 207.82782 \end{array}$$

To multiply by a power of 10, such as 10, 100, 1,000, move the decimal point in the multiplicand as many places to the right as there are zeros in the multiplier

Illustrations

a $2\ 8743 \times 10 = 28\ 743$
b $2\ 8743 \times 100 = 287\ 43$
c $2\ 8743 \times 1{,}000 = 2{,}874\ 3$
d $2\ 8743 \times 10\ 000 = 28\ 743$
e $2\ 8743 \times 100\ 000 = 287{,}430$

Locating decimal points in products

In the following problems, the digits of both multiplier and multiplicand remain the same As the decimal points in both multiplier and multiplicand are placed in different positions the digits of the product remain unchanged Can you properly locate the positions of the decimal points in the following products?

Given $2\ 387 \times 384 = 916\ 608$

Find
a $23\ 87 \times 38\ 4$
b $238\ 7 \times 384$
c $0\ 2387 \times 0\ 384$
d $0\ 02387 \times 38\ 4$
e $0\ 002387 \times 0\ 0384$
f $0\ 0002387 \times 384$
g $23{,}870 \times 0\ 00384$
h $23\ 87 \times 0\ 0000384$
i $238\ 7 \times 0\ 384$
j $0\ 02387 \times 38\ 400$
k $0\ 02387 \times 3{,}840{,}000$
l $2\ 387 \times 3\ 840$
m $2{,}387 \times 384$
n $238{,}700 \times 0\ 00384$
o $0\ 02387 \times 0\ 0384$
p $2\ 387 \times 3{,}840$
q $238\ 7 \times 0\ 0000384$
r $0\ 0002387 \times 0\ 000384$
s. $23{,}870 \times 38\ 4$
t $0\ 2387 \times 3{,}840$

Estimated product of decimals

The method presented to determine estimated products of integers applies to decimals as well, if both numbers are greater than 1.

Illustration: $38.274 \times 5.43 = 207.82782$ (exact answer)
$40 \times 5 = 200$ (estimated answer)

If both numbers are decimals, a similar method can be used.

Illustration: $0.0582 \times 0.227 = 0.0132114$ (exact answer)
$0.06 \times 0.2 = 0.012$ (estimated answer)

If one number is greater than 1 and the other is a decimal less than 1, an estimation of the product can sometimes be facilitated by dividing one factor, and multiplying the other by a power of 10. Thus to estimate the product when one is required to multiply 927×0.00397, the first step would be to round the numbers to 900 and 0.004. If 900 is divided by 100 the quotient is 9. If 0.004 is multiplied by 100 the product is 0.4. The estimated product can then be found as the product of 9×0.4 or 3.6. The same estimated product would be obtained by multiplying 900×0.004.

Since multiplication by a power of ten results in a shift to the left of the decimal point, the general rule in estimating the product of decimals is as follows:

If one number is greater than 1 and the other is less than 1, the decimal points can be shifted in opposite directions so as to make one of the numbers used in estimating the product lie between 1 and 10.

Illustration: $384.6 \times 0.00582 = 3.\overline{84}.6 \times 0.\overline{00}.582$
$= 3.846 \times 0.582 = 2.238372$
or
$= 0.\overline{384}.6 \times 0.\overline{005}.82$
$= 0.3846 \times 5.82 = 2.238372$

The estimated product is $4 \times 0.6 = 2.4$; or $0.4 \times 6 = 2.4$.

There is a grave chance that a decimal point will be improperly located, particularly when inexperienced persons use mechanical methods of calculating. The method just outlined of estimating a product permits one to locate the decimal point in the exact product with greater confidence. The procedure is somewhat as follows: first, determine the estimated product (here 2.4), then multiply the integer 3,846 by the integer 582 (giving 2,238,372); since the product must be about 2.4, the decimal point in the product must lie between the first and second digits from the left. Therefore the answer is 2.238372.

Note the use of estimated products in the following illustrations

Illustrations

a $38\ 42 \times 52\ 6 = ?$

The estimated product is $40 \times 50 = 2{,}000$ (4×5 with two zeros following)

$$3{,}842 \times 526 = 2{,}020{,}892$$

Since the estimated product is 2,000, the decimal point in the exact answer must lie between the fourth and fifth digits from the left Therefore the exact product is 2,020 892

b $0\ 00582 \times 0\ 0387 = ?$

The estimated product is $0\ 006 \times 0\ 04 = 0\ 00024$ (Put down $6 \times 4 = 24$, and count off five places from the right since there are three places to the right in 0 006 and two places to the right in 0 04)

$$582 \times 387 = 225{,}234$$

Since the estimated product has three zeros between the decimal point and the 2 (first digit not zero), the exact product is 0 000225234

c $4{,}874\ 6 \times 0\ 00825 = ?$

The estimated product is $5{,}000 \times 0\ 008 = 5\ 000 \times 0\ 008 = 5 \times 8 = 40$

$$48{,}746 \times 825 = 40{,}215{,}450$$

Since there are two digits to the left of the decimal point in the estimated product, the exact product is 40 215450

Significant numbers

Numbers obtained by counting such as the number of automobiles manufactured or the number of dollars and cents in a bank, are *exact numbers* Numbers obtained by measurement, such as the number of miles between two cities or the number of cubic inches in a brick are *approximate numbers*, since no measurement, even when the smallest measuring unit is used, is absolutely accurate

A measurement written as 8 feet indicates that the distance is more than $7\frac{1}{2}$ feet and less than $8\frac{1}{2}$ feet The measurement is said to be correct to the nearest foot If a measurement is written as 8 000 feet, it indicates that the distance lies between 7 9995 and 8 0005 feet A measurement of $9\frac{1}{4}$ inches shows that the distance is between $9\frac{1}{8}$ inches and $9\frac{3}{8}$ inches long As a decimal, the measurement of 8 feet is written 8 feet, and $9\frac{1}{4}$ inches is written as 9 25 inches

In dealing with approximate numbers, digits known to be correct are

called *significant digits.* Zeros are significant if they occur between two nonzero digits, or if the zero is the final digit to the right of the decimal point. For example, 10.05 inches has 4 significant digits; similarly 10.5 inches has 3 significant digits; it indicates only that the distance measured is between 10.45 and 10.55 inches. On the other hand, a measurement of 10.50 inches indicates that the distance is between 10.495 and 10.505 inches, and consequently it is more accurate than 10.5 inches. All 4 digits in 10.50 are considered significant.

A zero used merely to locate the decimal point is not considered a significant digit. Thus the measure of the distance of $\frac{1}{100}$ of an inch, 0.01 inch, has only 1 significant digit. Even so, 0.01 inch is considered as having the same precision as a measurement of 10.52 inches, which has 4 significant digits, since numbers are said to have the same *precision* if they are given to the same number of decimal places.

A zero at the end of a number on the left of the decimal point may or may not be significant. In considering an amount of $2,500, all 4 digits are significant if the figure is based on an exact count. If it is a mere estimate of the value it may have 1 or 2 significant digits. Similarly 2,500 miles may have 2, 3, or 4 significant digits, depending on the accuracy of the measurement.

In the problems cited up to this point, all the numbers used have been considered as exact numbers and all the answers as exact answers. In many instances, however, the factors used are only approximate numbers. Let us consider the accuracy of an answer under such circumstances. Suppose 17, an approximate number, is to be multiplied by 37.2, an approximate number.

From our knowledge of approximate numbers we know that 17 is greater than 16.5 but less than 17.5, and that 37.2 is greater than 37.15 but less than 37.25.

Using all possible values of each factor, we get the following products from low to high:

$$\begin{aligned}
16.5 \times 37.15 &= 612.975\\
16.6 \times 37.16 &= 616.856\\
16.7 \times 37.17 &= 620.739\\
16.8 \times 37.18 &= 624.624\\
16.9 \times 37.19 &= 628.511\\
17. \quad \times 37.20 &= 632.400\\
17.1 \times 37.21 &= 636.291\\
17.2 \times 37.22 &= 640.184\\
17.3 \times 37.23 &= 644.079\\
17.4 \times 37.24 &= 647.976\\
17.5 \times 37.25 &= 651.875
\end{aligned}$$

These figures show that there is a range of possible values from 612.975 to 651 875 From this illustration you can understand the following two important rules *

1 In multiplication and division, the number of significant digits in the product or quotient is the same as that in the least significant of the two figures to form the product or quotient Thus 37 2 multiplied by 17 is 632 4, but having only two significant digits would be written 630 in the text (The product 630 is called the approximate product) If the 17 is an integer and hence good to an infinite number of places, the correctly written figures would be 632, since 37 2 has three significant digits An example of significant digits in a quotient is 118 3 − 12 1 = 9 78

2 In addition and subtraction, the result is significant only to the last place of the least accurate figure An illustration of proper rounding follows

136 421	136 4	
28 3	28 3	
321	321	
68 243	68 2	
17 482	17 5	
	571 4	Answer is 571

The sum, having only three significant digits, ought to be written 571 in text Since the least accurate number, 321, is good only to a whole unit it is proper to set the problem up with each number carried to one more place This protects against rounding errors The number 571 is called the *approximate sum*

Illustrations

$52\ 6 \times 38\ 42 = 2\ 020\ 892$ (exact answer)
$= 2{,}020$ (approximate answer)
$0\ 00582 \times 0\ 387 = 0\ 000225231$ (exact answer)
$= 0\ 000225$ (approximate answer)

EXERCISE 4 2

State the number of significant figures in each of the following approximate numbers

1. 57
2. 5 71
3. 57 001
4. 57 000
5. 0 0057
6. 0 0570
7. 100 0057
8. 100
9. 100 1
10. 90

*Quoted from Bureau of Agricultural Economics, U S Department of Agriculture, *The Preparation of Statistical Tables A Handbook*, December, 1937

Round to 4 significant digits.

11. 416.17
12. 0.0064195
13. 0.00372100
14. 2,451.7
15. 0.083845
16. 16,187,451
17. 3,785,023
18. 82,315
19. 0.000828285
20. 3.00752

Find the approximate sum.

21.	**22.**	**23.**	**24.**
321.1	18.382	48.813	3.8742
457.08	5.27	7.29	12.332
892.171	123.8	11.1162	8.07
429	13	5.3	4.138

Find the approximate product.

25. 3.824×5.17
26. 428×7.32
27. $28.7 \times 4{,}382$
28. 16.362×4.118
29. 8.3882×15.61
30. 33.27×53.8
31. 0.05873×0.00326
32. 0.000681×0.0272
33. 0.3327×0.00804
34. 0.286×0.00042
35. 0.061734×0.003147
36. 8.8284×0.5352
37. 384.62×0.0517
38. $0.00827 \times 5{,}184$
39. 6.824×0.00518
40. 18.304×0.5263

Contracted multiplication of decimals

When the answer desired need be correct only to a specified number of decimal places—that is, an approximate product—the amount of work can be reduced, and a satisfactory answer obtained by a method called "contracted multiplication." It is a gradual contraction or abbreviation of the multiplicand as the product is determined. Some judgment must be exercised in determining when its use is feasible.

To find the product of two numbers such as 38.42 and 52.6 by this method, determine the location of the decimal point by estimating the product ($40 \times 50 = 2{,}000$, the estimated product), and proceed as follows:

$$
\begin{array}{rl}
38\cancel{4}\cancel{2} & \\
526 & \\
\hline
19210 & (3{,}842 \times 5 = 19{,}210) \\
768 & (384.2 \times 2 = 768.4) \\
230 & (38.4 \quad \times 6 = 230.4) \\
\hline
20208 &
\end{array}
$$

Steps of solution by contracted multiplication

1 Multiply the multiplicand (3 842) by the *left* digit of the multiplier (5) Here the product is 19 210

2 Strike out the *right* digit of the multiplicand (2)

3 Retaining the discarded digit (2) as a decimal multiply the contracted multiplicand (381 2) by the second digit from the left in the multiplier (2) Here the product is 768 4 Round 768 4 to 768

4 Strike out the second digit from the right of the multiplicand (4)

5 Retaining the discarded digit (4) as a decimal multiply the contracted multiplicand (38 4) by the third digit from the left in the multiplier (6) Here the product is 230 4 Round 230 4 to 230

6 Add the products together This gives 20 208 in this example

Since the estimated product for 38 42 × 52 6 is 40 × 50 = 2 000, the *approximate product is 2,020*

The process of exact multiplication is carried out as follows

$$\begin{array}{r} 38\,42 \\ 52\,6 \\ \hline 23052 \\ 7684 \\ 19210 \\ \hline 2{,}020\,892 \end{array}$$

Since the less significant of the two numbers being multiplied has only 3 significant digits, the approximate product is rounded to 3 significant digits 2 020, which is the same as the product found by contracted multiplication

EXERCISE 4 3

Find the approximate answer by contracted multiplication

1 482 7 × 35 4
2. 83 372 × 53 37
3 0 7372 × 0 384
4. 0 003872 × 0 0537
5 279 4 × 0 0628
6 387 42 × 0 3772
7. 18 008 × 8 34
8. 40 072 × 6 063
9 18 375 × 2 435
10 258 4 × 1 83

Division of decimals

In multiplication the value of the product is not changed if one factor is multiplied by a number and the other factor divided by the same number The effect of applying such a principle gives rise to the rule that

in multiplication the decimal points in the multiplier and multiplicand can be moved the same number of places in opposite directions without affecting the product.

In dealing with fractions we saw that the value of a fraction is not changed if both the numerator and the denominator are either divided or multiplied by the same number. Since a fraction is an indicated quotient, we can write any division problem in the form of a fraction. Thus $38.275 \div 6.14$ can be written $\frac{38.275}{6.14}$. If we choose, we can multiply the numerator and the denominator by 100, getting $\frac{3{,}827.5}{614}$; or if we divide both numerator and denominator by 100 we have $\frac{.38275}{.0614}$. In either case the quotient is the same. These relationships form the basis for the general rule that in division, decimal points can be moved in the *same* direction an equal number of places in both the dividend and the divisor without affecting the quotient. This fact is important in locating the decimal points in quotients.

Locating decimal points in quotients

If the divisor is a whole number, the decimal point in the quotient is directly above the decimal point in the dividend.

Illustration: $1{,}873.92 \div 488.$

$$\text{Solution:}\quad 488\,\overline{)\,1{,}873.92}^{\;3.84}$$

If the divisor is not a whole number, move the decimal point in the divisor to the right of the last digit in the divisor and move the decimal point in the dividend to the right an equal number of places adding zeros if necessary. After these changes have been made, locate the decimal point in the quotient directly above the new decimal point in the dividend.

Illustration: $18.7392 \div .0488$

$$\text{Solution:}\quad .0488.\,\overline{)\,18.7392.}^{\;3{,}840.}$$

In the following problems, the digits of both the dividend and the divisor remain the same. As the decimal points of the dividend and the divisor are placed in different positions, the digits of the quotient remain unchanged. Can you locate the decimal points in the following quotients?

Given $527\overline{)\,2\ 309{,}314}$ (quotient: 4 382)

Find

a $52\ 7\overline{)\,230\ 9314}$

b $527\overline{)\,2{,}309\ 314}$

c $52\ 7\overline{)\,2\ 309314}$

d $0\ 527\overline{)\,230\ 9314}$

e $5\ 27\overline{)\,0\ 02309314}$

f $52\ 7\overline{)\,0\ 0002309314}$

g $0\ 0527\overline{)\,0\ 002309314}$

h $0\ 527\overline{)\,230\ 9314}$

i $0\ 00527\overline{)\,2\ 309314}$

j $5{,}270\overline{)\,23\ 09314}$

k $5\ 27\overline{)\,23\ 09314}$

l $52{,}700\overline{)\,2{,}309\ 314}$

m $527\overline{)\,0\ 002309314}$

n $5\ 27\overline{)\,23{,}093\ 14}$

o $0\ 0527\overline{)\,230\ 9314}$

p $527{,}000\overline{)\,2{,}309\ 314}$

dividend than in the divisor For example 38 275 ÷ 6 14 is written 38 28 ÷ 6 14

Illustration 5 284 ÷ 3 743 = ?

```
              1412
    37̸4̸3̸ ⟌ 5284
              3743   (1 × 3,743)
              ----
              1541
              1497   (4 × 374 3)
              ----
                44
                37   (1 × 37 4)
                --
                 7
                 7   (2 × 3 7)
                 -
                 0
```

Steps of solution by contracted division

1 Determine by trial the first digit of the quotient (1 × 3 743 is less than and 2 × 3,743 is greater than 5,284) Put down 1 in the quotient

2 1 × 3,743 equals 3,743 Subtract 3,743 from 5,284, giving 1,541

3 Instead of bringing down a digit (or 0) from the dividend, cancel the right digit (3) in the divisor Determine by trial the second digit of the quotient (4 × 374 is less than and 5 × 374 is greater than 1,541) Put down 4 in the quotient

4 4 × 374 3 (retaining the discarded 3 as a decimal) equals 1,497 2 Subtract 1,497 from 1,541, giving 44

5 Cancel the next digit from the right (4) in the divisor Determine by trial the third digit of the quotient (1 × 37 is less than and 2 × 37 is greater than 44) Put down 1 in the quotient

6 1 × 37 4 (retaining the discarded 4 as a decimal) equals 37 4 Subtract 37 from 44, giving 7

7 Cancel the next digit from the right (7) in the divisor Determine by trial the fourth digit of the quotient (2 × 3 is less than and 3 × 3 is greater than 7) Put down 2 in the quotient

8 2 × 3 7 (retaining the discarded 7 as a decimal) equals 7 4 Subtract 7 from 7, giving 0

Thus the approximate quotient is 1 412 The position of the decimal point can be determined from the estimated quotient

In the next illustration, the steps of contracted division are shown in brief form

Illustration: $38.28 \div 6.14 = ?$

$$\begin{array}{r r l}
 & 623 & \\
6\cancel{1}\cancel{4} & \overline{)\,3828} & \\
 & \underline{3684} & (6 \times 614 = 3684) \\
 & 144 & \\
 & \underline{123} & (2 \times 61.4 = 122.8) \\
 & 21 & \\
 & \underline{18} & (3 \times 6.1 = 18.3) \\
 & 3 &
\end{array}$$

Since 3, the last difference, is less than one-half of 6.1, the approximate answer is 6.23.

EXERCISE 4.4

Find the approximate quotient by long division.

1. $18.62 \div 3.8$	**7.** $478.6 \div 12.7$	**13.** $38.27 \div 2.13$
2. $56.82 \div 31.8$	**8.** $538.2 \div 2.78$	**14.** $23.64 \div 3.47$
3. $473.2 \div 0.734$	**9.** $0.8264 \div 12.3$	**15.** $859 \div 16.7$
4. $0.7734 \div 0.0432$	**10.** $13.816 \div 0.0523$	**16.** $438.6 \div 23.5$
5. $3{,}874 \div 1.86$	**11.** $37.82 \div 0.82$	**17.** $6.6678 \div 0.5556$
6. $0.07274 \div 0.4384$	**12.** $0.00569 \div 238$	**18.** $82.772 \div 3.145$

Find the approximate quotient by contracted division:

19. $82.74 \div 55.83$	**20.** $43.27 \div 6.17$	**21.** $787.32 \div 208.15$
22. $127.16 \div 82.32$	**23.** $438.27 \div 8.773$	**24.** $32.873 \div 0.5034$
25. $0.6273 \div 41.81$		

Changing decimal fractions to common fractions

To change a decimal fraction to a common fraction, leave out the decimal point and write the decimal as the numerator of the fraction. The denominator will be 1 followed by as many zeros as there are places to the right of the decimal point in the fraction. Thus 0.1 would $\frac{1}{10}$; 0.002 would be $\frac{2}{1000}$, or (reducing the fraction to lowest terms) $\frac{1}{500}$.

Illustrations:

a. Change 0.27 into a fraction.
   $0.27 = \frac{27}{100}$ (This is the final answer since it cannot be reduced.)

b. Change 0.005 into a fraction.
   $0.005 = \frac{5}{1000} = \frac{1}{200}$

c Change $0\ 09\frac{1}{3}$ into a fraction

$$0\ 09\tfrac{1}{3} = \frac{9\frac{1}{3}}{100} = \frac{9\frac{1}{3} \times 3}{100 \times 3} = \frac{28}{300} = \frac{7}{75}$$

Changing common fractions to decimal fractions

To change a common fraction to a decimal fraction, divide the numerator of the fraction by the denominator

Illustration Change $\frac{3}{8}$ into a decimal

$$\frac{3}{8} = \frac{3\ 000}{8} = 0\ 375$$

A fraction whose denominator contains factors other than 2 or 5 will not come out an exact decimal, but comes out as a digit or set of digits repeating without end—such as $\frac{1}{3} = 0\ 333$, or $\frac{2}{11} = 0\ 1818$ This kind of decimal fraction is called a *repeating decimal* or a *circulating decimal* The three dots mean and so on and are commonly used to indicate a repeating decimal, although such a decimal can be written in more exact form as $0\ 333\frac{1}{3}$ or $0\ 1818\frac{2}{11}$

Illustration Change $\frac{12}{35}$ into a decimal

$\frac{12}{35} = 0\ 34\frac{2}{7}$ or $0\ 342857$ (Answer obtained by long division)

EXERCISE 4 5

Change the following decimals into their fraction equivalents

1. 0 48	6. 0 00625	11. 0 0444
2. 0 035	7. 0 0425	12. $0\ 25\frac{1}{8}$
3. 0.0075	8. 0.016	13. $0.06\frac{3}{7}$
4. 0 0006	9. 0 5625	14. 0 000115
5. 0 875	10. $0\ 12\frac{1}{2}$	15. 0 007

Change the following fractions into their decimal equivalents

16. $\frac{7}{12}$	21. $\frac{9}{16}$	26. $\frac{1}{40}$
17. $\frac{9}{40}$	22. $\frac{4}{15}$	27. $\frac{1}{20}$
18. $\frac{7}{80}$	23. $\frac{8}{25}$	28. $\frac{3}{5}$
19. $\frac{1}{250}$	24. $\frac{1}{2}$	29. $\frac{4}{25}$
20. $\frac{7}{8}$	25. $\frac{2}{5}$	30. $\frac{1}{16}$

Fractional parts

Multiplication or division by 10 or by any power of 10 is not difficult. In multiplying or dividing numbers the amount of calculation can sometimes be reduced if the number being used can be related to 10, 100, or 1,000. For example, if 150 is multiplied by $33\frac{1}{3}$ the product is found to be 5,000. Instead of multiplying by $33\frac{1}{3}$, however, we might have changed $33\frac{1}{3}$ to $\frac{100}{3}$. The multiplication then would be so simple that it would not be necessary to set any work down on paper.

Such a method is known as the "fractional-parts method." There are two types of fractional parts—*aliquot* and *aliquant*. An aliquot part of a number is defined as any divisor of that number which gives an integral number as a quotient. When $33\frac{1}{3}$ is divided into 100 the quotient is 3. Thus $33\frac{1}{3}$ is an aliquot part of 100. Since $66\frac{2}{3}$ is $\frac{2}{3}$ of 100, $66\frac{2}{3}$ is an aliquant part of 100.

To find whether a number is an aliquot part of another, divide the second number by the first. The first number is an aliquot part of the second number if the quotient is an integral number.

Illustrations:

a. Is 20 an aliquot part of 100?

Yes, since $100 \div 20 = 5$. That is, 20 is $\frac{1}{5}$ of 100.

b. Is $16\frac{2}{3}$ an aliquot part of 100?

Yes, since $100 \div 16\frac{2}{3} = 100 \div \frac{50}{3} = \frac{300}{50} = 6$. That is, $16\frac{2}{3}$ is $\frac{1}{6}$ of 100.

EXERCISE 4.6

Determine whether each of the following is a fractional part of 100.

1. 50
2. $33\frac{1}{3}$
3. $16\frac{2}{3}$
4. $12\frac{1}{2}$
5. $11\frac{1}{9}$
6. $8\frac{1}{3}$
7. $6\frac{2}{3}$
8. $6\frac{1}{4}$
9. $4\frac{1}{6}$
10. $3\frac{1}{3}$
11. $2\frac{1}{2}$
12. $1\frac{2}{3}$
13. $\frac{2}{3}$
14. $62\frac{1}{2}$
15. 64
16. $66\frac{2}{3}$
17. 75
18. 80
19. $82\frac{1}{2}$
20. $87\frac{1}{2}$

Little time can be saved by the use of the fractional-parts method if it is necessary to compute each time whether one of the numbers is a fractional part of a base number such as 10, 100, or 1,000. Anyone who can count money knows many fractional parts of 100. A price of 25 cents is commonly referred to as a quarter—that is, $\frac{1}{4}$ of a dollar. In some sections 25 cents is referred to as "two bits." The term bit came into use at the time the Spanish dollar or *real* was used; a bit was $\frac{1}{8}$ of a dollar. Thus $12\frac{1}{2}$ is $\frac{1}{8}$ of 100.

Much time will be saved in computation if you memorize the following fractional parts of 100

Fractional Part	*Amount*	*Fractional Part*	*Amount*
$\frac{1}{2}$	50	$\frac{1}{12}$	$8\frac{1}{3}$
$\frac{1}{3}$	$33\frac{1}{3}$	$\frac{1}{15}$	$6\frac{2}{3}$
$\frac{1}{4}$	25	$\frac{1}{16}$	$6\frac{1}{4}$
$\frac{1}{5}$	20	$\frac{1}{20}$	5
$\frac{1}{6}$	$16\frac{2}{3}$	$\frac{1}{25}$	4
$\frac{1}{8}$	$12\frac{1}{2}$	$\frac{1}{30}$	$3\frac{1}{3}$
$\frac{1}{9}$	$11\frac{1}{9}$	$\frac{1}{40}$	$2\frac{1}{2}$
$\frac{1}{10}$	10	$\frac{1}{50}$	2

The following fractional or aliquant parts of 100 are used so often that they too should be memorized

Aliquant Part	*Amount*	*Aliquant Part*	*Amount*
$\frac{3}{16}$	$18\frac{3}{4}$	$\frac{5}{8}$	$62\frac{1}{2}$
$\frac{5}{16}$	$31\frac{1}{4}$	$\frac{2}{3}$	$66\frac{2}{3}$
$\frac{3}{8}$	$37\frac{1}{2}$	$\frac{3}{4}$	75
$\frac{7}{16}$	$43\frac{3}{4}$	$\frac{7}{8}$	$87\frac{1}{2}$

Multiplication by a fractional part

To multiply one number by another when one of the factors is an aliquot part or an aliquant part of a basic number (such as 10, 100 or 1,000) the first step is to find the product of the multiplicand and the basic number, the second step is to multiply this product by the fractional part

Illustrations

a Multiply 960 by 25

25 is $\frac{1}{4}$ of 100 $960 \times 100 = 96\,000$ $96\,000 \times \frac{1}{4} = 24\,000$

b Multiply 400 by 375

$37\frac{1}{2}$ is $\frac{3}{8}$ of 100, therefore 375 must be $\frac{3}{8}$ of 1000 $400 \times 1000 = 400\,000$ $400{,}000 \times \frac{3}{8} = 150\,000$

c Multiply 5.672 by 12.5

12.5 is $\frac{1}{8}$ of 100 $5{,}672 \times 100 = 567{,}200$ $567{,}200 \times \frac{1}{8} = 70\,900$

d Multiply 12.840 by 0.875

$87\frac{1}{2}$ is $\frac{7}{8}$ of 100, therefore 0.875 must be $\frac{7}{8}$ of 1 $12.840 \times \frac{7}{8} = 11.235$

In these illustrations each step has been recorded After some practice you will do much of the work mentally and find no notations necessary

EXERCISE 4.7

Multiply by the use of the fractional-parts method.

1. $66 \times 66\frac{2}{3}$	5. $1{,}641 \times 0.33\frac{1}{3}$	9. 625×800
2. $640 \times 12\frac{1}{2}$	6. 270×60	10. $634 \times 8\frac{1}{3}$
3. $9{,}760 \times 8\frac{1}{3}$	7. $1{,}850 \times 25$	11. $1{,}440 \times 0.0625$
4. $1{,}641 \times 33\frac{1}{3}$	8. $1{,}624 \times 37\frac{1}{2}$	12. $6{,}125 \times 8.80$

When unit costs or unit selling prices are aliquot or aliquant parts of a dollar, calculation can be shortened.

Illustrations:

a. Find the cost of 160 pounds of celery at $12\frac{1}{2}$ cents a pound.
Since $12\frac{1}{2} = \frac{1}{8}$ of 100, then $12\frac{1}{2}¢ = \frac{1}{8}$ of \$1.
Rather than multiply $160 \times 12\frac{1}{2}¢$, it can be stated as

$$160 \times \tfrac{1}{8} \text{ of } \$1 = \$20$$

b. Find the cost of 180 units at $16\frac{2}{3}$ cents each.

$$16\tfrac{2}{3}¢ = \tfrac{1}{6} \text{ of } \$1.$$
$$180 \times \tfrac{1}{6} \text{ of } \$1 = \$30.$$

c. Find the cost of 120 handkerchiefs at 75 cents each.

$$75¢ = \tfrac{3}{4} \text{ of } \$1.$$
$$120 \times \tfrac{3}{4} \text{ of } \$1 = \$90.$$

A common business symbol is @. For example, 8 pounds of butter at 75¢ *per pound* is recorded: 8 pounds @ 75¢; 25 feet of hose at 30¢ *per foot* is written: 25 feet @ 30¢; and 48 dozen oranges at 40¢ *per dozen* is written: 48 dozen @ 40¢.

In any problem in multiplication the multiplier and the multiplicand can be interchanged without affecting the product. In using the fractional-parts method of multiplication it is sometimes easier if the amount and price are interchanged in making the computation. Thus 25 feet @ 36¢ gives the same total cost as 36 ft @ 25¢; $83\frac{1}{3}$ pounds @ 72¢ can be computed as 72 pounds @ $83\frac{1}{3}¢$.

EXERCISE 4.8

Find the cost of each of the following.

1. 36 feet @ 25¢	7. $87\frac{1}{2}$ grams @ 56¢
2. 48 pounds @ $18\frac{3}{4}¢$	8. $37\frac{1}{2}$ feet @ 40¢
3. 54 ounces @ $16\frac{2}{3}¢$	9. 30 yards @ 80¢
4. $12\frac{1}{2}$ feet @ 64¢	10. $41\frac{2}{3}$ feet @ 24¢
5. 24 quarts @ $33\frac{1}{3}¢$	11. 96 pounds @ $56\frac{1}{4}¢$
6. 42 ounces @ 75¢	12. $83\frac{1}{3}$ pounds @ 72¢

13. 60 pounds @ $58\frac{1}{3}$¢
14. 128 yards @ $37\frac{1}{2}$¢
15. 60 pints @ $66\frac{2}{3}$¢
16. 50 pounds @ $63\frac{1}{2}$¢
17. $33\frac{1}{3}$ feet @ 24¢
18. 27 yards @ 60¢
19. 200 feet @ $8\frac{1}{3}$¢
20. 30 feet @ $58\frac{1}{3}$¢

Division by a fractional part

If 600 is divided by 25 the quotient is 24. The same result is obtained by dividing 600 by 100 and multiplying the quotient by 4, that is, $\frac{600}{100} \times 4 = 24$.

To carry out division by the use of fractional parts the following procedure is used:

1. Find the quotient of the dividend divided by the basic number—that is, 1, 10, 100, or 1,000, or whatever number the divisor is a fractional part of.

2. Divide this quotient by the fractional equivalent. If the fractional equivalent is an aliquot part, as in the preceding example, the second step amounts to multiplying the quotient by the denominator of the aliquot part. This is true since in division by fractions, the procedure is to invert the divisor and multiply.

Illustrations

a. $640 \div 12\frac{1}{2} = ?$

$12\frac{1}{2} = \frac{1}{8}$ of 100, $640 \div 100 = 6.40$, $6.40 \div \frac{1}{8} = 6.40 \times 8 = 51.2$

b. $33.6 \div 16\frac{2}{3} = ?$

$16\frac{2}{3} = \frac{1}{6}$ of 100, $33.6 \div 100 = 0.336$, $0.336 \div \frac{1}{6} = 0.336 \times 6 = 2.016$

c. $2{,}490 \div 37\frac{1}{2} = ?$

$37\frac{1}{2} = \frac{3}{8}$ of 100, $2{,}490 \div 100 = 24.9$

$$24.9 \div \frac{3}{8} = 24.9 \times \frac{8}{3} = 8.3 \times 8 = 66.4, \quad \text{or} \quad = \frac{199.2}{3} = 66.4$$

EXERCISE 4.9

Divide by using the fractional parts method

1. $75 \div 25$
2. $90 \div 6.25$
3. $483 \div 66\frac{2}{3}$
4. $63.90 \div 8\frac{1}{3}$
5. $875 \div 62.5$
6. $4{,}383 \div 75$
7. $6{,}490 \div 18.75$
8. $864.28 \div 0.025$
9. $984 \div 50$
10. $833 \div 3\frac{1}{3}$
11. Divide 1,440 by $4\frac{1}{6}$, $6\frac{1}{4}$, $8\frac{1}{3}$, $12\frac{1}{2}$, $16\frac{2}{3}$
12. Divide 720 by $31\frac{1}{4}$, $37\frac{1}{2}$, $62\frac{1}{2}$, $66\frac{2}{3}$, 75
13. Divide 800 by 3.125, 62.5, 0.025, 0.125, 8.33
14. Divide 1,500 by 18.75, 3.75, 0.75, 1.66 , 4.166
15. Divide 1,001 by 43.75, $68\frac{3}{4}$, 81.25, 8.75, $13\frac{3}{4}$

5

Percentage and Discounts

Introduction

The fundamental principles of common fractions and decimal fractions have their principal application in business mathematics in problems dealing with percentages. When a smaller number is being compared to a larger number, the relationship is often expressed as a percentage. *Per cent* means "of" or "by the hundred," and is expressed by the symbol %. Thus 75% means 0.75, or $\frac{75}{100}$; the per cent symbol in this case is merely a substitute for the decimal point in the decimal fraction or the 100 in the denominator of the common fraction.

In ordinary usage, the word percentage may be used either as defined, or as synonomous with the rate. There is nothing fundamentally new in problems of percentage. Similar relationships have been discussed in the preceding chapters, but not in exactly the same terminology.

Changing a fraction to a per cent

To express a common fraction as a per cent, change the common fraction into a decimal fraction, and express as a per cent by shifting the decimal point two places to the right and affixing the % symbol.

Illustrations:

a. Express $\frac{7}{16}$ as a per cent.

$$7 \div 16 = 0.4375 = 43.75\%$$

b. Express $\frac{5}{12}$ as a per cent.

$$5 \div 12 = 0.4166\ldots = 41.66\ldots\%; \quad \text{or} \quad 0.41\tfrac{2}{3} = 41\tfrac{2}{3}\%$$

Sometimes it appears simpler to change a fraction to a per cent by multiplying the numerator of the fraction by 100% and dividing the product by the denominator. Since multiplication by 100% amounts to adding two zeros and a per cent sign to the numerator, this is an easy method to apply.

Illustration Express $\frac{1}{250}$ as a per cent

$$\frac{1}{250} = \frac{100\%}{250} = \frac{2}{5}\% = 0\ 4\%$$

That is, $\frac{1}{250}$ is $\frac{2}{5}$ of 1%, or 0 4 of 1%

EXERCISE 5.1

Find the decimal and per cent equivalents of the following common fractions

1. $\frac{1}{100}$	7. $\frac{5}{16}$	13. $\frac{1}{6}$	19. $\frac{7}{64}$	25. $\frac{11}{16}$	31. $\frac{7}{40}$
2. $\frac{6}{100}$	8. $\frac{2}{3}$	14. $\frac{7}{15}$	20. $\frac{31}{32}$	26. $\frac{4}{15}$	32. $\frac{5}{32}$
3. $\frac{3}{5}$	9. $\frac{7}{8}$	15. $\frac{11}{600}$	21. $\frac{1}{4}$	27. $\frac{1}{3}$	33. $2\frac{1}{2}$
4. $\frac{1}{2}$	10. $\frac{19}{200}$	16. $\frac{7}{12}$	22. $\frac{2}{5}$	28. $\frac{13}{30}$	34. $\frac{1}{200}$
5. $\frac{3}{8}$	11. $\frac{1}{12}$	17. $\frac{9}{32}$	23. $\frac{1}{16}$	29. $\frac{5}{24}$	35. $\frac{13}{400}$
6. $\frac{3}{4}$	12. $\frac{1}{8}$	18. $\frac{125}{200}$	24. $\frac{5}{16}$	30. $\frac{3}{16}$	36. $\frac{5}{5000}$

Finding the rate

There are three basic types of problems dealing with percentage One type is involved in converting a common fraction into a per cent Stating that $\frac{3}{4}$ is equivalent to 75% is in effect saying that 3 is 75% of 4 The same principle used in changing a fraction into a per cent is involved in finding what per cent one number is of another For example, the query 8 is what per cent of 40, means 8 is how many hundredths of 40? Written as $\frac{8}{40}$, and reduced to $\frac{1}{5}$, leaves the simple problem of converting $\frac{1}{5}$ into a decimal fraction of 0 20 and then into 20% Other similar types of problems rise frequently in business where it is desired to express relationships as per cents

Illustrations

a The Jeffrey Lewis Company has a policy of retiring all employees at age 65 If 36 of its 900 employees are now 64, what per cent of its employees should it plan to retire during the next year?

This is equivalent to saying 36 is what per cent of 900? 36 − 900 = 0 04 That is, 4% of the employees will be 65 within one year.

b If gross sales last year were $200,000 and sales returned amounted to $16,400, what was the per cent of sales returned?

Total sales = $200,000, Sales returned = $16,400

$16,400 − $200,000 = 8 2% sales returned

EXERCISE 5.2

Solve the following.

1. 6 is what per cent of 60?
2. 16 is what per cent of 240?
3. 20 is what per cent of 360?
4. 4 is what per cent of 600?
5. 15 is what per cent of 80?
6. 42 is what per cent of 60?
7. 5.4 is what per cent of 132?
8. 0.56 is what per cent of 27?
9. 13.6 is what per cent of 85?
10. 52.4 is what per cent of 471.6?
11. \$7 is what per cent of \$168?
12. \$91 is what per cent of \$156?
13. \$60 is what per cent of \$192?
14. \$25.50 is what per cent of \$2,040?
15. \$245 is what per cent of \$8,400?
16. \$900 is what per cent of \$160,000?
17. \$77.25 is what per cent of \$21,600?
18. \$43 is what per cent of \$970?
19. \$48.60 is what per cent of \$2,500?
20. \$81.50 is what per cent of \$384.40?
21. In a class there are 24 men and 12 women. What per cent of the class enrollment are men?
22. A house worth \$18,000 is insured for \$5,000. The insurance is what per cent of the value?
23. A suit that costs \$45 is sold for \$60. What per cent profit is made on cost? On selling price?
24. During a baseball season a team won 120 games and lost 40 games. What per cent of the games was won?
25. The net profit of a store amounted to \$18,560 the year before last; \$21,450 last year; \$19,740 this year. Find the per cent increase or decrease from year to year.

Finding a per cent of a number

A second basic type of problem dealing with per cent that occurs again and again is that of finding a percentage of a number. Such problems are basically problems in multiplication. Thus to find 18% of 450 is to find the product of the two numbers, the base and the rate.

Illustrations

a. Find 18% of 450

450 × 18% = 450 × 0.18 = 81. That is, 81 is 18% of 450.

b. Find 0.3% of 7,200

7,200 × 0.3% = 7,200 × 0.003 = 21.6. That is, 21.6 is 0.3% of 7,200.

Frequently a person is confused when it is necessary to find a per cent greater than 100%. The problem is still one in multiplication, and the general rule holds that in changing a per cent to a decimal you move the decimal point two places to the left and multiply by the resulting decimal fraction.

Illustration. Last year's production was 9,000 units. If production this year will be 200% of last year's, what is the expected production this year?

200% of a number = 2.00 times the number. 9,000 × 2 = 18,000. Thus the expected production this year will be 18,000 units.

EXERCISE 5.3

Find the following:

1. What is 3% of 15?
2. What is 10% of 85?
3. What is 6% of 1,940?
4. What is $3\frac{1}{4}$% of 1,620?
5. What is $2\frac{1}{8}$% of 1,000,000?
6. What is 150% of 30?
7. What is $87\frac{1}{2}$% of 87.5?
8. What is 3.75% of 80?
9. What is 5% of $321?
10. What is $33\frac{1}{3}$% of $54.18?
11. What is 15% of $325.40?
12. What is $\frac{3}{4}$% of $120?
13. What is 0.82% of 427?
14. What is $\frac{3}{8}$% of 118?
15. What is $5\frac{1}{3}$% of $560?
16. What is 195% of 1,200?
17. What is 110% of 90?
18. What is 125% of 6,780?
19. What is 105% of 1,000?
20. What is $103\frac{1}{3}$% of $10,000?

21. A contractor estimates that a certain new home will cost $12,800. If 18% of this amount is for plumbing, 33% for the building materials and supplies, and 10% for labor, what is the estimated cost of each item? If the balance is evenly divided between overhead and profit, what per cent of the total cost (100%) does the contractor make and how much?

22. If the total sales of a company increased $12\frac{1}{2}$% each year for 3 years, find the total sales for the third year if the total sales for the first year mentioned were $38,725.

23. The amount of taxes collected in Radio City last year was \$1,842,562.84. If the tax rate will be $8\frac{1}{3}\%$ more next year, what will be the amount of taxes collected then?

24. A merchant pays \$180 for 12 dresses. He sells them $37\frac{1}{2}\%$ above cost. What is the selling price of each dress?

25. If between one year and the next, food prices increased $3\frac{1}{2}\%$, how much more would have to be spent in the second year if the average family food bill was \$824 the year before?

Changing a per cent to a fraction

Sometimes it is easier to deal with a simple fraction than with a per cent. To change a per cent to a common fraction, remove the per cent symbol, and either state the per cent as the numerator of a common fraction with a denominator of 100, or multiply the denominator by 100. Then reduce to lowest terms.

When the per cent is a mixed number or a fractional part of 1 per cent it can be changed to a fraction by writing it first as either a proper or an improper fraction, adding two zeros to the denominator when the per cent symbol is removed, and then reducing the fraction to its lowest terms.

Illustrations:

a. Change 75% into a common fraction.
$75\% = \frac{75}{100} = \frac{3}{4}$

b. Change $12\frac{1}{2}\%$ into a common fraction.
$12\frac{1}{2}\% = \frac{25}{2}\% = \frac{25}{200} = \frac{1}{8}$

c. Change $\frac{3}{8}\%$ into a common fraction.
$\frac{3}{8}\% = \frac{3}{800}$

d. Change $\frac{5}{16}\%$ into a common fraction.
$\frac{5}{16}\% = \frac{5}{1600} = \frac{1}{320}$

e. Change 0.25% into a common fraction.
$0.25\% = \frac{1}{4}\% = \frac{1}{400}$

EXERCISE 5.4

Find the fraction equivalents of the following per cents.

1. 10%	5. $3\frac{1}{2}\%$	9. $16\frac{2}{3}\%$	13. $\frac{3}{5}\%$	17. 0.125%
2. 5%	6. $7\frac{1}{2}\%$	10. $22\frac{1}{2}\%$	14. $\frac{7}{16}\%$	18. 0.45%
3. 3%	7. $8\frac{1}{3}\%$	11. 36%	15. $\frac{5}{12}\%$	19. 2.75%
4. $2\frac{1}{2}\%$	8. $5\frac{1}{3}\%$	12. $1\frac{1}{4}\%$	16. $\frac{5}{24}\%$	20. 4.375%

Use of fractional parts

In many problems of per cent a knowledge of fractional parts introduced in the preceding chapter may be employed expeditiously If for example, it is necessary to find $12\frac{1}{2}\%$ of 560 the solution may be achieved quickly if it is known that $12\frac{1}{2}\%$ is equal to $\frac{1}{8}$ of 100% The problem of multiplying 560 by 100% and then by $\frac{1}{8}$ does not appear so difficult as to multiply 560 by 0 125 The same principle is involved in finding 25% of any number readily divisible by 4, since 25% stated as a common fraction is $\frac{1}{4}$ of 100%

Any one who often deals with per cents soon learns the fractional equivalents of the per cents he commonly employs Ordinarily one should know at a glance that

$$12\tfrac{1}{2}\% = \tfrac{1}{8} \qquad 25\% = \tfrac{1}{4}$$
$$16\tfrac{2}{3}\% = \tfrac{1}{6} \qquad 33\tfrac{1}{3}\% = \tfrac{1}{3}$$
$$20\% = \tfrac{1}{5} \qquad 50\% = \tfrac{1}{2}$$

Under certain circumstances it is desirable to memorize those not so commonly used such as the following

$$1\tfrac{1}{3}\% = \tfrac{1}{75} \qquad 5\tfrac{5}{9}\% = \tfrac{1}{18}$$
$$2\tfrac{1}{2}\% = \tfrac{1}{40} \qquad 6\tfrac{1}{4}\% = \tfrac{1}{16}$$
$$3\tfrac{1}{3}\% = \tfrac{1}{30} \qquad 6\tfrac{2}{3}\% = \tfrac{1}{15}$$
$$4\% = \tfrac{1}{25} \qquad 8\tfrac{1}{3}\% = \tfrac{1}{12}$$
$$4\tfrac{1}{6}\% = \tfrac{1}{24} \qquad 11\tfrac{1}{9}\% = \tfrac{1}{9}$$

Finding aliquant parts when aliquot parts are known

Since the numerator in each aliquot part is 1, if the aliquot part is known, any aliquant part can be found by multiplying the aliquot part by the numerator of the aliquant part

Illustrations

a Given that $12\frac{1}{2}\%$ is $\frac{1}{8}$ of 100%, what is $\frac{3}{8}$ of 100%?

Since $\frac{1}{8} = 12\frac{1}{2}\%$, then $\frac{3}{8} = 3 \times 12\frac{1}{2}\% = 37\frac{1}{2}\%$

b Given that $16\frac{2}{3}\%$ is $\frac{1}{6}$ of 100%, what is $\frac{5}{24}$ of 100%?

Since $\frac{5}{24} = \frac{4}{24} + \frac{1}{24} = \frac{1}{6} + \frac{1}{24}$, and since $\frac{1}{24}$ is $\frac{1}{4}$ of $\frac{1}{6}$, then $\frac{5}{24} = 16\frac{2}{3}\% + \frac{1}{4}$ of $16\frac{2}{3}\% = 16\frac{2}{3}\% + 4\frac{1}{6}\% = 20\frac{5}{6}\%$

It would be wise to memorize the following aliquant parts of 100%

$$37\tfrac{1}{2}\% = \tfrac{3}{8} \qquad 83\tfrac{1}{3}\% = \tfrac{5}{6}$$
$$62\tfrac{1}{2}\% = \tfrac{5}{8} \qquad 41\tfrac{2}{3}\% = \tfrac{5}{12}$$
$$87\tfrac{1}{2}\% = \tfrac{7}{8} \qquad 58\tfrac{1}{3}\% = \tfrac{7}{12}$$
$$66\tfrac{2}{3}\% = \tfrac{2}{3} \qquad 31\tfrac{1}{4}\% = \tfrac{5}{16}$$
$$75\% = \tfrac{3}{4} \qquad 56\tfrac{1}{4}\% = \tfrac{9}{16}$$

EXERCISE 5.5

Find the following fractional parts of 100%.

1. $\frac{3}{16}$	3. $\frac{11}{12}$	5. $\frac{11}{24}$	7. $\frac{7}{20}$	9. $\frac{7}{40}$
2. $\frac{7}{16}$	4. $\frac{7}{24}$	6. $\frac{13}{16}$	8. $\frac{11}{30}$	10. $\frac{5}{32}$

Express the following as fractional parts of 1.

11. $21\frac{1}{4}\%$	13. $7\frac{1}{2}\%$	15. $18\frac{1}{2}\%$	17. 45%	19. $\frac{11}{24}\%$
12. $4\frac{1}{2}\%$	14. $4\frac{1}{3}\%$	16. $42\frac{1}{2}\%$	18. $\frac{5}{16}\%$	20. $14\frac{2}{3}\%$

Finding the base when the rate and percentage are known

Occasionally problems arise in which the rate and the percentage are known but the base is unknown. If, for example, a man's taxes amount to \$480, and he was taxed at the rate of 5% on his real property, the question is, what was the valuation of his real property?

If 5% of the valuation of his real property is \$480, then 1% must be equal to $\frac{1}{5}$ of \$480 or to \$96. The total value of the property would be 100% of the valuation, or 100 times as much as 1%. That is, $\$96 \times 100 = \$9{,}600$.

Stating this whole process in one step, we have $\frac{\$480 \times 100}{5}$, or simply $\frac{\$480}{5\%}$. That is, to get the base (\$9,600), divide the percentage (\$480) by the rate (5%).

In some phases of business this is the most important type of percentage problem. For example, in dealing with the valuation of property, if the income is known, the value of the property can be determined by assuming a rate of return—that is, a per cent that is reasonable.

Illustration: A preferred stock pays an annual dividend of \$4.00. How much would one be justified in paying for the stock assuming that a rate of return of 5% is desired?

Since the amount that would be paid for the stock is the base, and the rate of return is the rate (5%), and the annual dividend is the percentage (\$4.00), one would be justified in paying $\frac{\$4.00}{5\%} = \frac{\$4.00 \times 100}{5} = \$80.$

EXERCISE 5.6

Solve the following problems.

1. 20 is 40% of what number?
2. 15 is 20% of what number?
3. 18 is 1% of what number?
4. 19.5 is 50% of what number?
5. 8.2 is 0.54% of what number?
6. 14.4 is 90% of what number?
7. \$17.50 is 35% of what number?
8. $\frac{3}{4}$ is 75% of what number?
9. 220 is 0.9% of what number?
10. 108 is $22\frac{1}{2}\%$ of what number?

11. \$960 is $31\frac{1}{4}\%$ of what amount?
12. \$8,800 is $43\frac{3}{4}\%$ of what amount?
13. \$42 is $41\frac{2}{3}\%$ of what amount?
14. \$1,620 is $8\frac{1}{3}\%$ of what amount?
15. \$18 is $\frac{3}{8}\%$ of what amount?
16. \$132 is $\frac{7}{12}\%$ of what amount?
17. \$2,880 is $\frac{5}{8}\%$ of what amount?
18. \$38 70 is $62\frac{1}{2}\%$ of what amount?
19. \$42 85 is 12 375% of what amount?
20. \$75 is 58 33 % of what amount?

21. A family saved \$375 in one year If this was $8\frac{1}{3}\%$ of the total income, find the total income

22. A man paid 30% of his debt If he paid \$22 50, what was the total amount of the debt?

23. The state sales tax is $3\frac{1}{2}\%$ of total sales If a merchant pays \$842 60, what were his total sales? *Note* The sales tax is not included in computing sales, but is computed separately

24. If a salesman sells a car for \$1,500 and states that the price is $17\frac{1}{2}\%$ less than quoted price, what is the quoted price?

25. Cost of doing business is 35% of gross sales If the cost of doing business is \$18,500, what must be the gross sales?

26. A college senior is offered a position selling shares in the AAA Mutual Fund at the rate of 4% commission on net sales Before accepting the position he decided to make some calculations of the amounts he would have to sell to achieve different basic salaries Complete his tabulation

In order for me to receive weekly	*I must have weekly sales of*
\$ 75	?
100	?
125	?
150	?
200	?
250	?

27. Five years ago John Sterling accepted a position with a brokerage house He receives on an average 25% of the commissions paid by his customers on the purchase and sale of securities The average rate of commission paid by the customer is 5% of the purchase or sale If John's present income averages \$1,000 a month, what is his monthly volume of business?

28. Since Tom Linthicum was made sales manager 3 years ago, sales have increased 175% each year over the preceding year. How large were sales the third year in comparison with sales the year before he took the job?

29. The David-Perry Department Store reported an 8% decrease in sales this year in comparison with last year. If sales this year were $1,840,000, what were sales last year?

30. Sales in March of last year were $15,600, and this March $16,632. Last year there were 26 selling days in March compared with 28 this year. What was the per cent change in average daily sales?

Discounts

The printing of a catalog is an expensive operation. Wholesalers, manufacturers, and jobbers who deal in articles which are standardized but in continuous demand, minimize expenses by issuing catalogs infrequently. The prices included in such a catalog, called *list prices* or *catalog prices,* are usually much higher than the vendor expects to get. Along with the catalog, the seller includes a separate *discount sheet,* showing the percentage reductions which may be made from the catalog prices. These reductions from the list price, or from the amount of a bill of goods, are known as *trade discounts.*

The catalog need not be reprinted as price changes occur; instead only new discount sheets need to be issued. The net price can be found readily by deducting the discount from the list price. Often as new discount sheets are issued, or as larger quantities are purchased, several discounts may be given from one price. These are known as *series discounts, successive discounts,* or *chain discounts.* Usually the trade discounts are deducted before the invoice is made out, and consequently the invoice may show only *net price*—that is, the cost of the goods after all trade discounts have been deducted.

Single discount

The trade discount is stated as a per cent of the list price. To find the amount of trade discount, it is necessary only to multiply the list price by the discount rate. To find the *net price,* deduct the trade discount from the list price.

Illustration: The catalog of a wholesaler lists radios at $140. The discount is 40%. What is the net cost?

First solution: $\$140 \times 40\% = \$56$, the discount
$\$140 - \$56 = \$84$, the net cost

Second solution Since the discount is 40% of the list price, the net cost must be 60% of the list price That is

If	100%	= list price per cent
less	40%	= discount per cent
gives	60%	= net price per cent

The net price per cent is called the complement of the discount per cent—that is, the net price per cent plus the discount per cent equals 100%

So, to find the net cost, multiply the list price by the complement of the discount per cent Thus $140 × 60% = $84, the net cost

EXERCISE 5.7

Solve the following problems

1. The Acme Refrigerator Company lists a certain style of refrigerator at $380 What is the net cost if the discount rate is $33\frac{1}{3}$%?

2. In a certain catalog hammers are listed at $2 10 If a discount rate of $18\frac{1}{3}$% is allowed, what amount would be on an invoice for one dozen hammers?

3. An importer offers a discount of $37\frac{1}{2}$% on rugs What are the net prices for the following list prices $485, $750, $1,200?

4. *Find the deduction allowed and the amount due for a piano listed* at $925, less $22\frac{1}{2}$%

5. Men s shirts, listed at $27 50 a dozen, cost how much each if a trade discount of $16\frac{2}{3}$% is allowed?

6. What rate of trade discount is allowed if the net price of an article is $180 and the list price is $250?

7. What trade discount per cent is allowed if the net price of a lawnmower is $12 50 and the list price is $240 a dozen?

8. Find the deduction allowed and the amount due for a television set listed at $480, less 23%

9. The White Company offers heavy-duty water pumps at $298, with a discount of 25%, a similar product is listed by the Black Company at $312, less 30% Which is less expensive?

10. Sterling Motors sells a $\frac{3}{4}$-horse power motor at $29 75, less 10% A competitor has a similar motor at $40, less $33\frac{1}{3}$% Which is cheaper?

Series discounts

It should be emphasized that when more than one discount is given, each successive discount is computed on the net price after the preceding discount has been deducted, and not on the basis of the list price The

order in which the discounts are taken does not affect the result. Discounts of 20% and 10% result in the same net price as discounts of 10% and 20%. This fact is shown in the following illustration.

Discounts of 20% and 10%		*Discounts of 10% and 20%*	
Cost..................	\$100	Cost..................	\$100
Deduct 20%..........	20	Deduct 10%..........	10
	\$80		\$90
Deduct 10%..........	8	Deduct 20%..........	18
	\$72		\$72

When a series of discounts is given, the net price may be found in a manner similar to that shown in the preceding example. The discount is deducted from the cost; from the difference, the next discount is calculated and deducted; and so on, until all discounts have been deducted. Since in a given business or industry the same discounts are often allowed by the various competing companies, much time may be saved by finding one discount which is equivalent to a series.

Merchandisers, like other groups, acquire habits of speech and expressions to meet their particular needs. In their vernacular, for example, they refer to the complement of a series of discounts as the "on" percentages. That is, the percentages which when multiplied by the list price will give the net price.

If, in the preceding illustration, the cost had been considered 100%, deducting the discounts of 20% (100% − 20% = 80%), and 10% (10% of 80% = 8%, giving 80% − 8% = 72%), gives 72%, the "on" percentage. The discount deducted actually was 28% as the equivalent of 20% and 10%. The equivalent of any series of discounts and the "on" percentage can be found in a similar way.

Illustration: Find a single discount equivalent to discounts of 40%, 10%, and 5%.

List price per cent......	100%
Deduct 40%...........	40%
	60%
Deduct 10% of the 60%	6%
	54%
Deduct 5% of the 54%..	2.7
Net price per cent......	51.3%, the "on" percentage.

Since the single discount equivalent to a series of discounts is the list price per cent minus the net price per cent, the single discount per cent equivalent to discounts of 40%, 10%, and 5% is 100% − 51.3% = 48.7%.

Exactly the same procedure can be used for any number of discounts in a series An alternate method is to subtract each single discount per cent from 100% and find the product of the remainders The difference between this final product and 100% is the single discount per cent equivalent to the series of discount per cents

Illustration Find the single discount per cent equivalent to discounts of 40%, 10%, and 5% using this alternate method

	Shown as decimals
100% − 40% = 60%	1 − 0 40 = 0 60
100% − 10% = 90%	1 − 0 10 = 0 90
100% − 5% = 95%	1 − 0 05 = 0 95

60% × 90% × 95% = 51 3%, 0 60 × 0 90 × 0 95 = 0 513 = 51 3%

100% − 51 3% = 48 7%

which is the single discount per cent equivalent to the three discounts of 40%, 10%, and 5%

A short method which can be used to find the single discount per cent equivalent to any two discount per cents is to *subtract the product of the discounts from the sum of the two discounts*

Illustration Using the short method, find the single discount equivalent to two discounts of 20% and 10%

The sum of the two discount per cents	30%
The product of the two discount per cents	2%
Their difference	28%

Therefore a single discount of 28% is equivalent to the two discounts of 20% and 10%

When combinations of discounts are frequently used, it is logical to construct a table showing single equivalent rates Such a table could be constructed by using the methods just shown

EXERCISE 5.8

Solve the following problems

1. Find the single discount equivalent to the following series of trade discounts

a 20% and 15%	d $12\frac{1}{2}$% and 10%	g 25%, 16%, and 10%
b 40% and 5%	e 20%, 20%, and 5%	h $33\frac{1}{3}$%, $16\frac{2}{3}$%, and $8\frac{1}{3}$%
c $12\frac{1}{2}$% and 20%	f 40%, 15%, and 8%	i 25%, 20%, 15%, and 5%

2. Find the following net prices.
   a. \$2,450 less 40% and 10%
   b. \$280 less 30%, 5%, and 5%
   c. \$720 less $12\frac{1}{2}$% and 10%
   d. \$1,535 less 40% and 5%
   e. \$600 less 20%, 20%, and 20%
   f. \$1,480 less $33\frac{1}{3}$% and $6\frac{1}{4}$%
   g. \$320 less 25%, 10%, and 10%
   h. \$1,350 less 10%, $6\frac{1}{2}$%, and $2\frac{1}{2}$%
   i. \$8,500 less 40%, $33\frac{1}{3}$%, and 25%
   j. \$1,750 less $8\frac{1}{3}$%, $6\frac{1}{4}$%, and $3\frac{1}{2}$%

3. The catalog price of a chair is \$67.50. If discounts of 20% and 15% are allowed, what is the net price?

4. Find the deductions allowed and the amount due for a piano listed at \$1,250 less 15%, 10%, and $7\frac{1}{2}$%.

5. The Superior Cabinet Company lists a certain style of kitchen cabinet for \$87.50. What is the net cost to a contractor if discounts of $22\frac{1}{2}$% and $12\frac{1}{2}$% are allowed?

6. Men's ties, listed at \$14.50 a dozen, cost how much each if trade discounts of 20%, 5%, and 2% are allowed?

7. What is the invoice figure for an item listed at \$750 for a good customer who is allowed discounts of 20%, 10%, 10%, and 8%?

8. Find the deduction allowed and the amount due for a camera listed at \$325, less $33\frac{1}{3}$%, 20%, and $12\frac{1}{2}$%.

9. How much was paid for 45 crosscut saws at \$2.40 each, less discounts of 15%, $12\frac{1}{2}$%, and 10%?

10. A furniture dealer bought two dozen lamps listed at \$8.50 each, less discounts of 20% and $12\frac{1}{2}$%. If the total freight charges were \$10.78, what was the total cost?

11. If a certain item on which discounts of 20%, 10%, and 5% are allowed cost \$13.50, what must be the list price?

12. A bookcase is to be sold for \$42.50 by a furniture manufacturer. If he allows trade discounts of 25%, 10%, 5%, and 2%, what must be his list price?

Cash discount

It is a common practice in business transactions between manufacturers and wholesalers, or between wholesalers and retailers, to encourage prompt payment of bills by allowing a certain percentage reduction in the price of merchandise if payment is made immediately, or within a

specified time Such allowances called *cash discounts*, are found in almost all lines of trade generally ranging from 1% to 2% of the price of the merchandise

The rate of cash discount allowed is usually specified at the top of the invoice It is stated somewhat as follows *Terms* 2/10n/30 In this example the first number gives the discount rate, 2% the second number, 10, indicates the number of days during which the discount may be deducted, the letter *n* refers to net, and along with the 30, indicates that if not paid earlier the full purchase price is due 30 days following the date of the invoice

Illustration Find the amount necessary on June 25 to pay an invoice of $121 37, dated June 16, *Terms* 2/10n/30

It is due, without discount, on July 16 Since the calculation of the payment period starts with the date of the invoice unless the terms indicate otherwise, the last date on which a cash discount could be taken on this invoice is June 26 If paid any time before June 26, cash discount equal to 2% of the face amount of the invoice may be deducted If payment is made within the discount period, the amount to be paid is calculated as follows

Amount of invoice	$421 37
Discount of 2%	8 42
Net amount of payment	$412 95

It should be observed that cash discount is found by multiplying the amount of the invoice by the cash discount rate If the payment is made within the specified period, the amount of cash discount is deducted from the face amount of the invoice The number of days plays no part in determining the amount of cash discount In this illustration, the cash discount is the same whether payment is made on June 17 or June 25

Datings

Terms of sale differ greatly from one line of merchandise to another and from one industry to another End of Month dating, represented by the letters E O M,' means that the days for allowing discount are counted from the end of the month, and not from the date of the invoice Thus an invoice dated August 14, *Terms* 3/10 E O M, means that if the purchaser pays before September 10 he may deduct 3% from the amount of the invoice If an invoice is dated after the 25th of a month, terms based on E O M usually mean the end of the following month On an invoice dated July 27, *Terms* 3/10 E O M, discount may be taken if the bill is paid before September 10

Extra dating means that the discount may be taken for a specified number of days in addition to the number first indicated in the terms. Thus 3/10 60 Extra indicates that 3% cash discount may be deducted not only during the first 10 days following the date of the invoice but also for 60 additional days, or a total of 70 days from the date of the invoice.

EXERCISE 5.9

Find the amount paid in each of the following bills.

	Amount of Invoice	*Terms*	*Date of Invoice*	*Date of Payment*	*Cash Discount*	*Amount Paid*
1.	$925.34	3/10 n/60	Aug. 11	Aug. 20		
2.	$524.80	2/10 n/30	Feb. 14	Feb. 21		
3.	$1,284.56	1/20 n/90	Apr. 24	May 4		
4.	$684.12	2/30 n/60	Mar. 17	Apr. 15		
5.	$196.25	2/10 E O M	Nov. 14	Dec. 8		
6.	$582.68	3/15 E O M	May 18	June 7		
7.	$242.62	2/10 E O M	Apr. 28	June 1		
8.	$2,156.48	3/20 E O M	Dec. 27	Feb. 18		
9.	$327.75	2/10 60 Extra	June 17	July 30		
10.	$738.82	1/15 60 Extra	Feb. 26	May 6		

Per cent increase

Three basic types of problems dealing with percentage so far discussed include finding the rate, or the percentage, or the base. There are two applications of these three basic types of problems which are sufficiently dissimilar and frequent to justify separate discussion. Such problems involve either the addition of a percentage to a base, or the deduction of a percentage from a base.

When any stated percentage is added to a given base the resulting sum is called the *amount*. Thus if $1,000 is increased by 11%, the increase would be $110 and the total—that is, the amount—would be $1,110. No difficulties arise in solving problems of amount if the relationships are clearly stated and understood. When the $1,000 is increased 11% the amount is 11% greater than the base; that is, the amount is 111% of the base. If it is known that the amount is $1,110, and that this is 11% more than the base, the base can be found by dividing the amount by the sum of 100% and the per cent increase. Thus $1,110 ÷ 111% = $1,000. Business problems often arise in which the amount is known and it is necessary to find the base.

Illustration. A contractor knows that to be successful he must earn 15% for profit and overhead. If he finds that a house can be sold for $12,600, what is the maximum cost he may incur in building such a house?

The cost per cent is 100%, and the profit and overhead per cent is 15%. Therefore the selling price per cent—that is, the amount per cent—is 115%. Since 115% of the cost is $12,600, the cost is $12,600 ÷ 115% = $10,956.52. This is the maximum cost he can incur to be successful.

This problem can be represented by a dual diagram.

Base (100%)	Increase (15%)
Amount (115%)	

Base ($10,956.52)	Increase ($1,653.48)
Amount ($12,600.00)	

Problems dealing with percentage increase often occur in merchandising. The cost of an item may be considered the base, the difference between the cost and the selling price—called the markup—the percentage increase, and the selling price the amount. The following illustrations give two examples of the types of problems most frequently encountered.

Illustrations

a. A merchant sold an article for $3.00. If the selling price was 20% above his cost, what was his cost?

The cost is 100%, and the per cent increase is 20%. Therefore the selling price is 120% of cost. Since 120% of the cost is $3.00, the cost is $3.00 ÷ 120% = $2.50.

b. What amount increased by 25% is $75?

The number is the base (100%). The per cent increase is 25%. Thus $75 must be 125% of the base. Thus the base must be $75 ÷ 125% = $60. That is, $60 increased by 25% of itself is $75.

Per cent decrease

Frequently a problem in per cent involves finding a percentage of a number and then deducting the percentage from the base to obtain a figure called the *difference*. This can be represented graphically as follows.

Difference	Percentage
Base	

In this diagram the base is equivalent to the sum of the Difference and the Percentage, in contrast to the diagram previously considered, in which the Amount represented the sum of the Base and the Percentage. To facilitate comparison the diagram showing the relationship between (a) the Base and Percentage, and (b) the Amount, is shown below.

Base	Percentage
Amount	

Often the base is known and it is necessary to find the difference and the percentage. For example, if a partial payment of \$847.70 is made on a \$1,000 invoice during the discount period, how much credit should be given if the terms are 2/10n/30? The payment of the total invoice of \$1,000 during the discount period would have required only \$980 (\$1,000 less 2% of \$1,000). In other words, each 98 cents paid during the discount period applied as \$1.00 on the account. Hence for a partial payment of \$847.70 more credit than \$847.70 should be given. Since \$980 could have discharged a debt of \$1,000, the credit which should be given for a payment of \$847.70 can be found by dividing it by 98% (100% − 2%). Since \$847.70 ÷ 98% = \$865, the credit should be \$865. To verify this, compute 2% of \$865 to find the discount. This will be found to be \$17.30, which when deducted from \$865 leaves \$847.70, the amount of the payment. The problem and solution is shown in the accompanying diagram.

(98%) \$847.70	(2%) \$17.30
\$865.00 (100%)	

It takes practice to recognize what is given and what is to be found in many problems of this type. In business where shrinkages occur, or where parts may be defective, adequate but not excessive allowances must be made. The following illustrations show examples of such problems.

Illustrations:

a. In the manufacture of a certain type of steel casting, it was found that 4 out of each 50 were defective. The manufacturer received an order for 825 perfect castings. In computing his costs, what is the minimum number of castings he can wisely consider making?

If 4 out of 50 are defective, the per cent defective is 8%, or $\frac{4}{50}$. Then 100% − 8% = 92%, the difference. Now 92% of the number cast is 825, the number of perfect castings desired, and 825 ÷ 92% = 896.74, or 897. Therefore 897 is the number which must be cast to assure 825 perfect ones.

b A given type of cloth shrinks 10% when it is dyed If a piece 5 yards long is desired after dyeing how long should it be before dyeing?

Length of desired piece is 5 yards = 5 × 36 inches = 180 inches Rate of shrinkage is 10% Therefore 100% − 10% = 90%, the difference Before dyeing (the base) is 180 inches ÷ 90% = 200 inches

c What number decreased by 15% of itself is 527?

100% − 15% = 85%, 85% of the desired number = 527

The desired number is therefore 527 ÷ 85% = 620

EXERCISE 5.10

Solve the following

1. 29 is $12\frac{1}{2}$% more than what number?
2. 32 is $12\frac{1}{2}$% less than what number?
3. 100 is 500% more than what number?
4. What number increased by 100% of itself gives 8?
5. 8 is 20% more than what number?
6. 12 is 20% less than what number?
7. $\frac{3}{4}$ is 200% more than what number?
8. $\frac{5}{8}$ is 60% less than what number?
9. What number increased by 5% of itself gives 210?
10. What number decreased by 15% of itself gives 527?
11. \$750 is $6\frac{1}{4}$% less than what number?
12. \$1,050 is $37\frac{1}{2}$% less than what amount?
13. \$4,200 is $31\frac{1}{4}$% more than what amount?
14. What amount increased by $43\frac{3}{4}$% of itself gives \$1,725?
15. What amount decreased by $58\frac{1}{3}$% of itself gives \$750?
16. What amount decreased by $1\frac{1}{4}$% of itself gives \$2,370?
17. What amount increased by $1\frac{2}{3}$% of itself gives \$3,660?
18. \$95 is $5\frac{5}{9}$% more than what amount?
19. \$102 is $5\frac{5}{9}$% less than what amount?
20. \$4,600 is $4\frac{1}{6}$% less than what amount?

21. If it is the policy of a company to increase the wages of each unskilled worker 20% at the end of his first year to the maximum rate of \$1 80 per hour, what must be the starting wage?

22. If the price of gasoline was raised from $28\frac{8}{10}$ cents to $30\frac{6}{10}$ cents a gallon, what was the per cent increase?

23. A real estate agent who is to receive a 5% commission sells a house for \$8,250 How much commission should he receive?

24. An article that costs \$75 is sold for \$120. What was the per cent profit based on costs? Based on selling price?

25. If the assessed value of a house is \$2,850 and the assessed value of the lot is \$1,450, what is the tax bill if the tax rate is 6.382%?

26. The population of Centerville was 18,742 according to the last census. If they can anticipate a $22\frac{1}{2}$% increase during the decade, what should be the population at the time of the next census?

27. Which gives the higher per cent profit, to sell a machine for \$1,680 that costs \$1,280, or to sell a machine for \$1,600 that costs \$1,200? What is the per cent difference?

28. There is a state sales tax of 3% and a local sales tax of 1%. These taxes are added to the retail selling price of each article, but no differentiation is made at the time of the sale between the selling price and the taxes. If total receipts last month were \$18,200, what was the total amount of the taxes collected? Of state sales tax collected? Of local taxes collected?

29. The population of the city of Burbank increased 128.8% in the last 10 years. If the population is 78,577 now, what was it 10 years ago?

30. If for every 1,000 people living there are 24.1 births and 9.7 deaths each year, what per cent increase in population occurs each year?

31. One out of every 4 persons in the labor force is engaged in professional, semiprofessional, managerial, highly skilled, or technical occupations. Thirty per cent are employed as semiskilled workers, operatives, and kindred workers. The rest are unskilled. Out of every 500,000 persons, how many are in each labor force group?

32. The population of the largest city in a state was 1,970,358 last year. If 19% of the population of the state lived in this city, what was the population of the state?

33. A recent survey showed that on an average cotton farm in the Black Prairie area, the investment was \$14,872 in land and buildings and \$1,982 in machinery and livestock. What per cent of the total investment was in machinery and livestock?

34. It is anticipated that in the year 2000 only 26% of the population of the United States will be under 20 years of age. About 61% will be aged somewhere between 20 and 64. The rest will be in the age group 65 and over. If the total population at that time is 210 million, how many will there be in each age group?

35. The Crown City Supply Company went bankrupt. As a result each creditor received $42\frac{1}{2}$% of the amount due. How much was paid the following creditors: A. W. Jones, whose bill was \$438.27; and R. F. Allen, whose bill was \$856.82? If F. W. Smith received \$236.27, how much was owed him?

36. At a clearance sale, dresses that were selling for $27 50 sold for $21 75 The reduction in price was what per cent of the original selling price?

37. Mr A C Williams, an attorney, charges 50% for the collection of debts In a suit for $7,626, Mr Williams collected $2,287 50 His client received half of the sum collected What per cent of the total debt did his client receive?

38. The J N Wright building was allegedly sold for $1,264,000 If the cost of operating the building amounts to $69,160, what must the owners receive as rental income to furnish them a 6% return on their investment?

39. Art Tetrick is paid $4\frac{1}{2}$% commission on the first $100,000 sales and 5% on all sales over $100,000 Last year he received an advance of $150 a week during the year and the balance of his commission at the end of December If his total sales were $175 000, how much did he draw at the end of December?

40. The state levies a sales tax of 3% which is added to the price of goods If $2,266 was paid for goods including the tax, how much of the $2,266 was tax?

41. A share of stock was sold for $60 If the seller lost 25% on the sale, what was the cost?

42. A worker's scale of pay was decreased 5% to $3 08$\frac{3}{4}$ an hour What was his hourly wage before the reduction?

43. In a manufacturing process 3 out of every 120 items are rejected To assure 780 perfect parts, how many should be manufactured?

44. A new salary schedule is announced as $62 50 a week If this is a 15% increase over the old salary schedule, what was the old schedule?

45. If the price of meat is lowered $8\frac{1}{3}$%, what was the old price for a certain cut if the new price is $1 08 a pound?

The 100% statement

Percentage is often used in business and accounting to facilitate comparisons Extremely large figures and great differences in magnitude are difficult for the ordinary person to grasp and compare As companies have grown in size, the figures included in the balance sheet have tended to lose their significance for most people Consequently, when financial analyses are to be made, and—more recently—when figures are to be presented to the public, large companies have found it effective to show the items on their balance sheets not only in dollars and cents but also in per cent, with each asset shown as a per cent of total assets and each liability as a per cent of total liabilities

Credit men often use percentage figures in a similar way to determine the amount of credit which may be extended; security analysts use percentages in selecting corporate securities. For comparative purposes in credit analysis, it is often helpful to supplement the dollar figures with per cent figures. A balance sheet, or series of balance sheets, in which the items are shown as a per cent of the total, is known as a *100%* or *common basis statement*. In preparing such a statement, the total assets are equal to 100%, and are used as the base. The separate items are shown as a per cent of the total.

Illustration: Prepare a 100% or common basis statement from the following statements for the last two years.

	Year Before Last		*Last Year*	
Assets	*Amount*	*Per Cent of Total*	*Amount*	*Per Cent of Total*
Cash	\$ 2,000	5%	\$ 2,500	$3\frac{1}{3}$%
Receivables	8,000	20	15,000	20
Inventory	5,000	$12\frac{1}{2}$	7,500	10
Total current assets	\$15,000	$37\frac{1}{2}$%	\$25,000	$33\frac{1}{3}$%
Fixed assets	25,000	$62\frac{1}{2}$%	50,000	$66\frac{2}{3}$%
Total assets	\$40,000	100%	\$75,000	100%
Liabilities				
Current liabilities	\$ 6,000	15%	\$25,000	$33\frac{1}{3}$%
Mortgages	14,000	35	15,000	20
Capital stock	15,000	$37\frac{1}{2}$	30,000	40
Surplus	5,000	$12\frac{1}{2}$	5,000	$6\frac{2}{3}$
Total liabilities	\$40,000	100%	\$75,000	100%

The first year the amount of cash is \$2,000; the total assets are \$40,000, and cash is 5% of the total (\$2,000 ÷ \$40,000 = 5%). Accounts receivable make up 20% of the total assets (\$8,000 ÷ \$40,000 = 20%). The second year, the amount of cash has increased to \$2,500 and the total assets have increased to \$75,000. Cash has therefore shown a relative decline since it now makes up only $3\frac{1}{3}$% of total assets (\$2,500 ÷ \$75,000 = $3\frac{1}{3}$%). In calculating per cent the second year, the base is the new total of assets, \$75,000.

The 100% statement shows readily what accounts have increased and what accounts have decreased relative to the total. It does not reveal the percentage increase or decrease in each account. Large changes in actual dollar values may mean only small changes on a percentage basis. The

changes shown in per cent are easier to visualize and to comprehend The comparison of one company with another is greatly simplified when both are considered on a 100% basis For this reason, when a financial analysis is made to aid in selecting the stock of one company over another, it is common practice to substitute per cent figures for the dollar values of the balance sheets being compared, and to review the significant ratios from the 100% statements

Horizontal percentage trend analysis

The same type of statement is also helpful in reviewing the progress of a company from one year to the next in that it readily shows what accounts have increased or decreased relative to the total It does not reveal the percentage increase or decrease in each account When such information is desired, it may be presented in what is known as a *horizontal percentage trend analysis* Under such an analysis one year is selected as the base (or 100%) and the items for subsequent years are computed in per cent of the base year It is contended that the horizontal trend analysis draws more attention to the disproportionate changes in balance sheet items than does the 100% statement The changes, whether indicative of increased strength or weakness are more easily seen by the comparison

Illustration Using the information from the preceding illustration show the per cent in each account for last year, using the year before last as the base year

	Year Before Last		*Last Year*	
Assets	*Amount*	*Per Cent*	*Amount*	*Per Cent of Base Year*
Cash	$ 2,000	100%	$ 2 500	125%
Receivables	8,000	100%	15,000	$187\frac{1}{2}$%
Inventory	5,000	100%	7,500	150%
Total current assets	$15,000	100%	$25,000	$166\frac{2}{3}$%
Fixed assets	25,000	100%	50 000	200%
Total assets	$40 000	100%	$75,000	$187\frac{1}{2}$%
Liabilities				
Current liabilities	$ 6,000	100%	$25,000	$416\frac{2}{3}$%
Mortgages	14,000	100%	15,000	$106\frac{4}{7}$%
Capital stock	15,000	100%	30,000	200%
Surplus	5,000	100%	5,000	100%
Total liabilities	$40,000	100%	$75,000	$187\frac{1}{2}$%

Cash increased from \$2,000 in the base year to \$2,500. Since \$2,000 is considered the base, \$2,500 is equal to 125% $\left(\frac{\$2{,}500}{\$2{,}000}\right)$ of the base year. The percentage of each asset is calculated in this way. The total assets for the base year are \$40,000. The total assets rose to \$75,000, or the assets the second year were $187\frac{1}{2}\%$ $\left(\frac{\$75{,}000}{\$40{,}000}\right)$ of what they were in the base year. The percentage change in the total assets must be found in the same way in which the change in each separate asset is found, since the percentage changes in the individual assets cannot be averaged, or totaled, to find the percentage change in the total.

The horizontal percentage statement is commonly used to show per cent increase or decrease for short periods of time between hours worked, sales made, production in units, or any other comparison needed periodically. The base period is selected as equal to 100%. The numerical difference is found as an increase or decrease over the base period. The numerical difference divided by the base number shows the per cent increase or decrease over the base period.

Illustration: From the figures showing the departmental sales of the ABC Department Store for the first three months of this year, as well as those showing sales for the same period last year, find: (a) the increase or decrease in quarterly sales by each department, and (b) the per cent of increase or decrease over the same period last year.

	Sales 1st Quarter		*Amount of*		*Per Cent*	
Dept.	*Last Year*	*This Year*	*Increase*	*Decrease*	*Increase*	*Decrease*
A	\$12,331	\$16,411	\$ 4,080		33	
B	8,424	6,318		\$2,106		25
C	24,125	28,970	4,845		20	
D	15,500	20,150	4,650		30	
Total	\$60,380	\$71,849	\$11,469		19	

In Department A, there was an increase of \$4,080; sales during the base period in this department were \$12,331; therefore the increase was \$4,080 ÷ \$12,331 = 33%. Department B had a decrease in sales of \$2,106; sales in the base year were \$8,424; thus the decrease was 25% of base year sales. Total sales changed from \$60,380 to \$71,849, an increase of \$11,469. Total sales in the base year were \$60,380. Therefore, an increase of \$11,469 was equivalent to an increase of 19%.

Sales are often classified by days, by weeks, or by clerks, and presented on a percentage basis From such data it is possible to see readily the relative position of the items being compared If, in the course of a month total sales have increased 20% while the records indicate that one salesman has doubled the volume of his sales, a second salesman has a small per cent increase, and a third salesman has an actual decline, this situation may indicate a need either for remedial action or for further investigation Often the weekly or monthly sales of each clerk are compared with the average in his department

Illustration Find (a) the total monthly sales for Department A, (b) the average monthly sales for all clerks, and (c) the per cent of total sales made by each clerk

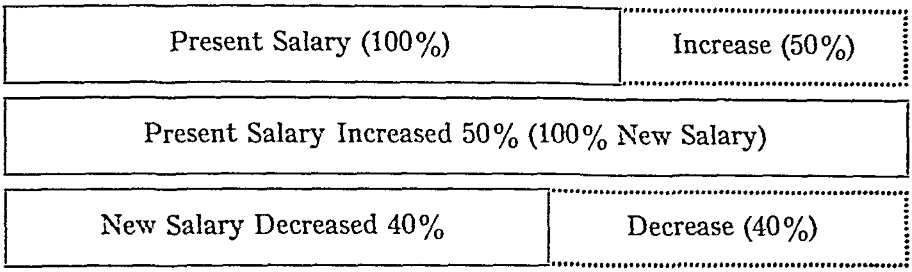
Clerk's Number	*Monthly Sales*
1	$ 3,303 85
2	1,852 85
3	3,065 62
4	2,678 45
Total	$10,900 77

The total sales were $10,900 77 Since there were 4 salesmen, the average was $10,900 77 ÷ 4 = $2,725 19 Of the total sales of $10,900 77 clerk number 1 accounted for $3,303 85, or 30 3% of the total $\left(\frac{\$3,303\ 85}{\$10,900\ 77}\right)$, clerk number 2 had sales of only $1,852 85, or 17 0% of the total $\left(\frac{\$1,852\ 85}{\$10,900\ 77}\right)$, clerk number 3 accounted for 28 1% of the total $\left(\frac{\$3,065\ 62}{\$10,900\ 77}\right)$, and clerk number 4 had the balance, or 24 6% of total sales in the department

Dangers to be avoided in the use of per cent

When dealing with percentage figures, people are prone to make misstatements if they do not distinguish carefully between the use of percentage in absolute and in relative terms Assume that one salesman makes 30% of the sales in a given department, and another makes 25% Speaking in absolute terms, we can say that one sold 5% more than the other, or speaking in relative terms, we can say that the second salesman needs to increase his total sales by 20% (since 20% of 25% is 5%) before his total sales will equal the total sales of the first salesman

A second possibility of error in using per cent is the failure to observe a change in the base on which the per cent is calculated. I would not be willing to receive a 50% increase in pay today and a 40% reduction tomorrow, since I then would actually lose because the base on which the two percentages is calculated is not the same. This fact is shown in the accompanying diagram.



The new salary decreased 40% is only 90% of the present salary. Since 100% + 50% = 150%, the new salary is 150% of the present salary. That is, the base in the first calculation was the present salary (100%). But, for the second calculation the base is the new salary (150%), and 150% − 40% of 150% = 90%.

Illustrations:

a. Robert Olds's salary is $450 a month. His employer increases his salary 40%. But after a few months his new salary is decreased 30%. How much does he now receive?

When his salary was increased he received 140% (100% + 40%) of his old salary. New salary is $450 × 140% = $630. Later this new salary was decreased 30%, so he received 70% of $630. His salary at present is thus $630 × 70% = $441.

b. During a period of prosperity, the number of employees in a given department increased from 20 to 100. What was the percentage increase?

In the ensuing depression, what per cent of the employees of the department would have to be laid off to return to the original number of employees?

$$100 - 20 = 80; \qquad \tfrac{80}{20} = 400\% \text{ increase}$$

$$\text{Decrease necessary} = 100 - 20 = 80; \qquad \tfrac{80}{100} = 80\% \text{ decrease.}$$

Unless one considers the change in base which has occured, he is not likely to realize that a reduction of only 80% in the number of employees is equivalent to a 400% increase.

EXERCISE 5.11

Solve the following problems

1. Prepare a 100%, or common basis, statement from the following statements for the last two years Usually such calculations are rounded to two decimal places

Assets	*One Year Ago*	*Today*
Cash	\$ 68,000	\$ 52,000
Accounts receivable	21,000	48,000
Inventory	37,000	80,000
Total current assets	\$126,000	\$180,000
Fixed assets	\$174,000	\$200,000
Total assets	\$300,000	\$380,000
Liabilities		
Current liabilities	\$ 45,000	\$ 60,000
Long-term liabilities	65,000	120,000
Capital stock	160,000	160,000
Surplus	30,000	40,000
Total liabilities	\$300,000	\$380,000

2. Refer to the balance sheet in Problem 1 Using last year as the base year, show the per cent in each account today

3. Calculate the percentage increase or decrease which an investor gained on each of the following five stocks

Stock of Company	*Number of Shares Owned*	*Cost Per Share*	*Present Market Price*	*Per Cent + or −*
A	1,000	$2\frac{1}{2}$	$3\frac{1}{2}$	
B	100	120	130	
C	50	$38\frac{1}{4}$	35	
D	200	168	170	
E	100	23	28	

4. On the basis of the data in the preceding problem, find the total dollar amount of gain or loss and the per cent of gain or loss for the five stocks

5. A speculator bought 2,000 shares of stock at $3\frac{1}{2}$ The price increased 100%, and then declined from the higher level by 60% of the higher level What was the total market value of his stock after both changes had taken place?

6. Complete the following report. Calculate each per cent to the nearest 0.1%.

Assets	*This Year*	*Last Year*	*Net Change Decrease*	*Net Change Increase*	*Per Cent of Change Decrease*	*Per Cent of Change Increase*
Cash	\$ 4,244	\$ 3,847				
U.S. Gov't securities	7,927	4,320				
Accounts receivable	6,040	8,660				
Inventory	28,549	25,925				
Property, plant and equipment	16,559	16,456				
Total assets	\$63,319	\$59,208				
Liabilities						
Current liabilities	\$17,670	\$20,101				
Long-term debt	10,000	2,000				
Preferred stock	none	8,000				
Common stock	8,000	7,500				
Earnings retained in business	27,649	21,607				
Total liabilities	\$63,319	\$59,208				

7. Complete this comparative statement of income and expenditures by filling in the per cent of net sales column. Calculate each per cent to the nearest 1%.

COMPARATIVE STATEMENT OF INCOME AND EXPENDITURES

	This Year		*Last Year*	
	Amount	*Per Cent of Net Sale*	*Amount*	*Per Cent of Net Sale*
Net sales	\$165,710	100%	\$135,150	100%
Selling, general and admin. expense	30,650		24,643	
Interest	1,017		494	
Federal income tax	10,474		13,815	
Other taxes	1,050		3,785	
Net profit	7,931		8,620	

8. Sales of the third largest public utility company last year totaled \$127,270,104. The sales are classified as follows.

Domestic	$46,090,138
Agricultural	9,213,205
Commercial	26,979,337
Industrial	32,022,327
Public authorities	8,295,018
Sales for resale	2,224,447
Railways	1,041,861
Other	1,403 771

Compute the percentage of sales made in each category

9. The Western Utility Company is financed as follows

Bonds	$268 000 000
Preferred stock	130,442,325
Common stock	150,410,122

If these represent the total securities issued by the company, what per cent of the total is represented by each?

10. A company operates five retail outlets The sales of each store for the last two years are shown in the following table Calculate the amount of increase or decrease in sales and the per cent increase or decrease, for each outlet and for all outlets

Outlet	*Sales First Year*	*Sales Second Year*
A	$123,496	$246,992
B	222,490	200,241
C	337,800	295,575
D	500,000	529,789
E	144,500	100,050
Total		

11. A bank charges $\frac{3}{4}\%$ for selling travelers checks This is equivalent to how many cents for each $100 worth of travelers' checks?

12. A bank pays an annual premium to the Federal Deposit Insurance Corporation equal to $\frac{1}{12}\%$ of its deposits If a national bank has deposits of $1,778,280, what is the annual premium?

13 An architect charges 8% for plans and specifications and 1% for supervising the construction of a building which costs, excluding the architect s fees $75,000 Find the amount of the architect s fees What is the total cost of the building?

14. In collecting a debt of $1,840 for a client, a lawyer compromised and accepted $87\frac{1}{2}$ cents on the dollar If the lawyer got 15% of the amount collected, how much did the client receive?

15. The Central Illinois Railroad has 1,358,000 shares of stock outstanding. If the officers and directors of the road own a total of 545 shares, what per cent of the shares outstanding do they own?

16. Last year the Pacific Atlantic Trading Corporation had net sales of \$121,000. The accountant estimated that on the basis of past experience $2\frac{1}{2}\%$ of the amount due from the sales could not be collected. Find the estimated amount of loss on bad debts.

17. In a particular industry past records indicate that $3\frac{1}{4}\%$ of all charge sales are uncollectible. If 50% of all sales are charge sales, what should be the anticipated loss on bad debts if sales last year were \$278,442?

18. Last year the total sales of the Beta Company were \$247,200. Advertising expenses totaled \$4,800. Find what per cent advertising was of sales, and how many cents were expended in advertising for each \$1 of sales.

19. In the community chest drive \$117,725 was raised. This was 97% of the quota. Find the quota.

20. The net sales of a department store last year amounted to \$2,416,000. The cost of the merchandise sold was \$1,500,000, commissions and salaries were \$40,220, heat, light, and electricity \$20,000, advertising \$28,000. Each item was what per cent of sales?

REVIEW PROBLEMS

Chapters 3, 4, and 5

Perform the indicated operations

1. $2\frac{3}{8} + 1\frac{7}{12} + 3\frac{1}{6} + \frac{9}{16} + 1\frac{3}{4}$
2. $1\frac{2}{5} + 4\frac{5}{8} + 2 + 7\frac{11}{20} + 2\frac{7}{40}$
3. $7\frac{1}{3} + 3\frac{3}{8} + 1\frac{5}{9} + 4\frac{7}{18} + 1\frac{11}{24}$
4. $1\frac{2}{9} + 2\frac{1}{3} + 6\frac{3}{5} + \frac{4}{15} + \frac{2}{45}$
5. $3\frac{3}{5} + 1\frac{7}{12} + 2 + 8\frac{1}{4} + 4\frac{7}{30}$
6. $8\frac{5}{16} - 5\frac{7}{8} + 3\frac{1}{2} + 1\frac{3}{4}$
7. $3\frac{7}{12} - 6\frac{5}{8} + 3\frac{5}{6} + 2\frac{2}{3}$
8. $4\frac{3}{5} - 3\frac{1}{3} - \frac{1}{15} - \frac{5}{12}$
9. $7\frac{7}{18} - 2\frac{1}{6} - 4\frac{3}{4} + 1\frac{4}{9}$
10. $9\frac{13}{24} - 7\frac{3}{8} - 1\frac{1}{6} + 4\frac{5}{12}$

Perform the indicated operations

11. $\frac{3}{4}$ of 48
12. $\frac{5}{8}$ of 56
13. $\frac{11}{15}$ of 90
14. $\frac{2}{3}$ of $1\frac{1}{2}$
15. $\frac{4}{5}$ of $3\frac{3}{4}$
16. $\frac{7}{8} \times \frac{16}{21}$
17. $\frac{1}{6} \times \frac{3}{5} \times \frac{15}{16}$
18. $\frac{3}{7} \times \frac{14}{15} \times \frac{5}{8} \times \frac{3}{4}$
19. $3\frac{3}{4} \times \frac{2}{5}$
20. $6\frac{2}{3} \times 3\frac{3}{5}$
21. $4\frac{3}{8} \times 3\frac{3}{7}$
22. $9\frac{3}{8} \times 4\frac{4}{9}$
23. $1\frac{3}{7} \times 2\frac{2}{5} \times \frac{2}{3}$
24. $2 - \frac{4}{9}$
25. $8 - 1\frac{3}{5}$
26. $\frac{3}{8} - \frac{21}{32}$
27. $3\frac{1}{2} - 5\frac{1}{4}$
28. $5\frac{5}{8} - 2\frac{1}{4}$
29. $28\frac{4}{5} - 9\frac{3}{5}$
30. $2\frac{5}{8} \times 3\frac{3}{7} - 3\frac{3}{5}$

Find the exact, the estimated, and the approximate products of the following

31. 23 416 × 46 28
32. 0 3712 × 51 8
33. 55 5423 × 8 225
34. 809 392 × 6 8723
35. 327 82 × 0 0622
36. \$2,836 70 × 0 125
37. \$863 82 × 1 036
38. \$23,805 25 × 0 86645
39. \$82,550 65 × 0 07634
40. \$236,506 80 × 0 008675

Find the estimated and the approximate quotients of the following

41. 478 282 ÷ 6 827
42. 5,068 112 ÷ 82 57
43. 82 432 ÷ 187 2
44. 16 456 ÷ 0 3285
45. 82 918 ÷ 0 04782
46. \$328 56 ÷ 2 135
47. \$118 42 ÷ 0 4378
48. \$578 30 ÷ 0 04382
49. \$2,356 80 ÷ 125
50. \$8 37 ÷ 0 00785

Find the other values in each of the following problems.

Problem	*Fraction Form*	*Decimal Form*	*Fraction % Form*	*Decimal % Form*
51.	$\frac{3}{16}$	—	—	—
52.	$\frac{11}{16}$	—	—	—
53.	$\frac{5}{12}$	—	—	—
54.	$\frac{7}{12}$	—	—	—
55.	$\frac{5}{32}$	—	—	—
56.	—	0.325	—	—
57.	—	0.0875	—	—
58.	—	0.00875	—	—
59.	—	$0.033\frac{1}{3}$	—	—
60.	—	0.833...	—	—
61.	—	—	$2\frac{1}{4}\%$	—
62.	—	—	$7\frac{1}{2}\%$	—
63.	—	—	$\frac{1}{4}\%$	—
64.	—	—	$22\frac{2}{9}\%$	—
65.	—	—	$8\frac{1}{3}\%$	—
66.	—	—	—	0.375%
67.	—	—	—	6.75%
68.	—	—	—	2.45%
69.	—	—	—	4.66...%
70.	—	—	—	0.0533...%

Solve the following percentage problems.

71. 3% of 48
72. 8% of 125
73. 5% of 180
74. 15% of 40
75. 32% of 45
76. 18% of 75
77. 22% of $36
78. $3\frac{1}{2}\%$ of $1,000
79. $\frac{1}{3}\%$ of $360
80. 200% of 12

81. $12\frac{1}{2}\%$ of 96
82. $2\frac{1}{4}\%$ of 320
83. $37\frac{1}{2}\%$ of 64
84. 13% of 50
85. $8\frac{1}{3}\%$ of 240
86. 10% of 132
87. 1% of 48.6
88. $\frac{3}{5}\%$ of 125
89. 1,000% of 3.2
90. 225% of 6

91. $1\frac{1}{2}\%$ of 60
92. $6\frac{1}{2}\%$ of 32.6
93. $\frac{1}{2}\%$ of 2.25
94. $\frac{1}{4}\%$ of 800
95. 0.45% of 500
96. 0.75% of 144
97. 0.16% of 483
98. 0.06% of 483
99. 2.66...% of $24.90
100. 13.85% of $72.48

101. 8 is what per cent of 40?
102. 12 is what per cent of 96?
103. 42 is what per cent of 300?
104. 4 is what per cent of 9?
105. $\frac{1}{4}$ is what per cent of 2?

106. 18 is what per cent of 80?
107. 150 is what per cent of 1,200?
108. 450 is what per cent of 6,000?
109. $\frac{3}{4}$ is what per cent of 8?
110. 16 is what per cent of 24?

111. 3 is what per cent of 120?
112. 5 is what per cent of 600?
113. 25 is what per cent of 800?
114. $\frac{1}{2}$ is what per cent of 15?
115. 0 4 is what per cent of 5?
116. 0 15 is what per cent of 0 5%?
117. 32 is what per cent of 400?
118. 27 is what per cent of 20?
119. 40 is what per cent of 25?
120. $200 is what per cent of $750?
121. 320 is 40% of what number?
122. 64 is 8% of what number?
123. 8 25 is 1% of what number?
124. 9 6 is 24% of what number?
125. $48 is 12% of how much?
126. 3 is 0 5% of what number?
127. 36 is 0 09% of what number?
128. 65 is $16\frac{2}{3}$% of what number?
129. $8 40 is $3\frac{1}{3}$% of what amount?
130. $32 is $8\frac{1}{3}$% of what amount?

131. What number increased by 15% gives 135 7?
132. What number increased by 40% gives 203?
133. What number decreased by $12\frac{1}{2}$% gives 161?
134. What number decreased by $8\frac{1}{3}$% gives 1,650?
135. 135 is what per cent more than 120?
136. 32 is what per cent less than 40?
137. What is $12\frac{1}{2}$% more than 60?
138. What is $16\frac{2}{3}$% less than 960?
139. What number increased by 20%, then increased by $12\frac{1}{2}$%, gives 270?
140. What number increased by $22\frac{1}{2}$%, then decreased by $6\frac{1}{4}$% gives 441?

Find the cost of the following purchases

141. 32 pounds of cheese @ $87\frac{1}{2}$¢
142. 45 pounds of candy @ $33\frac{1}{3}$¢
143. 20 yards of cloth @ $83\frac{1}{3}$¢
144. $16\frac{2}{3}$ yards of silk @ $4 80
145. 96 feet of rope @ $12\frac{1}{2}$¢
146. 75 gallons turpentine @ $66\frac{2}{3}$¢
147. $62\frac{1}{2}$ pounds of meat @ 50¢
148. 24 yards of rayon @ $1.75
149. $87\frac{1}{2}$ yards of rubber tubing @ $1 20
150. 21 quarts of milk @ $21\frac{2}{3}$¢
151. 56 pounds of coffee @ $87\frac{1}{2}$¢
152. 48 pounds of flour @ $7\frac{1}{2}$¢
153. 72 yards of ribbon @ $83\frac{1}{3}$¢
154. $16\frac{2}{3}$ gallons of gasoline at 27¢ per gallon
155. 60 pounds of sugar at $8\frac{1}{3}$¢ per pound
156. $83\frac{1}{3}$ gallons of solvent at $1 20 per gallon

157. $87\frac{1}{2}$ acres at \$560 per acre
158. 48 feet of hose at $6\frac{1}{4}$¢ per foot
159. 32 tons of coal @ \$12.50
160. 64 grams of a drug @ $18\frac{3}{4}$¢

Find the unknown in each of the following.

Problem	*Rate %*	*Base*	*Percentage*
161.	?	\$1,610	\$230
162.	0.45%	?	\$27
163.	$6\frac{1}{4}$%	\$2,720	?
164.	?	\$3,458	\$1,642.55
165.	39%	\$3,666.67	?
166.	$8\frac{1}{3}$%	?	\$520
167.	$22\frac{1}{2}$%	\$10,428	?
168.	150%	?	\$7,200
169.	$3\frac{1}{2}$%	\$2,400,000	?
170.	$3\frac{3}{4}$%	?	\$1,862,500

171. A speculator bought 50,000 bushels of corn at \$1.16$\frac{3}{4}$ per bushel. The commission charged on the purchase was $\frac{3}{8}$ of a cent per bushel. Soon he sold 20,000 bushels at \$1.18$\frac{3}{8}$ per bushel, and the remaining 30,000 bushels at \$1.17$\frac{7}{8}$ per bushel. The commission paid on the sale amounted to \$15 for each 5,000 bushels. Find his net gain.

172. An investor has 1,000 shares of stock which were purchased several years earlier at a total cost of \$7,651.50. No dividends have been paid on the stock, and since the company is on the verge of bankruptcy the market price has declined to $\frac{1}{64}$ (that is, the market price per share is $\frac{1}{64}$th of a dollar). In order to take advantage of his loss for income tax purposes, the seller sells the 1,000 shares at the market price. The seller is required to pay the federal tax of 5 cents per share, the state tax of 1 cent per share, and the broker's commission of $\frac{1}{2}$ cent per share. Find the investor's total loss.

173. A salesman is paid 2 cents per gallon as a commission for selling an oil spray. There are $7\frac{1}{2}$ gallons in a cubic foot. What should be the salesman's commission if he sells the entire contents of a tank $\frac{5}{6}$ full if the tank has a total capacity of $36\frac{1}{2}$ cubic feet?

174. The price of wheat dropped from \$3 per bushel to \$0.46 per bushel. Find the per cent decrease. From the low price of \$0.46 the price increased during the next 5 years to \$1.38. What was the per cent increase?

175. Past records indicate that in a particular shop $4\frac{1}{2}\%$ of the castings were defective If the operator of this shop is to deliver 1,000 perfect castings what is the smallest number on which he should calculate his costs?

176. A merchant spent \$12 800 for advertising His advertising expenditure was 8% of his sales How much were his sales?

177. In a certain charity drive \$378,000 was raised This was $94\frac{1}{2}\%$ of the quota What was the quota?

178. A merchant feels that by an intensive advertising campaign he can increase his sales each year by 10% over the previous year s sales If his sales were \$130,000 the year before the campaign started and he accomplished his annual increase each year, what were his sales the fifth year? What per cent gain in sales did the campaign produce?

179. A car which cost \$2,350 when it was purchased $3\frac{1}{2}$ years ago is worth \$460 today Find the average monthly decline in value What per cent of the cost was the average monthly decline in value?

180. A train ran 14 8 miles in 9 minutes Find its rate per hour

181. In running a certain business past records indicate that $4\frac{1}{3}\%$ of all charge sales are uncollectible If 65% of all sales are charge sales, what should be the anticipated loss on bad debts if total sales last year were \$1,327,834 67?

182. The net sales for a going concern for a certain year were \$847,268 If \$38,127 06 was spent for advertising, what per cent of net sales was used for advertising?

183. If the maximum wage in an agreed-upon wage scale is to be \$1 32 per hour, and the base wage is to be increased 20% each year for 3 years to reach this scale, what is the base pay per hour?

184. If the sales for a retail outlet increased from \$123,496 to \$246,992 in one year, what was the per cent increase?

185. If the sales for a retail outlet decreased from \$337,800 to \$295,575 in one year, what was the per cent decrease?

Find the net gain or loss on each of the following transactions

	Number of Shares	*Purchase Price per Share*	*Commission per Share*	*Selling Price per Share*	*Commission per Share Paid by Seller*	*Taxes per Share Paid by Seller*
186.	100	$58\frac{5}{8}$	27 16 cents	$61\frac{3}{4}$	27 94 cents	10 cents
187.	200	107	30 cents	$107\frac{1}{2}$	30 cents	10 cents
188.	400	196	35 cents	197	35 cents	7 cents

189. An independent florist rents 600 square feet of floor space with the understanding that he will pay 6% of sales on the first $50,000 of sales and 5% on sales over that amount. His shop is open 6 days a week. His total annual sales amount to $92,000. Find his annual rental and his average daily sales.

190. Find the cost of laying a sidewalk 4 feet wide by 85 yards if the contract cost is 28 cents per square foot.

191. The owner of a building has an opportunity to rent it to an automobile supply store at $3\frac{1}{2}$% of annual gross sales, or to an independent hardware store at 5% of annual gross. The estimated sales of the two are $175,000 and $120,000, respectively. Which would pay the larger rent?

192. In the English monetary system 12 pence (d) are equal to one shilling (s) and 20 shillings are equivalent to a pound sterling. Express the following as the decimal equivalent of a pound sterling.

s	*d*	*Decimal of a Pound*
0	6	
1	0	
3	8	
14	6	
18	9	

193. In estimating costs for a sewer construction job, the hourly wage scales to be paid were as follows: cement mason, $2.70; sewer-pipe layer, $2.36; driver of dump truck, $2.29; trenching-machine operator, $2.73. Find the daily wage payment to each for an 8-hour day.

194. In one large housing unit, there are 60 apartments which rent for $55 a month and 150 apartments which rent for $78 a month. The unit is for sale at $6\frac{1}{2}$ times the gross annual rental income. Assuming that all apartments are fully rented, find the gross annual rental income and the selling price.

195. An investor is considering purchasing a hotel with 40 rooms. The estimated weekly income per room is $15. He can lease the entire hotel to an operator for 25% of the gross annual income. If his estimations of income are correct, what average monthly rental should he receive?

196. A building leased to a J. C. Penney store at $2\frac{1}{2}$% of gross sales produced a total rental of $23,250 last year. Find the gross sales of the store.

197. A commercial building 40 by 140 feet, with an estimated net income of $6,290 is offered for sale for $85,000. If the estimate of income is correct, find the per cent return on the investment.

198. A building 66 by 132 feet rents for $21,562 20 Find the annual rental per square foot

199. A building which produces $17,837 59 a year net income is valued at $310,000 What is the ratio between the value and the net income?

200. Department A carries an average inventory of $7,495 26 If sales for the year were 4 63 times the average inventory, what was the amount of sales?

201. One mile is 5 280 feet In one square mile there are 640 acres How many square feet are there in an acre?

202. The total volume of business done last year by the Harwood Construction Company was $540,000 The net worth of the company was $60,000 Find the ratio of sales to net worth

203 The net profit of the H and P Construction Company last year was $24,000 The net worth of the company was $60,000 Find the per cent earned on net worth

204. On a map ½ inch is used to represent 10 miles The distance between two cities on the map is 3¼ inches How many miles apart are they?

205. Concrete weighs 144 pounds per cubic foot Find the weight of a concrete retaining wall 6 feet high, 120 feet long, and 8 inches thick

206 Last year the combined city, county, and school tax rate for Fulton was $67 60 per $1,000 of assessed valuation The tax was computed on an assessed valuation representing 60 per cent of actual value What was the property tax bill for house and lot worth $12,000?

207. The state tax on tobacco is 4 cents for each package of 20 cigarettes If a man smokes one package a day, how much will he have paid in state tobacco taxes by the end of the year?

208. The state levies a gasoline tax of 6½ cents per gallon The owner of a car which averages 12 miles to the gallon drives 7,200 miles per year How much will he have paid in state gasoline taxes by the end of the year?

209. Gordon Hiller's sales increased 10% the second year, 15% the third year, and 25% the fourth year over his first year sales His total sales for the 4 years were $180,000 What were his sales for each year?

210. An automobile which costs $2,400 is assumed to depreciate in value 30% from the beginning of the year to the end of the year What is its depreciated value at the end of the third year?

211. A salesman is paid $30 a week as a base salary plus 4% of his weekly sales in excess of $750 If his weekly sales averaged $1,800 what is his average weekly income?

212. The district representative of the Fidelity Corporation receives a salary of \$4,000 plus commissions of $\frac{1}{2}\%$ on the first \$200,000 of sales, $\frac{3}{4}\%$ on sales from \$200,000 to \$500,000, and 1% on sales over \$500,000. Last year the sales in his district were \$550,000. What were his average monthly earnings?

213. A student arranged to sell tickets for a travel agency. He is to retain 8% of his total sales as his commission. He sold 18 tickets at \$28.00 each and 15 tickets at \$12.50 each. How much was his commission?

214. Willard Gear bought a house for \$18,500. He spent \$2,500 in repairs and listed it for sale as \$27,500. He paid the broker who sold it a commission of 5% of the selling price. What per cent profit did Mr. Gear make on his investment?

215. James Hale lives in a state with a $3\frac{1}{2}\%$ retail sales tax. His income is \$4,200 per year. If 51% of his income is spent on goods subject to the sales tax, what per cent of his income will he pay in sales tax?

6

Fundamentals of Algebra

Introduction

In the arithmetical operations only arabic numbers are used. In algebra, the arabic numbers are supplemented by the letters of the alphabet. The letters used, since they may have different numerical values assigned to them, are called *literal or general numbers*.

The four fundamental operations used in arithmetic—addition, subtraction, multiplication, and division—have the same meaning when applied to general numbers.

The symbols used in arithmetic to indicate addition or subtraction are used in the same way with literal numbers, $a + b$ indicates the sum of a and b, while $x - y$ indicates that the y is subtracted from x. Since the sign of multiplication ($\times$) might be confused with the letter x, frequently used to indicate an unknown quantity, multiplication is generally indicated by simply writing the literal numbers together, such as ab, cd, axy. The product of an arabic number and a general number is indicated by writing them together, thus $3 \times a$ is written $3a$, and $4 \times a \times b$ is written $4ab$. Sometimes a center dot is used to indicate multiplication, such as $a \cdot b$, $2 \cdot c \cdot d$, $b \cdot c \cdot x$.

Numbers such as a, b, and x, as well as products such as $3a$ and cy, are called *algebraic terms*. When written in combination, such as $3a + cy$, or $3a - cy$, the combination is referred to as an *algebraic expression*. An algebraic expression of one term ($3a$) is called a *monomial*, one of two terms ($3a + 4y$) is called a *binomial*. Any expression of two or more terms ($3a + 4y - 2x$) can be called a polynomial.

Since $3a$ indicates the product of 3 and a, it is logical to refer to both 3 and a as factors of $3a$. Either factor is called a *coefficient*. The 3, being an arabic numeral, is called the *numerical coefficient* of a, and the a, being a general number, is called the *literal coefficient* of 3. If 1 is multiplied by a,

the product is $1a$; but since a alone has exactly the same meaning, the coefficient 1 is not written. That is, $1a = a$. Algebraic terms with the same literal coefficient are called *like terms* or *similar terms.*

Numerical values of algebraic expressions

The indicated addition in the algebraic expression $3a + 4y$ cannot be carried out until definite numerical values have been assigned to the letters. If $a = 3$ and $y = 5$, $3a + 4y$ can be combined. Then $3a = 3 \times 3 = 9$; $4y = 4 \times 5 = 20$; then $3a + 4y = 9 + 20 = 29$. That is, if $a = 3$ and $y = 5$, then $3a + 4y = 29$. If other numerical values are assigned to a and y, other values will be found for the algebraic expression $3a + 4y$.

EXERCISE 6.1

If $a = 4$, $b = 2$, $c = 5$, $x = 1$, and $y = 3$, determine the numerical value of each of the following algebraic expressions.

1. $2a + 5b$
2. $4b + 2c$
3. $5x + 8y$
4. $4c - 3y$
5. $8x - 3b$
6. $12x + 3a - 7b$
7. $8c - 5y - 3a$
8. $5b + 3a + 2c$
9. $4a + 3b + 2c$
10. $5x + 7y - 6a$
11. $axy - b$
12. $bcx + 3a$
13. $\dfrac{4a}{y}$
14. $\dfrac{6c}{b}$
15. $\dfrac{ac}{b}$
16. $\dfrac{4a + 3b}{2y}$
17. $\dfrac{7a - 4c}{2b}$
18. $\dfrac{6y}{4a + b}$
19. $\dfrac{4a}{b} - 5x$
20. $\dfrac{6c}{5x} - \dfrac{2c}{b}$

Addition and subtraction of algebraic terms

Algebraic terms can be added together or subtracted from each other if the literal coefficients are the same. In combining similar terms the literal coefficient does not change, while the numerical coefficients are combined exactly as in arithmetic. Thus $3a + 7a = 10a$. That is, since $3 + 7 = 10$, then $3a + 7a = 10a$.

This relationship may be illustrated by an arithmetical example It is known that $22 + 33 = 55$ Suppose that the two numbers 22 and 33 were written as $2 \times 11 + 3 \times 11$ Then under the rule for combining similar terms the numerical coefficients 2 and 3 would be added, and the result would be $2 \times 11 + 3 \times 11 = 5 \times 11$ Since $5 \times 11 = 55$, it can be seen that the rule is also true of arabic numbers Thus $2a + 3a = 5a$ When $a = 11, 2a = 22, 3a = 33$, and $5a = 55$ The relationship described, however, holds for any value of a

It must be emphasized that only similar terms may be added, although $3a + 7a + a = 11a$, since a is equivalent to $1a$ If the terms are not similar they cannot be combined Thus $3a + 7a + b$ can be shown as $10a + b$, while $3a + 7a + 2$ would be shown as $10a + 2$ In an algebraic expression, such as $3a + 7a + 2b + 5b$, the terms containing a can be combined and the terms containing b can be combined That is $3a + 7a + 2b + 5b = 10a + 7b$

EXERCISE 6.2

Perform the indicated operations

1. $7x + 12x = ?$
2. $8y - 4y = ?$
3. $9a - 5a = ?$
4. $15cd - 12cd = ?$
5. $9ab + 4ab = ?$
6. $8m - 5m = ?$
7. $27ab + 6ab = ?$
8. $3cd + 27cd = ?$
9. $132f - 118f = ?$
10. $14b - 8b = ?$
11. $5a + 3a + 2c + 6c = ?$
12. $5ab + 8ab - 6ab + 4cd = ?$
13. $6xy - 4xy + 7ab - 2ab = ?$
14. $11pq - 5pq - 4pq + 5p = ?$
15. $8r + 5s - 3s - 6r + 2r = ?$
16. $3w + 7 + 4w - 3 - 5w = ?$
17. $14ab + 9 + 3ab - 6 - 16ab = ?$
18. $12x + 15y + 5 - 12y - 3x - 2 = ?$
19. $4e + 7f + 4f - 9f - e - 2f = ?$
20. $2a + 3b + 5c + 1a - b - 3c = ?$

In problems of business and finance it is seldom necessary to add many algebraic expressions, but in tax work and statistical projection it is sometimes necessary to do so The trained mathematician does not need to change the form to add algebraic expressions quickly and accurately The beginning student, however, or the person subject to many interruptions may find that the problem of adding, or combining like terms, is often simplified by rewriting the problem so that like terms are arranged in vertical columns and finding the total of each column

Illustration Find the sum of $8r + 5s + 4x$, $3r + 2x$, and $5r + 4s + 5x$

Rearranging these,

$$\begin{array}{r}8r + 5s + \ 4x\\ 3r \qquad + \ 2x\\ 5r + 4s + \ 5x\\ \hline 16r + 9s + 11x\end{array}$$

Or, if you desire, simply give the result as

$$8r + 5s + 4x + 3r + 2x + 5r + 4s + 5x = 16r + 9s + 11x$$

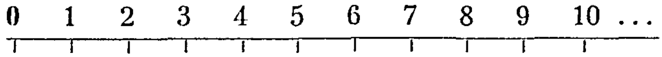
EXERCISE 6.3

Add the following algebraic expressions.

1. $3x + 5y + 8z;\quad 2x + 3y + z;\quad 4x + 2y + 3z$
2. $7a + 5b + 3c;\quad 2a - 3b + c;\quad a + 2b - 2c$
3. $2ab + 3bc + 2cd;\quad 4ab + 2bc + 5cd;\quad ab + bc - cd$
4. $7a + 2b + 8;\quad 2a + 3b - 5;\quad 2a - b + 2$
5. $4xy + 3ab + 7;\quad 2xy - ab - 4;\quad xy - ab + 2$
6. $3x + 5y + 8;\quad 2x - 3y - 4;\quad x + y + 1$
7. $14a + 9b + 11c;\quad a + 7b + 8c;\quad a - 10b - 15c$
8. $7w + 27v + 18q;\quad 2w - 15v - 11q;\quad w - 8v - 3q$
9. $7x + 3y + 2z;\quad 5y - 3x + z;\quad 2z - 4y + x$
10. $5ab + 3cd + 2ef;\quad 4ef + cd - 2ab;\quad 2cd - 3ef - ab$

Graphic representation of real numbers

An understanding of the addition and subtraction of numbers may be improved by considering a graphic representation of them. Customarily this is done by drawing a line, and selecting a point 0, which is called the *origin*. To the right of the origin place a succession of points such that any two consecutive points are the same distance apart. These points are numbered 1, 2, 3, 4, 5, etc. on the accompanying diagram.



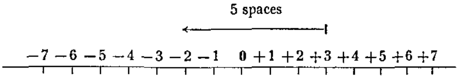
To perform addition by using this diagram, refer to the points on the line, or more simply, refer to the number corresponding to the points. To find $4 + 2$, begin at the point 4 and move 2 spaces to the *right*. The answer is 6. By taking other examples it is apparent that such addition corresponds to arithmetical addition illustrated in Chapter 1. From studying such illustrations the general rule has been developed that *to add any two integers a and b, begin at the point corresponding to a and move b spaces to the right to find the answer.*

Subtraction with integers may also be represented on the same line To subtract 2 from 6, begin with 6 and move 2 spaces to the *left* to the point 4 In arithmetic, however, the size of the subtrahend cannot exceed the minuend, or the result has no arithmetic meaning

Signed numbers

If in using the number scale previously developed, an attempt is made to subtract 5 from 3, the movement of 5 spaces to the left of the point 3 takes one beyond the point 0 into a section of the line where no units are marked

This difficulty can be overcome by marking spaces to the left of the point of origin, **0**, the same as to the right If the numbers to the right of the origin are designated as $+1$, $+2$, $+3$, and so on, the numbers to the left of the origin are designated as -1, -2, -3, and so on Then $3-5$ computed in this manner would show a point represented by 2 spaces to the left of the origin, namely the point which is designated -2 Labeled in this way the diagram appears as follows



This graphic representation illustrates what is called *the algebraic number scale*

Thus in algebra the plus (+) and minus (−) signs are given new significance As well as being used to indicate addition and subtraction they are used as signs of *quality* The numbers to the right of **0** are called *positive numbers*, and may be written $(+1)$, $(+2)$, $(+3)$, $(+4)$, etc The + sign, called a sign of quality, indicates that the number is positive, in contradistinction to the negative numbers, which may be written (-1), (-2), (-3), (-4), etc

Since the numbers used in arithmetic are always positive, no signs of quality are used Thus $3+5$ has the same meaning as $(+3)+(+5)$ Thus, when signed numbers are used, the symbols (+ or −) within the parentheses indicate the quality of the numbers, and the symbols between the parentheses denote the fundamental operation In algebra, as in arithmetic, all *unsigned* numbers are considered positive, so that even in algebraic form $(+3)+(+5)$ would probably be written simply as $3+5$, and $(+5)-(+3)$ would be written $5-3$

Addition of signed numbers

The addition of signed numbers is the same as the addition in arithmetic if the numbers are all positive. Thus $(+3) + (+2) = (+5)$ is the same as $3 + 2 = 5$.

In solving the problem $(-3) + (+5)$, consistency in reasoning would lead one to begin at the point -3 and move 5 spaces to the right, ending at $+2$, since in adding a positive number to a number the rule is to move to the right.

Suppose, however, that the numbers had been reversed and the problem was stated not as $(-3) + (+5)$, but rather as $(+5) + (-3)$, would the answer be the same? From an understanding of arithmetic it is known that the sum of two or more numbers is independent of the order in which they are added. In dealing with absolute values, the numbers of arithmetic, $3 + 5 = 5 + 3$. If the problem is written as $(+5) + (-3)$, to be consistent, the count should begin at $+5$. If the addition of a positive number indicates movement to the *right*, should the addition of a negative number indicate movement to the *left*? Moving to the left 3 spaces from $+5$ one arrives at $+2$, the same as in the example $(-3) + (+5) = +2$.

One other type of addition problem can be illustrated by $(-3) + (-2)$. Following the same rules, begin with -3. Since the addition of a positive number indicates movement to the right, and the addition of a negative number indicates movement to the left, move to the left 2 spaces. The answer is -5. If the numbers are interchanged to read $(-2) + (-3)$, the same answer is found.

If you understand these relationships you will have no difficulty in applying the rules for the addition of signed numbers, which may be summarized as follows.

1. If all the numbers to be added have the same sign, find the sum of their absolute values and prefix the common sign.

Illustrations:

a. $(+4) + (+7) = +11$
b. $(-5) + (-3) = -8$

2. If one number is positive and the other is negative, find the difference of their absolute values and prefix the sign of the greater.

Illustrations:

a. $(+8) + (-5) = +3$
b. $(+5) + (-12) = -7$

Letters may be attached to signed numbers for example, $3a$, $-7x$, $5ab$, $12xy$, and $-8ax$ Similar terms can be added together

Illustrations

a $(-4b) + (-5b) = -9b$
b $(+3a) + (-7a) = -4a$
c $(+8xy) + (-5xy) = +3xy$

EXERCISE 6.4

Add the following

1. $(+5) + (-9)$
2. $(-3) + (-12)$
3. $(-8) + (+5)$
4. $(-5) + (+14)$
5. $(+8) + (-12)$
6. $(-14) + (7)$
7. $(-13) + 10$
8. $21 + (-31)$
9. $-43 + 89$
10. $-346 + 279$
11. $(+7x) + (+12x)$
12. $(-14y) + (-12y)$
13. $(+\frac{2}{3}d) + (+\frac{4}{9}d)$
14. $(+\frac{3}{11}z) + (-\frac{5}{11}z)$
15. $(+\frac{3}{2}a) + (-\frac{1}{2}a)$
16. $(+0\ 9w) + (-0\ 17w)$
17. $(-5\ 1c) + (-7\ 6c)$
18. $3\ 68y + (-2\ 41y)$
19. $-4\ 25a + 2\ 875a$
20. $-21\ 25cd + 24\ 5cd$

Subtraction of signed numbers

In discussing subtraction as a arithmetic concept it was pointed out *that subtraction is the inverse of addition To find the difference between* 9 and 3, that is, $9 - 3$, the 3 could be taken away from 9, or one could consider how much would have to be added to 3 to get 9 The subtraction of signed numbers may be easier to understand if the latter concept is used The subtraction of one positive number from a larger positive number is the process already explained in discussing the arithmetical process Consider, however, the problem of $3 - 5$ Restated the question is, how much must be added to 5, to obtain 3? From the previous discussion of the addition of negative numbers it is seen that the answer is -2

To solve the problem of $3 - 5$ on the algebraic number scale, begin at the point $+3$ and move to the *left* 5 spaces as in the previous cases of subtraction The answer is found to be -2

Suppose, however, that the problem is $(+3) - (-5)$ Restated, it would be, What number must be added to -5 to get 3? From a knowledge of addition it is seen that the answer is $+8$ Worked on the algebraic number scale the problem indicates a starting point at $+3$ If the number

being subtracted were positive, the movement would be to the left, but since the number to be subtracted is negative the movement is to the *right* 5 spaces. Moving five spaces to the right from the point $+3$ the answer is found to be $+8$.

The other type of problem is a negative number subtracted from a negative number, such as $(-3) - (-2)$. Restated, this problem would read, How much must be added to (-2) to get (-3)? The answer is -1. To solve it on the algebraic number scale begin with the point -3, and move from that point 2 spaces to the right to indicate subtraction of a negative number. This movement will end at the point marked -1.

These three problems may be illustrated as follows.

Move 5 spaces

$(+3) - (+5) = -2 \qquad -4\ -3\ -2\ -1\ \ 0\ +1\ +2\ +3\ +4$

Move 5 spaces

$(+3) - (-5) = +8 \qquad -1\ \ 0\ +1\ +2\ +3\ +4\ +5\ +6\ +7\ +8\ +9$

2 spaces

$(-3) - (-2) = -1 \qquad -6\ -5\ -4\ -3\ -2\ -1\ \ 0\ +1\ +2\ +3$

The important thing to understand is the relationships. Once the principles are understood it is easy to remember that *the general rule to observe in subtracting one signed number from another is to change the sign of the subtrahend and add to the minuend.*

Illustrations:

a. $(+8) - (+3) = (+8) + (-3) = +5$
b. $(+8x) - (-3x) = (+8x) + (+3x) = +11x$
c. $(-5y) - (+2y) = (-5y) + (-2y) = -7y$
d. $(-5z) - (-2z) = (-5z) + (+2z) = -3z$

EXERCISE 6.5

Perform the indicated operation.

1. $(+3) + (-5)$
2. $(-8) - (-15)$
3. $(-3) - (+8)$
4. $(+5) + (-4)$
5. $(+12) - (+7)$
6. $(+7) - (-8)$
7. $(-5) + (-3)$
8. $(-8) - (+7)$
9. $(-14) + (-12)$
10. $(+21) - (-18)$

11. $(+\frac{2}{3}) - (+\frac{4}{9})$
12. $(-\frac{5}{8}) + (+\frac{21}{32})$
13. $(-\frac{3}{11}) - (+\frac{5}{11})$
14. $(+\frac{3}{8}) - (-\frac{4}{9})$
15. $(+\frac{3}{2}a) + (-\frac{7}{2}a)$
16. $(-5\ 1x) + (-2\ 7x)$
17. $(-8f) - (+32f)$
18. $(-5a) - (+18a)$
19. $(-132f) + (+118f)$
20 $(-xz) - (-7xz)$

The same rules of subtraction apply to algebraic expressions Like terms may be subtracted by changing the sign of the subtrahend and adding to the minuend It is usually advisable to arrange the expressions so that the like terms are in a vertical column

Illustrations

a Subtract $(+4x) - (+3y)$ from $(-6x) + (+y)$

$$\begin{array}{r} (-6x) + (+y) \\ (+4x) - (+3y) \\ \hline (-10x) + (+4y) \end{array}$$

b Subtract $(-3x) - (+7y)$ from $(-2x) + (-7y)$

$$\begin{array}{r} (-2x) + (-7y) \\ (-3x) - (+7y) \\ \hline (+x) \end{array}$$

EXERCISE 6 6

Carry out the following subtractions

1. $(+3x) + (+5y)$ from $(+5x) + (+8y)$
2. $(+4x) - (-3y)$ from $(-3x) + (-2y)$
3. $(+8x) - (+5y)$ from $(+7x) - (-3y)$
4. $(-5x) + (-4y)$ from $(-2x) + (-4y)$
5. $(+8x) + (+5y)$ from $(+3x) - (-5y)$
6. $(+5x) + (+8y)$ from $(+3x) + (+5y)$
7. $(-3x) + (-2y)$ from $(+4x) - (-3y)$
8. $(+7x) - (-3y)$ from $(+8x) - (+5y)$
9. $(-2x) - (+4y)$ from $(-5x) + (-4y)$
10. $(+3x) - (-5y)$ from $(+8x) + (+5y)$

Algebraic sum of signed numbers

The sum of two or more signed numbers is called the algebraic sum even though the answer is negative If more than two numbers with different signs are involved, the process may be carried out (1) step by

step, using the rules used to find the sum of signed numbers; or (2) by finding the sum of all the positive numbers and the sum of all the negative numbers, and then adding these two sums by finding the difference of their absolute values and prefixing the sign of the greater.

Illustration: Find the sum of the following.

$$(+5) + (-8) + (+7) - (+12) + (+5)$$

By the first method, the solution is

$$\begin{aligned} (+5) + \ (-8) &= -3 \\ (-3) + \ (+7) &= +4 \\ (+4) - (+12) &= -8 \\ (-8) + \ (+5) &= -3 \end{aligned}$$

By the second method, the solution is

Step 1: The sum of all positive values,	$5 + 7 + 5 =$	17
Step 2: The sum of all negative values,	$(-8) + (-12) =$	-20
Step 3: The sum of these two sums,		-3

Since an unsigned number is presumed to be positive, the example can be written: $5 + (-8) + 7 + (-12) + 5 = ?$

It has already been observed that $5 + (-8) = -3$. Since the sum $5 + (-8)$ is the same as the difference $5 - 8$, the plus signs preceding the minus 8 can be omitted and it can be written as $5 - 8 = -3$. Similarly the plus sign preceding the minus 12 can also be omitted and the problem written: $5 - 8 + 7 - 12 + 5 = ?$ Written in this manner it may be more readily seen that the answer is -3.

Suppose, however, that the problem $(+5) - (-8)$ is to be written with a minimum number of signs. The plus 5 can be rewritten as 5. The difference between 5 and minus 8 is, according to the law of subtraction of signed numbers, $(+13)$. Therefore the $5 - (-8)$ can be rewritten as $5 + 8 = 13$. Therefore, the rules observed in omitting signs of quality are:

1. If like signs appear together, they can be rewritten as plus (+).
2. If unlike signs appear together, they can be rewritten as minus (−).

Illustrations:

$+(+8) = +8; \quad +(-8) = -8; \quad -(+8) = -8; \quad -(-8) = +8.$

EXERCISE 6.7

Find the algebraic sum of the following

1. $(+5) + (-9) + (+12) + (-4)$
2. $(+18) + (-6) + (+7) + (-12)$
3. $(-19) + (-5) + (-17) + (+28)$
4. $(+7) + (+12) + (-8) + (-5)$
5. $(-3) + (-8) + (+13) + (-15)$
6. $(+5) + (-19) - (+8) - (-17)$
7. $-7 + 8 - (-6) - 15 - (-8)$
8. $-32 - (-43) + 8 - 13 - (-8)$
9. $-3 + 5 - 21 - 15 + 8 + 41$
10. $21 + 37 - 82 - 127 + 44 - 32$
11. $-0\ 9 - (-0\ 17) - 5\ 1$
12. $7\ 3 + (-6\ 9) - (+0\ 31)$
13. $-2\ 42 + 3\ 68 - (-9\ 57)$
14. $\frac{2}{5} - (-\frac{4}{5}) + (+\frac{1}{6})$
15. $4\frac{1}{4} + 2\frac{7}{8} - (-1\frac{3}{8})$
16. $8wz - (-5wz) - 4wz$
17. $-12w - (-21w) + (-5w)$
18. $-xz - (-7xz) - 6xz$
19. $27cd - 32cd + 12cd - 18cd$
20. $-8f + 12g - 32f - (-9g)$

Multiplication of signed numbers

When positive numbers are multiplied together, the product is always positive Thus $(+4) \times (+3)$ is considered as $4 + 4 + 4 = 12$ It can readily be seen that according to the rules of addition of signed numbers the product must be positive That is, $(+4) \times (+3) = +12$

Suppose that the problem is to multiply $(-4) \times (+3)$ Then the multiplication can be considered as $(-4) + (-4) + (-4) = -12$ According to the rules for the addition of signed numbers the product in this case is negative That is, $(-4) \times (+3) = -12$ Since the commutative law of multiplication states that the product of two numbers is the same in whatever order they are multiplied, the conclusion is reached that the product of two numbers with unlike signs is always negative

Since $(+4) \times (+3)$ can be thought of as adding 4 three times then $(+4) \times (-3)$ can be thought of as subtracting $+4$ three times. That is, $(+4) \times (-3) = -(+4) - (+4) - (+4) = -12$ Now consider the multiplication of $(-4) \times (-3)$ As above, think of -3 as meaning to subtract -4 three times

$$(-4) \times (-3) = -(-4) - (-4) - (-4) = ?$$

According to the rule developed in an earlier section, two like signs coming together can be written as $+$ Thus the indicated multiplication can be written as $4 + 4 + 4 = 12$

A general rule of algebra is that if a negative number is multiplied by a negative number the product is positive

The rules for the multiplication of signed numbers may be summarized as follows

1 If both have the same sign, the product is positive

Illustrations:

a. $(+3) \times (+2) = +6$
b. $(-3) \times (-2) = +6$

2. If one number is positive and the other is negative, the product is negative.

Illustrations:

a. $(+5) \times (-2) = -10$
b. $(-5) \times (+2) = -10$

3. If more than two signed numbers are multiplied together, the sign of the product is positive if there is an even number of negative factors; and the sign of the product is negative if there is an odd number of negative factors.

Illustrations:

a. $(-5) \times (-6) \times (+3) \times (-2) = -180$
b. $(-5) \times (-6) \times (+3) \times (+2) = +180$
c. $(+a)(+b)(-c) = -abc$
d. $(-a)(-b)(+c) = abc$
e. $(-a)(-b)(-c)(-d) = abcd$

EXERCISE 6.8

Multiply the following.

1. $(+3) \times (-4)$
2. $(+7) \times (+5)$
3. $(-7) \times (+3)$
4. $(-4) \times (-6)$
5. $(+8) \times (-5)$
6. $(-7) \times (-6)$
7. $(-3) \times (+8)$
8. $(-5)(-6)$
9. $(+6)(+9)$
10. $(+8)(-6)$
11. $(-7)(+4)$
12. $(-5)(-8)$
13. $(-3)(-2)(+5)$
14. $(+5)(-4)(+6)$
15. $(-8)(+3)(-7)$
16. $(-4)(-2)(-6)$
17. $(+8)(-5)(-4)$
18. $(-3)(+7)(-5)$
19. $(-3)(-4)(+5)(-6)$
20. $(-3)(-3)(-3)(+3)$
21. $(-6)(-3)(+8)(-2)$
22. $(-4)(-3)(-7)(-12)$
23. $(-2)(-5)(-4)(-8)$
24. $(-6)(+6)(-3)(-3)$
25. $(-2)(-3)(+5)(-8)$
26. $(-1)(-3)(+8)(+4)$
27. $(-5)(+3)(-4)(-2)$
28. $(-1)(+4)(+1)(-7)$
29. $(-8)(+3)(-3)(+4)$
30. $(+3)(-4)(+5)(-6)$

Division of signed numbers

In the division of positive numbers the product of the divisor and quotient is equal to the dividend. If the general numbers of algebra are used to illustrate this it is seen that if $a \div b = c$, then $cb = a$. When this same relationship is applied to signed numbers, it is seen that if positive 9 is divided by positive 3 the answer is 3, and the sign of the quotient must be such that when it is multiplied by a positive 3 the product must be a positive 9. Thus $9 \div 3 = 3$. If 9 is divided by -3, the answer will still be 3 but it must bear such a sign that the product of it and -3 is $+9$. Since a -3 can be multiplied only by a -3 to get a $+9$, the sign of the quotient must be negative. That is, $9 \div (-3) = -3$.

The rules for the division of signed numbers may be summarized as follows:

1. If the dividend and divisor have like signs, the quotient is positive.

Illustrations

a. Since $(+3)(+3) = +9$, then $(+9) \div (+3) = +3$
b. Since $(+3)(-3) = -9$, then $(-9) \div (-3) = +3$

2. If the dividend and divisor have unlike signs, the quotient is negative.

Illustrations

a. Since $(-3)(-3) = +9$, then $(+9) \div (-3) = -3$
b. Since $(-3)(+3) = -9$, then $(-9) \div (+3) = -3$

If put in fraction form, parentheses are not needed:

$$\frac{+9}{+3} = +3, \quad \frac{-9}{-3} = +3, \quad \frac{+9}{-3} = -3, \quad \frac{-9}{+3} = -3$$

Signs of a fraction

There are three signs in any fraction with signed numbers—the sign of the numerator, the sign of the denominator, and the sign in front of the fraction. In the study of fractions in Chapter 3 it was seen that the value of a fraction is not changed if the numerator and denominator are multiplied or divided by the same number. Since this is true, the numerator and denominator of a fraction can be multiplied by $+1$ or -1 and the value of the fraction will not be changed. If, however, the numerator or denominator alone is multiplied by -1, the sign before the fraction must be changed to keep its value unaltered. Thus the rule is developed that any two of the three signs of a fraction may be changed without changing the value of the fraction. Thus

$$+\frac{+9}{+3} = +\frac{-9}{-3} = -\frac{+9}{-3} = -\frac{-9}{+3}$$

If there is more than one factor in the numerator or denominator, the sign of the fraction must be changed if the signs of an odd number of factors are changed, but the sign of the fraction is not changed if the signs of an even number of factors are changed.

Illustrations:

a. $\frac{(+3)(+8)}{+6} = \frac{(-3)(+8)}{-6} = \frac{(-3)(-8)}{+6} = \frac{24}{6} = 4$

b. $\frac{(+3)(+8)}{-6} = \frac{(-3)(-8)}{-6} = -\frac{(+3)(+8)}{+6} = -\frac{24}{6} = -4$

c. $-\frac{(+a)(-b)(-c)}{(-d)(+e)} = +\frac{(+a)(+b)(+c)}{(+d)(+e)} = \frac{abc}{de}$

EXERCISE 6.9

Carry out the indicated operations.

1. $(+8) \div (+2)$
2. $(-49) \div (+7)$
3. $(+144) \div (-12)$
4. $(-72) \div (-8)$
5. $(+56) \div (-8)$
6. $(+75) \div (+15)$
7. $(-88) \div (-11)$
8. $(-45) \div (+9)$
9. $(-35w) \div (-5w)$
10. $(+42a) \div (-7a)$
11. $(+28bc) \div (+4bc)$
12. $(-54d) \div (-6d)$

13. $\frac{+35}{-7}$
14. $\frac{+42}{+6}$
15. $\frac{-63}{-7}$
16. $\frac{-121}{+11}$
17. $\frac{+72}{-8}$
18. $\frac{+27}{+3}$
19. $\frac{-48a}{+16a}$
20. $\frac{-125xy}{-5xy}$
21. $\frac{(-8)(+3)}{-6}$
22. $\frac{(+9)(-6)}{-18}$
23. $\frac{(+6)(-3)}{+9}$
24. $\frac{(+27)(-4)}{12}$
25. $\frac{(+16)(-3)}{+8}$
26. $\frac{(-5)(-12)}{-15}$
27. $\frac{+72}{(-3)(+4)}$
28. $\frac{-48}{(+6)(+3)}$
29. $\frac{+51}{(-17)(+1)}$
30. $\frac{(+8)(+6)}{(-4)(+3)}$

Powers and roots

When a number, either arabic or general, is multiplied by itself, it is said to be squared, or raised to the second power. The number itself is called the *base*, and the power to which the base is to be raised is indicated by a number called an *exponent*, written to the right and slightly above the number to be raised

Thus $a \times a$, or aa, is written a^2, and is read "a square," $a \times a \times a$, or aaa, is written a^3, and is read "a cube," $a \times a \times a \times a$, or $aaaa$, is written a^4, and is read "a to the fourth power"

When no exponent is expressed, the exponent must be regarded as 1. That is, $a = a^1$ When a^2 is multiplied by a, the product is a^3, and when a^3 is multiplied by a^2, the product is a^5 The reasonableness of these conclusions can readily be seen if the numbers are written without exponents Thus $a^2 \times a$ becomes $aa \times a$ or aaa, $a^3 \times a^2$ becomes $aaa \times aa$ or $aaaaa$ These examples illustrate a general rule of mathematics that *the exponent of a product equals the sum of the exponents of its factors if the bases are the same*

Since any number raised to the first power is equal to itself, a and a^1 are the same The general rule of exponents shows that if a^2 is multiplied by a, the product is a^{2+1} or a^3, and that if a is multiplied by a, the product is a^{1+1} or a^2

It should be recalled from Chapter 4 that the symbol $\sqrt{\ }$ is called a radical sign, and is used to indicate a root of a number The symbol $\sqrt{9}$ indicates the square root of 9 If a cube root or a higher root is indicated, a small number called the *index* or *order* is shown Thus the symbol $\sqrt[4]{64}$ indicates the 4th root of 64 The number appearing under the sign of the radical is called the *radicand* In the expression $\sqrt[3]{64} = 4$, the index is 3, the radicand is 64, the root is 4

Radicals with the same index, or of the same order, which have the *same radicand are called similar or like radicals, and may be added or* subtracted as like terms If radicals are not similar, they may not be combined, although their algebraic sum or difference may be indicated

Illustrations

a Combine $6\sqrt{2} + 5\sqrt{2}$

$$6\sqrt{2} + 5\sqrt{2} = (6 + 5)\sqrt{2} = 11\sqrt{2}$$

b Simplify $4\sqrt[3]{6} + 5\sqrt[3]{6} - 7\sqrt{2}$

$$4\sqrt[3]{6} + 5\sqrt[3]{6} - 7\sqrt{2} = (4 + 5)\sqrt[3]{6} - 7\sqrt{2} = 9\sqrt[3]{6} - 7\sqrt{2}$$

EXERCISE 6.10

Simplify the following.

1. $x \cdot x^2 \cdot x^3$
2. $a^3 \cdot a^2 \cdot a^4$
3. $y^2 \cdot y^5 \cdot y^7$
4. $a^2b \cdot ab^3$
5. $x^2y \cdot x^2y^3$
6. $x^5 \div x^3$
7. $y^8 \div y^6$
8. $\dfrac{x^5}{x^8}$
9. $\dfrac{w^5}{w^5}$
10. $\dfrac{w^3}{w^7}$
11. $5\sqrt{3} + 3\sqrt{3}$
12. $8\sqrt{17} - 5\sqrt{17}$
13. $2\sqrt[3]{15} + 3\sqrt[3]{15}$
14. $27\sqrt{14} - 18\sqrt{14}$
15. $2\sqrt[4]{9} - 3\sqrt[4]{9} + 4\sqrt[4]{9}$
16. $7\sqrt{6} + 8\sqrt{7} - 5\sqrt{7}$
17. $17 - 8 + 2\sqrt{2} + 7\sqrt{2}$
18. $5\sqrt[3]{4} + 3\sqrt[3]{12} - 2\sqrt[3]{4}$
19. $3\sqrt{5} + 7\sqrt[3]{5} - 2\sqrt{5} - 4\sqrt[3]{5}$
20. $8\sqrt[3]{2} + 5\sqrt[3]{3} - 5\sqrt[3]{2} + \sqrt[3]{3}$

Multiplication of monomials and polynomials

To multiply two monomials it is necessary to find the product of the numerical coefficients, and the product of the literal factors. If the literal factors are the same, the exponent of the product is the sum of the exponents of the factors. Unless the literal factors are the same they may not be combined.

Illustrations:

a. Multiply $5x^2$ and $4x$.

Since $5 \times 4 = 20$, and $x^2 \times x = x^3$, then $5x^2 \times 4x = 20x^3$.

b. Multiply $6ax^2y$ and $4by^2$.

Since $6 \times 4 = 24$, and $ax^3y \times by^2 = abx^2y^3$, then $6ax^2y \times 4by^2 = 24abx^2y^3$.

If a quantity is to be considered as a single number it is ordinarily inclosed in parentheses ( ), brackets [ ], or braces { }. Thus $(20 + 7)$ written within parentheses has the same meaning as 27. In algebraic expressions, the quantities inclosed usually cannot be combined.

In the study of arithmetic it is seen that if the sum of two or more numbers is to be multiplied by another number, the product of each of the numbers and the multiplier may be obtained and the products added. Thus $27 \times 4 = 108$, or $4(20 + 7) = 80 + 28 = 108$. This fact provides the basis for finding the product of a monomial and a polynomial.

Illustrations:

a. Multiply $2x^3 + 7x^2y - 4$ by 2.

Multiply each term of the polynomial by 2. Thus

$$2(2x^3 + 7x^2y - 4) = 4x^3 + 14x^2y - 8$$

b Multiply $a^2 + 8a^2b^2 - b^2$ by $4ab$ Then

$$4ab(a^2 + 2ab - b^2) = 4a^3b + 8a^2b^2 - 4ab^3$$

Vertical arrangement often facilitates such multiplication

$$\begin{array}{r} a^2 + 2ab - b^2 \\ 4ab \\ \hline 4a^3b + 8a^2b^2 - 4ab^3 \end{array}$$

EXERCISE 6.11

Carry out the indicated operations

1. $7x \quad 8x^2$
2. $5a^2 \quad 3a^3$
3. $2a \quad 5a^3$
4. $4y^3 \quad 3y$
5. $2ab^2 \quad 3a^2b$
6. $5x^2y^3 \quad 8xy$
7. $7w^3 \quad 5xw^2$
8. $9a^4 \quad 5a^2b$
9. $27a^5 \quad 2b^2$
10. $15x^2y \quad 3xz^3$
11. $7(4x^2 + 3x - 8)$
12. $12(x^3 + 5x^2 - 3)$
13. $4(x^3 + 3x^2 + 2x)$
14. $3(7a^3 - 5a + 3b)$
15. $2x(5x^2 - 3x + 2)$
16. $7x^2(12x^3 - 8x^2 + 4x - 2)$
17. $3ab(5ab^2 + 8a^2b - 4)$
18. $4x^2y(3xy^2 - 2x^2y - 3y)$
19. $4a^2b^2(5ab - 8a^2b + 2ab^2)$
20. $3w^2z^3(-4wz^2 + 3w^2 + 2z)$

Multiplication of binomials

If 27, written $(20 + 7)$, is to be multiplied by 15, written $(10 + 5)$, the product can be found by multiplying each term of one quantity by each term of the other Thus

$$\begin{array}{r} 20 + 7 \\ 10 + 5 \\ \hline 7 \times 5 = 35 \\ 20 \times 5 = 100 \\ 10 \times 7 = 70 \\ 10 \times 20 = 200 \\ \hline 405 \end{array}$$

That is, $27 \times 15 = 405$

In the past a knowledge of the principles involved in the multiplication of one binomial by another has been of little practical significance in solving business problems The growing size and complexity of business units, however, has placed greater emphasis on statistical methods of control More and more persons will find employment in the managerial

and executive aspects of business. A familiarity with the common forms of products aids in an understanding of the solution of more complex equations—called quadratic equations—by a process known as factoring. The multiplication of binomials and polynomials can generally be easily mastered, especially if the terms of the product are set vertically. Then the multiplication may be carried out by using the same procedure used when mixed numbers are multiplied together.

Illustrations:

a. Multiply $a + b$ by itself.

$$\begin{array}{lll} a + b & & \\ \underline{a + b} & & \\ a^2 + ab & & \\ \quad\quad ab & + b^2 & \\ \hline a^2 + 2ab & + b^2 & \end{array}$$

b. Multiply $3x + 4y$ by $3x - 4y$.

$$\begin{array}{lll} 3x & + 4y & \\ 3x & - 4y & \\ \hline 9x^2 & + 12xy & \\ & - 12xy & - 16y^2 \\ \hline 9x^2 & & - 16y^2 \end{array}$$

In this second illustration, which entails finding the product of the sum and difference of two numbers, the answer is the square of the first number minus the square of the second. This relationship always prevails.

EXERCISE 6.12

Find the following products.

1. $(x - 3)^2$
2. $(3x + 2y)^2$
3. $(2x - 5y)^2$
4. $(4a - b)^2$
5. $(3a - 2b)^2$
6. $(5x - 3y)^2$
7. $(a + 5)(a - 5)$
8. $(2x + 3y)(2x - 3y)$
9. $(2a - 3b)(2a + 3b)$
10. $(3a - 2c)(3a + 2c)$
11. $(2x + 5y)(2x - 5y)$
12. $(3x + 4)(3x - 4)$
13. $(x + 3)(x + 4)$
14. $(x - 5)(x + 3)$
15. $(x + 3)(x - 7)$
16. $(x - 4)(x - 2)$
17. $(x - 1)(x + 4)$
18. $(x + 5)(x - 3)$
19. $(2x + 3)(3x - 4)$
20. $(4x - 3)(2x - 1)$
21. $(3a + 2b)(2a - 5b)$
22. $(5x + 2y)(3x - 4y)$
23. $(2x + 3y)(3x - 2y)$
24. $(x - y)(2x + 5y)$
25. $(2x + y)(x + 2y)$
26. $(2x + y)(3x - y)$
27. $(3x - 2y)(2x - 5y)$
28. $(3a + 4b)(2a + 3b)$
29. $(5a - 3b)(2a + 3b)$
30. $(7w + 5z)(3w - 4z)$

Symbols of grouping

When the indicated operation in an algebraic expression, such as $2(x+3)$, has been carried out, the parentheses are no longer needed, and they are said to be removed Parentheses and other symbols of grouping may be inserted or removed if the following rules are observed

1 A pair of symbols of grouping preceded by a plus (+) sign may be removed or may be inserted without changing the sign of any term between the pair of symbols

Illustrations

a $5+(4+9)=5+4+9$
b $a+(b+c)=a+b+c$
c $7+x+y=7+(x+y)$

2 A pair of symbols of grouping preceded by a minus (—) sign may be *removed or may be inserted provided that every term between the pair* of symbols has its sign changed

Illustrations

a $18-(7-2)=18-7+2$
b $x-(a+b)=x-a-b$
c $x-5a+6b=x-(5a-6b)$

This second rule can be understood if it is observed that in the expression $x-(a+b)$, the numerical coefficient of $(+a+b)$ is -1 To remove the parentheses is in effect to carry out the multiplication $-1\times(+a+b)$ $=-1\times(+a)+(-1)\times(+b)=-a-b$

If the coefficient of a quantity is some number or letter other than $+1$ or -1, each term of the quantity must be multiplied by the number or letter when the pair of symbols of grouping is removed

Illustrations

a $5x+3(2x-y)=5x+6x-3y=11x-3y$
b $3c-4(-2c+5)=3c+8c-20=11c-20$
c $a-b(c-d)=a-bc+bd$

When a pair of symbols of grouping is inserted in an expression, any factor common to all the terms in the quantity can be removed from these terms if it is put down as the coefficient of the quantity

Illustrations

a $27+55-33=27+11(5-3)$
b $4x-9y+6w=4x-3(3y-2w)$
c $a+bc+dc-ec=a+c(b+d-e)$

When one set of groupings is enclosed within another set, first remove the innermost pair according to the preceding rules.

Illustration: $x - [3a + 2(2a + b) - 7b] + y$
$= x - [3a + 4a + 2b - 7b] + y$
$= x - [7a - 5b] + y$
$= x - 7a + 5b + y$

EXERCISE 6.13

Remove all symbols of grouping and collect terms.

1. $7x - (3x - 2)$
2. $-3a + (4a + 9)$
3. $(2c - 3d) - (3c - 5d)$
4. $-(5w + 3) + (7w - 10)$
5. $8 - (3w + 2s - 5)$
6. $(4x - 9) + (3 - 2x)$
7. $(3a + b - 2c) - (2a - b)$
8. $(4x - 7y + 7) - 2(3x - 2y)$
9. $3x - (4x + 9) - (-2x + 5)$
10. $7a - (3a - 4) - (-3a + 7)$
11. $(2x - 3y + 7) - (3x + y - 4)$
12. $(a - 7b + c) - (-4a + b + 2c)$
13. $7x + 2(3x - 4) - (8x - 13)$
14. $3w - 3(2w + x - 4) - 2(w - x + 2)$
15. $2y - (5x + 7) - 2(3y - 2x + 1)$
16. $x - [3x - (4x + 3)]$
17. $8x + [5x - 4(3x - 1)]$
18. $(4a - b) - [(2a - b) - (3a + 2b)]$
19. $2(x - 2y) - [(2x + y) - (3x - 5y)]$
20. $7 - 3\{(4x - 1) - [3x - 2(2x + 3)]\}$

Insert symbols of grouping in the following.

21. $3x + 21$
22. $2x - 8$
23. $4x - 10$
24. $-2x + 4$
25. $-5x - 15$
26. $8x + 4y - 20$
27. $6a + 8b - 4c$
28. $3a + 6b - 5c - 10d$
29. $-4a + 8b + 3x - 9y$
30. $ax + av + bw - 2bv$

Factoring

The process of determining the component parts or factors of an algebraic expression is called *factoring*. A review of the nature of the

products most generally found in algebra may assure facility in factoring The following types of products occur often

1 $a(x + y) = ax + ay$ Thus a product of the nature $ax + ay$, may be factored as $a(x + y)$ If every term in an algebraic expression contains a common factor, that factor may be removed

Illustrations

a Factor $7x - 14$

Each of the terms contains the common factor 7, so

$$7x - 14 = 7(x - 2)$$

b Factor the expression $4a^2x^3 - 20abx^2 - 24bx$

Each term contains the common factors 4 and x, hence

$$4a^2x^3 - 20abx^2 - 24bx = 4x(a^2x^2 - 5abx - 6b)$$

EXERCISE 6.14

Factor the following

1. $a^2x - ax^2$
2. $8x^2 - 12x$
3. $3x^3 + 6x^2y$
4. $9y^2z - 3yz^2$
5. $14x^2 - 7x^2y$
6. $7 - 14x^2$
7. $4x^3 + 8x^2y^2 - 12xy^3$
8. $5x^2 + 10x^2y - 20x^2y^3$
9. $12a^2b^2c^2 - 8abc - 4a^2b$
10. $3w^3 - 5w^2z - 4wz^3 + w$

2 $(x + a)(x - a) = x^2 - a^2$ Thus a product which appears as the difference of two squares, such as $x^2 - a^2$, may be factored as the sum and difference of two numbers $x^2 - a^2 = (x + a)(x - a)$

Illustration Factor $16x^2 - 9y^2$

$$16x^2 = (4x)^2,\ 9y^2 = (3y)^2, \text{ hence}$$
$$16x^2 - 9y^2 = (4x + 3y)(4x - 3y)$$

EXERCISE 6.15

Factor the following

1. $x^2 - 25$
2. $4x^2 - y^2$
3. $9a^2 - 16b^2$
4. $16x^2 - 25y^2$
5. $36a^2 - 25b^2$
6. $9x^2 - 4a^2$
7. $9x^4 - 16y^2$
8. $4a^2b^2 - 9c^2$
9. $16x^2y^2z^2 - w^2v^2$
10. $a^2b^2 - 4c^2d^2$

3. $(x + a)^2 = x^2 + 2ax + a^2$ A perfect square is recognized by the fact that the middle term is equal to twice the product of the square roots of the first and third terms.

Illustration: Factor $x^2 + 6x + 9$.

Since the middle term, $6x$, is twice the product of x, the square root of x^2, and 3, the square root of 9, then $x^2 + 6x + 9 = (x + 3)^2$.

EXERCISE 6.16

Factor the following.

1. $x^2 + 4x + 4$
2. $x^2 - 8x + 16$
3. $x^2 - 10x + 25$
4. $x^2 + 2x + 1$
5. $x^2 - 12x + 36$
6. $4x^2 - 12x + 9$
7. $9x^2 + 6x + 1$
8. $16x^2 - 24x + 9$
9. $4x^2 + 20x + 25$
10. $36x^2 + 12x + 1$

4. $(x + a)(x + b) = x^2 + (a + b)x + ab$. Here the first term of the polynomial is the square of x, the second term is the sum of the products of the two outer terms and the two inner terms, and the last term is the product of the second term of each.

Thus if an expression such as $x^2 + 3x + 2$ is to be factored, it is readily seen that it is the product of $(x + ?)(x + ?)$. The two unknown factors must have a sum of 3 and a product of 2. Since $2 + 1 = 3$ and $2 \times 1 = 2$, the unknown terms are 2 and 1. That is: $x^2 + 3x + 2 = (x + 1)(x + 2)$.

If the product is $x^2 - 3x - 18$, the problem is to find two factors for -18 such that their sum is a -3. Factors of 18 are 9 and 2, 6 and 3, and 18 and 1. Thus the factors of -18 which have a sum of -3 are -6 and $+3$. Thus the factors of $x^2 - 3x - 18$ would be $(x + 3)(x - 6)$.

Illustrations:

a. Factor $x^2 + 7x + 12$.

The problem is to find two numbers whose sum is 7, and whose product is 12. The factors of 12 are 12 and 1, 3 and 4, 2 and 6. Only 3 and 4 satisfy the requirement. Hence

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$

b. Factor $x^2 - 7x - 8$.

The problem is to find two numbers whose difference is 7, and whose product is 8. Factors of 8 are 8 and 1, and 2 and 4. Only 8 and 1 satisfy the requirement. Hence

$$x^2 - 7x - 8 = (x - 8)(x + 1)$$

EXERCISE 6.17

Factor the following

1. $x^2 + 8x + 15$
2. $x^2 - 8x + 12$
3. $x^2 + 4x + 3$
4. $x^2 - 7x + 10$
5. $x^2 + x - 6$
6. $x^2 - x - 20$
7. $x^2 - 5x - 24$
8. $x^2 + 3x - 28$
9. $x^2 + 13x + 40$
10. $x^2 - 12x + 35$

5 $(ax + b)(cx + d) = acx^2 + (cb + da)x + bd$ This is the general quadratic form To factor an expression such as $12x^2 - x - 6$, it is necessary to find first the possible factors of the coefficient of x^2, and the possible factors of the last term -6, which can be combined in such a manner as to give a sum of -1 Sometimes many trials must be made before the right combination is found Possible factor combinations of 12 are 3 and 4, 6 and 2, 12 and 1, and possible factor combinations of 6 are 3 and 2, and 6 and 1 Therefore the possible factors for $12x^2 - x - 6$ are $(4x + 3)(3x - 2)$, $(4x - 3)(3x + 2)$ $(12x + 1)(x - 6)$, and so on By trial the correct product can be found The second one is the correct product, since $3 \times (-3) + 4 \times 2 = -9 + 8 = -1$ That is, $12x^2 - x - 6 = (4x - 3)(3x + 2)$

Illustrations

a Factor $2x^2 - 15x + 7$

The factors of 2 are 1 and 2, and the factors of 7 are 1 and 7 Hence

$$2x^2 - 15x + 7 = (2x - 1)(x - 7)$$

b Factor $2x^2 - 9x + 7$

The factors of 2 are 1 and 2 and the factors of 7 are 1 and 7 Hence

$$2x^2 - 9x + 7 = (2x - 7)(x - 1)$$

EXERCISE 6.18

Factor the following

1. $2x^2 - 7x + 6$
2. $2x^2 - x - 6$
3. $2x^2 + 7x + 6$
4. $2x^2 + x - 6$
5. $3x^2 - 17x + 10$
6. $3x^2 + 13x - 10$
7. $3x^2 - x - 10$
8. $6x^2 - 25x + 4$
9. $6x^2 - 11x - 2$
10. $6x^2 + 13x + 2$

EXERCISE 6.19

Factor the following.

1. $3x + 6y$
2. $5x^2 - 15x$
3. $P + Pi$
4. $2a^2 + 6a - 4a^3$
5. $a^2 - 4b^2$
6. $16x^2 - 9y^2$
7. $27^2 - 24^2$
8. $x^2 + 4x + 4$
9. $x^2 - 2xy + y^2$
10. 32^2 and 35^2
11. $4a^2 - 20ax + 25x^2$
12. $x^2 - 7x + 12$
13. $x^2 + x - 12$
14. $x^2 + 3x - 10$
15. $x^2 - 7x + 10$
16. $2x^2 - 3xy - 2y^2$
17. $4x^2 + 8xy + 3y^2$
18. $12x^2 + 5x - 3$
19. $6x^2 + 7xy + 2y^2$
20. $6x^2 - 7x - 3$

7

Equations and Their Solutions

Introduction

The primary objective of studying elementary algebra is to develop facility in the solution of equations In subsequent sections the principles you have previously learned will be reviewed, and enough problems given to permit you to gain proficiency in the mechanics of solution

If a problem in business recurs regularly with simple variations *only* in the numbers used, one timesaving device is to develop a chart or *table which shows the answers With such a chart or table the solution* can be regularly found by anyone whether or not he has the mental facilities to understand how the computation was made originally Simple tables showing cash discounts, sales taxes, and payroll withholding taxes, are in general use while much more complicated tables have been developed to meet particular needs

Training in mathematics helps to develop facility with numbers It *may also help to develop a type of reasoning and understanding* which is a contributing factor to success in business The objective of this section is to help you develop an inquisitiveness as to the relationships existing in particular problems and to show you how they may be expressed algebraically

Types of equations

In algebra simple arithmetic relationships are generalized by expressing them in general numbers Letters such as a, b, c, and d—that is, letters at the beginning of the alphabet—are customarily substituted for the known values, and letters such as x, y, and z—that is, letters at the end of the alphabet—are customarily used for unknown quantities

The algebraic statement of $a + a + a = 3a$ holds true for all the values that may be assigned to a, or $x + x = 2x$ is true for all values

assigned to x. Any such statement of equality between two expression—or members—is called an *equation*. An equation, such as the two examples just given, which holds true for all values of the letter involved is called an *identical equation* or an *identity*.

An equation that is true for only a certain value of the letter, called the *unknown*, or a specific set of values of the unknown, is called a *conditional equation*. In the equation $x + 1 = 5$ it is apparent that 4 is the only number which when added to 1 gives 5; thus the equation is true only if x is equal to 4. Any number which, when substituted for the unknown in an equation reduces both sides to the same number is called the *root* of the equation and is said to *satisfy* the equation. To *solve* an equation is to find the root or roots, or the value or values for the unknown which reduces both sides of the equation to the same number.

EXERCISE 7.1

Which of the following are identical and which are conditional equations?

1. $2a + 3a = 5a$
2. $7x = 14$
3. $5x - 5 = 0$
4. $3x + 8x = 11x$
5. $x + 2 = 7$
6. $4x + 5 = 3x + x + 5$
7. $2x - x = 3x - 2x$
8. $3x + 11 = 20$
9. $x^2 + 6x + 9 = (x + 3)^2$
10. $(x - 2)(x + 2) = x^2 - 4$

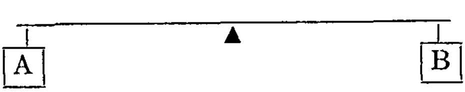
Axioms of equality

The solution of equations entails largely the application of certain axioms which have been learned in your study of arithmetic, geometry, and logic. The axioms are:

1. If equals are added to equals, the sums are equal.
2. If equals are subtracted from equals, the differences are equal.
3. If equals are multiplied by equals, the products are equal.
4. If equals are divided by equals, the quotients are equal.

Operations that can be performed on equations

Since an equation represents an equality between two sides, it is similar to a balanced pair of scales where the two arms are of the same



length. If objects of equal *weight* are added to or subtracted from both sides of the scale, the balance will not be destroyed. In algebra, as in

arithmetic, only like objects can be added or subtracted from each other An equation, however, is not necessarily concerned with like objects, but rather with terms of equal values To carry further the analogy of the scale, it matters not whether the two sides represent lead and gold or gold and potatoes If equal weights are added or deducted the scale will remain in balance In the application of the axioms which follow, numbers are used to make it obvious that the value on the left side of the equal sign balances the value on the right side of the equal sign Few persons experience difficulty in understanding the fundamental principles of the axioms of equality when they are viewed in this light

1 *Addition* When arabic numbers are used, it is easily understood that

Given $5 = 5$
Adding 3 to each side gives $5 + 3 = 5 + 3$
Simplifying $8 = 8$

Practice in dealing with general numbers makes it equally easy to see that

Given the equation $x - a = b$
Adding a to each side gives $x - a + a = b + a$
Simplifying $x = b + a$

Illustration Solve for x in the equation $x - 5 = 2$

Given the equation $x - 5 = 2$
Adding 5 to each side gives $x - 5 + 5 = 2 + 5$
Simplifying, we have $x = 7$

2 *Subtraction* Using arabic numbers, we accept the fact that

Given $5 = 5$
Subtracting 2 from each side gives $5 - 2 = 5 - 2$
Simplifying $3 = 3$

Using general numbers, the same principle may be illustrated

Given the equation $x + a = b$
Subtracting a from each side gives $x + a - a = b - a$
Simplifying $x = b - a$

Illustration Solve for x in the equation $x + 3 = 7$

Given the equation $x + 3 = 7$
Subtracting 3 from each side gives $x + 3 - 3 = 7 - 3$
Simplifying, we have $x = 4$

3. *Multiplication.* Using arabic numbers, it is not difficult to see that

Given $5 = 5$

Multiplying each side by 3 gives $5 \times 3 = 5 \times 3$

Simplifying $15 = 15$

Using general numbers

Given the equation $\frac{x}{a} = b$

Multiplying each side by a gives $\frac{x}{a} \times a = b \times a$

Simplifying $x = ba$

Illustration: Solve for x in the equation $\frac{x}{4} = 2$.

Given the equation $\frac{x}{4} = 2$

Multiplying each side by 4 gives $\frac{x}{4} \times 4 = 2 \times 4$

Simplifying, we have $x = 8$

4. *Division.* Using arabic numbers we see that

Given $5 = 5$

Dividing both sides by 2 gives $\frac{5}{2} = \frac{5}{2}$

Simplifying $2\frac{1}{2} = 2\frac{1}{2}$

Using general numbers,

Given the equation $ax = b$

Dividing both sides by a gives $\frac{ax}{a} = \frac{b}{a}$

Simplifying $x = \frac{b}{a}$

Illustration: Solve for x in the equation $4x = 15$.

Given the equation $4x = 15$

Dividing each side by 4 gives $\frac{4x}{4} = \frac{15}{4}$

Simplifying, we have $x = 3\frac{3}{4}$

The rule of transposition

In reviewing the results of adding numbers to or deducting numbers from both sides of an equation, it is noticed that in one illustration the original equation was $x - a = b$, while the final equation was $x = b + a$. That is, the minus a on the left-hand side disappeared, and a plus a appeared on the right-hand side. Actually an a had been added to both sides, but

the ultimate result was as if the a had been transposed and its sign changed

In the second illustration, $x + a = b$ became $x = b - a$ when a was deducted from both sides Again one is left with the impression that the a was transposed and the sign changed If one understands what actually has happened he may safely use the *rule of transposition* which states that *any term may be moved from one side of the equal sign of an equation to the other side if the algebraic sign is changed*

Solutions of simple equations

In order to solve a given equation it is usually necessary to apply one or more of the fundamental operations discussed

Illustrations

a Solve the equation $x + 3 = 8$
Transpose the 3 $x = 8 - 3$
Collect terms $x = 5$

To verify this answer, the value found for x is substituted in the original equation

Original equation $x + 3 = 8$
Substitute $5 + 3 = 8$
Collect terms $8 = 8$

b Solve the equation $x - 5 = 7$
Transpose the 5 $x = 7 + 5$
Collect terms $x = 12$

Check $12 - 5 = 7$
$7 = 7$

c Solve the equation $4x = 32$
Divide both sides by 4 $\frac{4x}{4} = \frac{32}{4}$
Simplifying we have $x = 8$

Check $4 \times 8 = 32$
$32 = 32$

d Solve the equation $\frac{x}{3} = 7$
Multiplying by 3 $\frac{x}{3} \times 3 = 7 \times 3$
Simplifying we have $x = 21$

Check $\frac{21}{3} = 7$
$7 = 7$

All the preceding illustrations are intended to disclose the mental process entailed in the solution of simple equations. Actually, all these and some much more complex can be solved orally.

EXERCISE 7.2

Solve and check the following orally.

1. $x + 9 = 13$
2. $x + 13 = 18$
3. $x + 3 = 18$
4. $x + 8 = 9$
5. $x - 3 = 8$
6. $x - 24 = 1$
7. $x - 5 = 8$
8. $x - 2 = 7$
9. $x + 7 = 4$
10. $x + 12 = 7$
11. $x - 6 = -2$
12. $x - 3 = -8$
13. $\frac{x}{4} = 5$
14. $\frac{x}{3} = 2$
15. $\frac{x}{6} = -2$
16. $\frac{x}{2} = -8$
17. $3x = 27$
18. $4x = 18$
19. $2x = -9$
20. $5x = -24$

Solution of more complex equations

For most practical purposes an equation which involves only one operation can be solved as a problem in arithmetic. In a more complex equation it may be necessary to use more than one of the four basic methods illustrated. Taken step by step, such equations should present no difficulties.

Illustrations:

a. Solve the equation $2x + 5 = 21.$

Transpose the 5 $\quad 2x = 21 - 5$

Collect terms $\quad 2x = 16$

Divide by 2 $\quad \frac{2x}{2} = \frac{16}{2}$

Simplifying, we have $\quad x = 8$

Check: $2 \times 8 + 5 = 21$

$16 + 5 = 21$

$21 = 21$

b Solve the equation $4x - 5 = 31$
Transpose the 5 $4x = 31 + 5$
Collect terms $4x = 36$
Dividing by 4, we have $x = 9$

Check $4 \times 9 - 5 = 31$
$36 - 5 = 31$
$31 = 31$

c Solve the equation $5x - 7 = 2x + 5$
Transpose the 7 $5x = 2x + 5 + 7$
Transpose the $2x$ $5x - 2x = 5 + 7$
Collect terms $3x = 12$
Dividing by 3, we have $x = 4$

Check $5 \times 4 - 7 = 2 \times 4 + 5$
$20 - 7 = 8 + 5$
$13 = 13$

EXERCISE 7.3

Find the root in each of the following equations

1. $4x + 9 = 25$
2. $5x - 4 = 26$
3. $4x + 12 = 0$
4. $8x + 3 = -13$
5. $2x + 5 = -7$
6. $3x - 7 = 11$
7. $4x - 9 = -5$
8. $4y - 19 = 2y - 1$
9. $2v - 6 = 3v + 6$
10. $x - 2x = 15$
11. $3w - 12 = 2w + 20$
12. $3y - 9 = 11 - y$
13. $5z = 27 - 4z$
14. $24 = 33 - z$
15. $4x - 6 = x + 3$
16. $11x = 2 + 3x$
17. $x + 14 = 20 - 5x$
18. $7x - 10 = x + 2$
19. $1 - 3x = 5x + 4$
20. $7 + 4x = 6x - 3$
21. $2x - 21 = 4x - 2$
22. $7x + 3 = 3x - 1$
23. $2x = 15 - x$
24. $8x + 9 = 5x + 10$
25. $8 - 3w = 2w - 2$
26. $2 - 3x = 50$
27. $2y - 10 = 4y$
28. $3x + 20 = 7x$
29. $-10z + 21 = 6z + 5$
30. $8w + 3 = 6w$
31. $4x - 60 = 3x - 36$
32. $12 = 24 - 2x$
33. $7x - 1 = 3x + 1$
34. $3x - 9 + 2x + 7 = 6x + 4$
35. $8 - 2x + 5 = 7x + 1 - 3x$
36. $5x - 2x = 18 - 3x$
37. $8y + 3 - y = 23 - 3y$
38. $10x + 5 - 3x + 4 = 4x - 3$
39. $3x + 5 - 2x = 4x - 5$
40. $10 + 5x - 18 + 3x = 7x + 5$

Equations involving fractions

Some equations contain terms which are fractions. To solve such an equation, first eliminate the fraction by multiplying each term by the lowest common denominator (L. C. D.).

Illustrations:

a. Solve the equation $\frac{x}{3} + \frac{7x}{6} = 6$

Multiply each term by 6, the L. C. D.	$2x + 7x = 36$
Collect terms	$9x = 36$
Dividing by 9, we have	$x = 4$

Check: $\frac{4}{3} + \frac{28}{6} = 6$

$1\frac{1}{3} + 4\frac{2}{3} = 6$

$6 = 6$

b. Solve the equation $\frac{9}{x+1} = 6$

Clear fractions	$9 = 6x + 6$
Collect terms	$3 = 6x$, or $6x = 3$
Divide by 6	$x = \frac{1}{2}$

Check: $\frac{9}{\frac{1}{2} + 1} = 6$

$\frac{9}{\frac{3}{2}} = 6$

$6 = 6$

EXERCISE 7.4

Solve for the unknown and check.

1. $\frac{x}{2} + \frac{x}{3} = 5$
2. $\frac{x}{2} + \frac{x}{4} = 18$
3. $\frac{2x}{3} - \frac{3x}{5} = \frac{5}{6}$
4. $\frac{3x}{5} - \frac{5x}{9} = \frac{4}{15}$
5. $\frac{x}{3} - \frac{2x}{5} = \frac{2}{3}$
6. $\frac{x}{3} - 7 = -\frac{5x}{6}$
7. $\frac{8x}{3} - 10 = x$
8. $\frac{x}{4} + \frac{x}{7} = \frac{11}{7}$
9. $\frac{x}{2} = 7 - \frac{x}{5}$
10. $\frac{11}{2} - \frac{x}{8} = \frac{7x}{24}$
11. $\frac{7}{x+2} = 2$
12. $\frac{5}{x-3} = 3$
13. $\frac{x}{2x-5} = 1$
14. $\frac{3x+1}{4x-7} = 2$
15. $\frac{2}{x} = \frac{5}{x+2}$

Equations containing quantity symbols

If an equation contains quantity symbols, such as ( ), [ ], or { }, they must be removed before the equation can be solved Thus, in solving equations, the step-by-step procedure to be followed can be summarized as follows

1 Remove any quantity symbols which occur in the equation, if they contain the unknown, otherwise merely combine the numerical values within them

2 Clear any fractions which occur in the equation

3 Combine similar terms

4 Transpose all terms containing the unknown to one side of the equal sign, and all other terms to the other side of the equal sign

5 Collect like terms on each side

6 Divide both sides by the coefficient of the unknown

After carrying out step 5, the equation should be in a form equivalent to the equation $ax = b$, in which a is the coefficient of the unknown, x is the unknown—a number with no indicated exponent—and b is the value found from collecting the terms on the other side of the equation *An equation which may be reduced to the form $ax = b$ is an equation of the first degree in one unknown, or a linear equation in one unknown* Once the equation has been reduced to the form $ax = b$, the final solution, step 6, is to divide both sides of the equation by the coefficient of the unknown, so that $x = \frac{b}{a}$

After the root of an equation has been found the solution can be verified as follows

1 Substitute the value found for the unknown in the given equation

2 Remove the quantity symbols by combining the numerical values within them

3 Combine the terms of each side

4 Verify the fact that the left side equals the right side

Illustrations

a Find the value of the unknown in $2(x - 3) - 3(x + 2) = 8$

Removing parentheses, we have	$2x - 6 - 3x - 6 = 8$
Combine similar terms	$-x - 12 = 8$
Transpose like terms	$-x = 8 + 12$
Collect like terms	$-x = 20$
Divide both sides by -1	$x = -20$

Check

$$2(-20 - 3) - 3(-20 + 2) = 8$$
$$2(-23) - 3(-18) = 8$$
$$-46 + 54 = 8$$
$$8 = 8$$

b. Find the value of the unknown in $\dfrac{2(x+3)}{5} - \dfrac{3(x-1)}{4} = \dfrac{1}{5}$.

Multiply each term by 20 (L.C.D.)

$$8(x+3) - 15(x-1) = 4$$

Removing parentheses, we have $8x + 24 - 15x + 15 = 4$

Combine similar terms $-7x + 39 = 4$

Transpose like terms $-7x = 4 - 39$

Collect like terms $-7x = -35$

Divide both sides by -7 $x = 5$

Check: $\dfrac{2(5+3)}{5} - \dfrac{3(5-1)}{4} = \dfrac{1}{5}$

$$\frac{2 \times 8}{5} - \frac{3 \times 4}{4} = \frac{1}{5}$$

$$3\tfrac{1}{5} - 3 = \tfrac{1}{5}$$

$$\tfrac{1}{5} = \tfrac{1}{5}$$

c. Find the value of the unknown in $3 - \dfrac{14}{3x} = \dfrac{3x}{x+2}$.

Multiply each term by $3x(x+2)$, the L.C.D.

$$9x(x+2) - 14(x+2) = 9x^2$$

Remove parentheses $9x^2 + 18x - 14x - 28 = 9x^2$

Combine similar terms $9x^2 + 4x - 28 = 9x^2$

Transpose like terms $9x^2 - 9x^2 + 4x = 28$

Collect like terms $4x = 28$

Divide both sides by 4 $x = 7$

Check: $3 - \dfrac{14}{3 \times 7} = \dfrac{3 \times 7}{7+2}$

$$3 - \tfrac{14}{21} = \tfrac{21}{9}$$

$$3 - \tfrac{2}{3} = 2\tfrac{1}{3}$$

$$2\tfrac{1}{3} = 2\tfrac{1}{3}$$

EXERCISE 7.5

Solve for the unknown and check.

1. $3(x-2) + 2(x-5) = 19$
2. $4(x+3) - 3(x-2) = 20$
3. $2(x-5) - 3(2x-1) = 1$
4. $5(3x-8) = 2(3x+4) - 12$
5. $6(2x-1) = 4 - 2(x+5)$
6. $2(4x-3) = 2 - 8(x-2)$
7. $2(x+7) + 5(x-4) = 1$
8. $3(2x-5) - 2(2x+3) = 5$
9. $2(3x+4) = 4(x-3) + 5$
10. $5 - 3(2x-1) = 2(3x+1)$

11. $3 + 5(3x + 2) = 7(2x - 1)$

12. $4(2x - 1) = 9 - 2(3x - 4)$

13 $\dfrac{x-3}{2} + \dfrac{x+4}{5} = 7$

14. $\dfrac{2x-1}{3} - \dfrac{x+3}{4} = 1$

15 $\dfrac{3x-5}{2} - \dfrac{7-2x}{3} = \dfrac{5}{3}$

16. $\frac{1}{3}(2x + 1) = \frac{1}{4}(3x - 2)$

17. $\frac{1}{2}(4x - 3) = \frac{2}{3}(2x + 3)$

18. $\frac{3}{4}(2x - 5) = \frac{2}{3}(x - 3)$

19. $\frac{2}{5}(3 - x) = \frac{3}{2}(x - 3)$

20. $\dfrac{8}{x} + \dfrac{5}{2x} - \dfrac{7}{2}$

21. $\dfrac{5}{2x} - \dfrac{2}{x} = \dfrac{1}{2}$

22. $\dfrac{4}{3x} + \dfrac{3}{2x} = \dfrac{17}{6}$

23. $\dfrac{3}{4} - \dfrac{5}{x} = \dfrac{7}{4x}$

24. $\dfrac{x+3}{x} = \dfrac{5}{8}$

25. $\dfrac{x-2}{x} = \dfrac{7}{6}$

26 $\dfrac{x-1}{x+1} = \dfrac{5}{2}$

27. $7 - \dfrac{5}{x} = \dfrac{7x+1}{x+3}$

28. $\dfrac{2x-3}{x-1} = \dfrac{2x}{x+1}$

29. $\dfrac{6x+7}{3} + \dfrac{7x-13}{2x+1} = 2(x + 2)$

30. $\dfrac{9x+20}{36} = \dfrac{4(x-3)}{5x-4} + \dfrac{x}{4}$

Solving formulas

In the equations solved so far the root has been a specific number, except at the beginning of the chapter in the illustrations of the fundamental operations where *literal equations* were also used *A literal equation is defined as one which involves one or more literal numbers besides the one whose value is required*

Illustrations

a Given the equation $3x - 2a = 7a$, solve for x
Transposing $2a$, we have $3x = 9a$
Simplifying, we have $x = 3a$

Check $3 \times 3a - 2a = 7a$, or $7a = 7a$

b Given the equation $4x - 3a = a + 8b$, solve for x
Transposing $3a$, we have $4x = 4a + 8b$
Simplifying, we have $x = a + 2b$

Check $4(a + 2b) - 3a = a + 8b$, or $4a + 8b - 3a = a + 8b$, or $a + 8b = a + 8b$

For students of business and finance the solution of literal equations has its most practical application in the solution of *formulas.* Many relationships in business and industry fall into such precise patterns that they can be readily stated in algebraic symbols. *A statement of a principle or rule involving magnitudes expressed in form of an equation is called a formula.*

A formula may be solved for any particular letter by considering that letter to be the unknown. Thus, given the formula and its known quantities, the unknown quantity can be found by following the same steps as in the solution of numerical equations. One important difference will be noticed in such solutions—often the operation cannot be fully carried out since no particular values have been assigned to the known quantities.

Illustrations:

a. Given the formula $V = \dfrac{M - S}{r + 1}$, solve for M.

Think of every letter, except M, as a known value.

Multiply each term by $r + 1$, the L.C.D.	$V(r + 1) = M - S$
Remove the parenthesis	$Vr + V = M - S$
Transpose the S	$Vr + V + S = M$,
	or $M = Vr + V + S$

b. Given the formula $V = \dfrac{M - S}{r + 1}$, solve for r.

Think of every letter, except r, as a known value.

Multiply each term by $r + 1$, the L.C.D.	$V(r + 1) = M - S$
Remove the parenthesis	$Vr + V = M - S$
Transpose the term *not* containing r	$Vr = M - S - V$
Divide both sides by V	$r = \dfrac{M - S - V}{V}$

c. Given $D = Sdt$, solve for t.

The coefficient of the unknown is Sd. Therefore divide both sides by Sd.

$$\frac{D}{Sd} = t, \quad \text{or} \quad t = \frac{D}{Sd}$$

EXERCISE 7.6

Solve the following formulas for the letters indicated

1. $S = P + I$, for I
2. $S = P + I$, for P
3. $C = Pqr$, for r
4. $C = Pqr$, for q
5. $C = Pqr$, for P
6. $S = a + b + c$, for b
7. $S = a + b + c$, for c
8. $S = P(1 + i)$, for P
9. $S = P(1 + i)$, for i
10. $W = ab + c$, for c
11. $W = ab + c$, for a
12. $A = \frac{ab}{2}$, for a
13. $K = 2\pi RH$, for R
14. $a(b - 1) = 2a + b$, for a
15. $a(b - 1) = 2a + b$, for b
16. $W = \frac{m + n}{n}$, for m
17. $W = \frac{m + n}{n}$, for n
18. $V = \frac{a}{a - b}$, for a
19. $V = \frac{a}{a - b}$, for b
20. $D = \frac{cd}{c + d}$, for c
21. $D = \frac{cd}{c + d}$, for d
22. $C = \frac{E}{R + nr}$, for E
23. $C = \frac{E}{R + nr}$, for r
24. $C = \frac{E}{R + nr}$, for R
25. $\frac{a}{w} = \frac{b}{c}$, for c
26. $\frac{a}{w} = \frac{b}{c}$, for a
27. $\frac{a}{w} = \frac{b}{c}$, for w
28. $\frac{a}{A} = \frac{D}{360}$, for a
29. $\frac{a}{A} = \frac{D}{360}$, for A
30. $\frac{a}{A} = \frac{D}{360}$, for D

Word problems

A knowledge of arithmetic is essential for success in business and finance A knowledge of algebra may not be essential—that is, one who has little knowledge of algebra may succeed in business and finance—but in enterprises of every size, simple problems constantly arise which can be solved readily by application of algebraic principles In most business problems a knowledge of only the fundamentals of algebra is necessary

The problems which arise in business are not algebraic equations. Instead, from certain known facts it is possible to find a relationship which can be stated as an algebraic equation through the introduction of a symbol, such as x, for one of the unknown quantities. After the problem has been stated in equation form, it is possible to solve the problem readily.

To train a person to see and to be able to express algebraic relationships develops his capacity for business reasoning and his understanding of business processes. While an untrained person may be able to apply specific formulas to achieve the desired results, one who understands the wider implications of the procedures applied should be able to progress more rapidly toward his ultimate goal.

The ability to state relationships in algebraic symbols is the first requisite to success in the solution of problems. The ability to state problems in such form may be improved by concentrated attention on a few elementary relationships. The practical significance of such problems per se may not be great but they may be effective in training students to express relationships algebraically.

Traditionally the problems used for training students in algebra fall into one or more of the following categories: digit problems, relations between number problems, time-rate-distance problems, work problems, and problems involving mixtures. In business the algebraic type of problem often is some form of ratio and proportion, such as percentage, interest, markup, and markdown problems. Some business problems represent applications of more advanced mathematical principles.

Even though many of the traditional types of problems have limited direct use in business, their inclusion here seems necessary for two reasons. First, they are often found on placement tests of various kinds, civil service examinations, and intelligence tests. Second, they are good vehicles for training the student to express relationships.

In solving problems dealing with mathematical relationships between general numbers, words and expressions mean exactly the same as they do in arithmetic. The excess of 10 over 6 can be shown as $10 - 6$; the excess of a over b can be shown as $a - b$.

In the following illustration, certain ideas are expressed in words, and their algebraic equivalents are given. In many instances it is seen that seemingly widely different word expressions may be stated in the same algebraic expression. Read over the following English phrases, and their algebraic expressions. Then cover the second column and write the expression youself.

Illustration Express the following in algebraic terms

English Phrase	*Algebraic Expression*
1. Three more than x	$x+3$
2. x increased by 3	$x+3$
3. The sum of x and 3	$x+3$
4. Three greater than x	$x+3$
5. Five less than x	$x-5$
6. The excess of x over 5	$x-5$
7. x diminished by 5	$x-5$
8. Three more than twice x	$2x+3$
9. The number that is 1 more than half of x	$\frac{1}{2}x+1$
10. Three less than twice the product of x and y	$2xy-3$
11. The sum of the squares of x and y	x^2+y^2
12. The value of x nickels in dollars	$0.05x$
13. The sum of x and y	$x+y$
14. The value of x pounds at y cents a pound	xy cents
15. The price of x tickets at $1 per ticket	$\$x$

If x is the rate at which A rows and y is the rate of the current in a river

16. The rate at which A will move downstream is	$x+y$
17. The rate at which A will move upstream is	$x-y$
18. If B's rate is 6 greater than A's, B s speed in still water is	$x+6$
19. The distance B will travel downstream in 20 minutes (i e , $\frac{1}{3}$ of an hour) is	$\frac{x+y+6}{3}$

If A can do a job in 10 days, B in 5 days, and together they do it in x days

20. The part of the job completed by both in 1 day is	$\frac{1}{x}$
21. The part of the job completed by A in 1 day is	$\frac{1}{10}$
22. The part of the job completed by B in 1 day is	$\frac{1}{5}$

In converting written statements to algebraic terms, the procedure used is simply to state the unknown in terms of a symbol, such as x, and to add to, deduct from, multiply, or divide, this quantity to express the known relationships

EXERCISE 7.7

State briefly the following.

1. What number is 5 more than 10? 5 more than x?

2. If one number is x, what number is 1 more than 3 times as great? 2 less than twice as great?

3. If y is a number, write the product of y multiplied by itself.

4. What number is 10 more than the square of y?

5. If x is an odd number, what is the next higher number? Is it odd or even?

6. If x is an even number, what is the next higher even number? What is the number that is 5 less than the next higher even number?

7. If x is less than 100, what is the excess of 100 over x?

8. If the annual rate of interest on x dollars is 5 per cent, what is the amount of interest for 2 years?

9. If x exceeds $300, what is the excess of x over $300?

10. If the rate on the first $300 borrowed is 2 per cent per month, and the rate on the balance is $1\frac{1}{2}$ per cent per month, how much must be paid for x dollars borrowed for one month, assuming x is more than $300?

11. If annual interest at 2 per cent is paid on the amount in an account up to $10,000, how much is paid annually for x dollars when x is less than $10,000?

12. If annual interest is paid at the rate of 2 per cent on the first $10,000, and at the rate of 1 per cent on the amount over $10,000, how much annual interest is paid on the excess of x over $10,000? What is the total amount of interest paid annually on the x dollars?

13. If a piece of metal which weighs y ounces contains 40 per cent silver, how much does the silver weigh?

14. If 100 ounces of pure copper are added to an alloy which weighs y ounces and contains 40 per cent silver, what is the percentage of silver in the new alloy?

15. At a football game there were 16,000 paid admissions. If x students paid $1 each and the balance paid $2, how many paid $2 each?

16. If A can do a job in 5 days and B can do it in 8 days, what part of the job does A do in one day? What part of the job does B finish in one day? What part of the job do they both finish in one day?

17. If A can do a job in x days and B can do it in y days, what part of the job does A do in one day? What part of the job does B do in one day? What part of the job do they both do in one day?

18. If x is a sum of money to be divided equally among 5 persons, how much will each receive?

19. If x is a sum of money to be divided between 2 persons in such a manner that one will receive 4 times as much as the other, how much does each receive?

20. Divide x into 3 parts so that the first part is 4 times the second and the third is 3 times the second

The solution of stated problems

Most of the problems encountered in business are not clearly defined Therefore it is essential to develop definite procedures in attacking a problem in order to be sure, first, of what is known, and, second, of what is to be found Experience shows that most people are aided in accomplishing these two objectives by making some kind of graphic representation

Illustration A jobber has an opportunity to buy 1,000 gallons of insect spray The spray contains only a 25% solution of the actual killing agent By changing the spray from a 25% solution to a 40% solution, and then relabeling the can, the jobber can make a reasonable profit How many gallons of full-strength (100%) solution of the actual killing agent must be added to increase the solution to a 40% solution?

He now has

$$\boxed{\text{1,000 gallons}}$$

of which 25%, or 250 gallons, is full-strength insect spray.

$$\boxed{25\% \times 1{,}000 = 250}$$

He must add x gallons of full-strength (100%) solution,

$$\boxed{100\% \times x = x}$$

giving a total amount of

$$\boxed{\text{1,000 gallons}} + \boxed{x \text{ gallons}}$$

Algebraic solution

1 Let x = number of gallons of full-strength (100%) solution to be added

2 Then $(1{,}000 + x)$ = amount of new solution, and $40\%\,(1{,}000 + x)$ = total amount of pure insect spray in new solution

3. Since $25\% \times 1{,}000$ = total amount of pure insect spray in original solution, and since $100\% \times x$ = total amount of pure insect spray in solution added, the total amount of pure insect spray in the mixture is $25\% \times 1{,}000 + 100\% \times x$. Therefore, since the two total amounts of pure insect spray are the same, the following linear equation is formed:

$$40\% \times (1{,}000 + x) = 25\% \times 1{,}000 + 100\% \times x$$

4. Solve for x.

$$400 + 0.4x = 250 + x$$
$$0.6x = 150$$
$$x = 250$$

5. Since $x = 250$, then $(1{,}000 + x) = 1{,}250$. Therefore the jobber needs to add 250 gallons of full-strength solution of the actual killing agent to the 1,000 gallons of 25% solution of insect spray to change the spray to a 40% solution. He will have 1,250 gallons of this new solution to sell.

Check: 25% of 1,000 gallons + 100% of 250 gallons = 250 gallons + 250 gallons = 500 gallons
40% of (1,000 + 250) gallons = 40% of 1,250 gallons = 500 gallons

The procedure used in this illustration is a good one to follow in solving problems of this type. The procedure may be summarized as follows:

1. Show by a diagram what is given.
2. Let a symbol, such as x, represent the unknown quantity.
3. Express the other unknown quantities in terms of the same symbol.
4. Show graphically what is given and what is unknown.
5. Form the linear equation for the conditions expressed in the problem.
6. Solve the linear equation for the unknown symbol.
7. Determine the unknown quantities.

The problem illustrated is one example of a general type of problem know as a mixture problem, or alloy problem. Such problems are expressed in terms of quantities or values. They include such diverse problems as those dealing with investment income and the mixtures of drugs.

Illustration: An investor with \$100,000 wants to receive an over-all return of 4% on his investment. He can buy government bonds giving a return of $3\frac{1}{2}\%$ with a minimum of risk, or common stocks with a return

of 6%, but carrying a higher risk He wants to keep his investment in common stocks at a minimum To assure a return of 4%, how should he divide his funds between government bonds and common stocks?

His total investment ($100,000) is divided as follows

Amount in bonds $= x$
Balance is in common stock i e, $100,000 $- x$

The income he wants to receive is 4% on $100,000, or $4,000

Income on governments bonds $3\frac{1}{2}\% \times x = 0\,035x$
Income on common stock $6\% \times (\$100,000 - x) = \$6,000 - 0\,06x$

Algebraic solution

1 Let x = amount in bonds and $3\frac{1}{2}\% \times x$ = income on bonds

2 Then $(\$100\,000 - x)$ = amount in common stock, and $6\% \times (\$100,000 - x) = \$6,000 - 0\,06x$ = income on common stock

3 Since the total income desired is $4,000,

$$\$4,000 = 0\,035x + \$6,000 - 0\,06x$$

4 Solve for x

$$\$4,000 = \$6,000 - 0\,025x$$
$$0\,025x = \$2,000$$
$$x = \$80,000, \text{ the amount in government bonds}$$
$$\$100,000 - x = \$20,000, \text{ the amount in common stocks}$$

Check $3\frac{1}{2}\%$ of $\$80,000 + 6\%$ of $\$20,000 = \$4,000$

$$\$2,800 + \$1,200 = \$4,000$$
$$\$4,000 = \$4,000$$

The traditional type of mixture problem is given in the following illustration

Illustration A grocer desires to mix 75 cent coffee with some $1 50 coffee in order to have 20 pounds of coffee that will sell for 90 cents a pound How much coffee is needed at each price?

1. Let x = number of pounds of 150-cent coffee; then $150x$ is the value of the \$1.50 coffee used.

2. Then $(20 - x)$ = number of pounds of 75-cent coffee, and $75(20 - x)$ is the value of the 75-cent coffee used.

3. Since the sum of the values of the two amounts equals the value of the total mixture—that is, 20×90 cents = 1,800 cents—we have $150x + 75(20 - x) = 1{,}800$.

4. Solve for x.

$$\begin{aligned} 150x + 1{,}500 - 75x &= 1{,}800 \\ 75x &= 300 \\ x &= 4 \text{ pounds of \$1.50-coffee} \\ 20 - x &= 16 \text{ pounds of 75-cent coffee} \end{aligned}$$

Check: Since 20 pounds at 90 cents a pound is worth \$18, since 4 pounds at \$1.50 a pound is worth \$6, and since 16 pounds at 75 cents a pound is worth \$12, the problem checks (\$6 + \$12 = \$18). In checking such solutions, check the given problem, not the equation or formula set up for solution.

EXERCISE 7.8

Solve the following problems.

1. A widow is left insurance totaling \$50,000. She estimates her minimum income needs at \$3,000 a year. She can buy high-grade bonds with a minimum of risk which will give her a return of 4% a year. She can buy real estate mortgages which will furnish a return of 8% but require more time for supervision. To gain the maximum safety, and the minimum amount of supervision, how shall she divide her funds to get the desired income?
2. Four years ago a trust fund was established for Rickey Kenney. The fund was invested in $3\frac{1}{4}$% government bonds, and 4% preferred stock. The annual income from the fund is \$925. If the trust fund is \$25,000, how much is invested in each type of security?
3. A merchant has two grades of coffee selling at 75 cents and \$1.00 a pound. He wants 100 pounds of coffee selling at 80 cents a pound. How many pounds of each should be used in the mixture?
4. How much candy selling for 35 cents a pound must be added to 40 pounds of candy selling for 20 cents a pound in order to have a mixture that will sell for 25 cents a pound?
5. A confectioner has 25 pounds of candy worth 40 cents a pound. How much candy worth 60 cents a pound is to be added to make a mixture worth 55 cents a pound?

6. A dealer sells two grades of motor oil at 30 and 15 cents a quart. In what proportions should he mix them to produce 100 quarts of oil to sell at 35 cents per quart?

7. A druggist has 10 quarts of a 3% solution How many quarts of a 10% solution must be added to make a 5% solution?

8. How much pure silver must be added to 45 ounces of 75% alloy silver to make 82% alloy silver?

9. How many pounds of 70-cent tea and 90-cent tea must be mixed to make 35 pounds of 82 cent tea?

10. How many ounces of peanuts worth 32 cents a pound must be mixed with 81 ounces of mixed nuts worth 58 cents a pound in order to make a mixture worth 50 cents a pound? There are 16 ounces in 1 pound

11. How much water must be added to a quart of 80% pure alcohol to make it 75% pure?

12. The receipts from the sale of 80,000 tickets for a football game were $150,000 The price for admission was $1 for students and $2 for others How many tickets of each kind were sold?

13. Student tickets for a home football game sold for $1 each All others paid $1 50 If 635 tickets were sold in all and if $698 50 was taken in, how many students attended the game?

14. The manager of a branch bank is allowed $1,275 as weekly wages for the clerical staff of 20 If the average weekly wages paid are $60 for women and $75 for men, how many men and how many women may he employ?

15. If the radiator of a car holds 4 gallons, how much water and how much 80% pure alcohol should be used in order to have a 35% mixture?

16. A person who possessed $100 000 placed the greater part of it out at 5%, and the other part at 4% The interest which he received for the whole amounted to $4,640 How much is invested at each rate?

17. A man invested $12,000 in securities Part of the securities yielded 4% and the rest 3% If the incomes from the two securities were equal, how much was invested in each?

18. A photographer has a mixture of 40% photographic developer and 60% water How much of a mixture of 20% developer and 80% water must he use with the first mixture in order to make 10 ounces of a mixture of 35% developer and 65% water?

Value problems

One type of problem that appears often on examinations purely as a measure of abstract reasoning is the so-called value problem They are offered here only as exercises, since it is difficult to find examples of a practical application of problems of this type

Illustration:

a. A box contains \$2.75 in nickels and dimes. If there are 35 coins, how many are there of each kind?

1. Let x = number of nickels; then $5x$ = value of the nickels in cents.

2. Then $(35 - x)$ = number of dimes; and $10(35 - x)$ = value of the dimes in cents.

3. Since the total value of the coins is \$2.75 = 275 cents,

$$5x + 10(35 - x) = 275$$

4. Solve for x. $5x + 350 - 10x = 275$

$$75 = 5x$$

$$x = 15, \text{ number of nickels}$$

$$35 - x = 20, \text{ number of dimes}$$

Check: Since 15 × 5 cents = 75 cents and 20 × 10 cents = \$2, the problem checks (75 cents + \$2 = \$2.75).

b. One type of yarn costs $1\frac{1}{2}$ times as much as another. If a buyer pays \$15.20 for 12 skeins of the first type and 20 skeins of the second type, what is the price per skein of each type?

1. Let x = price of the cheaper yarn per skein; then $1.5x$ = price of the more expensive yarn per skein.

2. If the more expensive yarn is the first type, $1.5x \times 12 = 18x$ = value of the first type; and $20x$ = value of the second type. Therefore $18x + 20x = 1{,}520$ cents.

3. Solve for x. $38x = 1{,}520$

$$x = 40, \text{ cents per skein – cheaper yarn}$$

$$1\tfrac{1}{2}x = 60, \text{ cents per skein – other yarn}$$

Check: Since 20 skeins at 40 cents a skein = \$8, and 12 skeins at 60 cents a skein = \$7.20, the problem checks (\$8 + \$7.20 = \$15.20).

EXERCISE 7.9

Solve the following problems.

1. A purse contains 38 coins, consisting of dimes and quarters. If the coins are worth \$5.90, how many dimes are there?

2. A register contains 32 coins, consisting of nickels and dimes, worth \$2.25. How many of each are there?

3. In \$14.50 worth of dimes and quarters, there are two more quarters than there are dimes. How many of each are there?

4. A real estate salesman sold 10 houses and 5 vacant lots for \$110,000 The average price per house was 5 times as great as the average price of the lots What was the average price of each?

5. Richard Whitlo is paid 10% commission on each used car sold, and 5% on each new car The average value of a new car is \$2,400 and of the used car \$800 Last year, on sale of 32 cars, his commission totaled \$3,200 How many of each did he sell?

6. A cashier needs to have 24 dollar bills changed so that she will receive twice as many pennies as dimes If she requests pennies and dimes only, how many of each should she have in her request?

7. A counter on a cash drawer shows 86 sales at 5 cents or 10 cents each If the total receipts were \$5 90, how many 5-cent sales were made?

8. Easter cards are sold at 15 cents and 25 cents apiece During the week before Easter 148 cards were sold and the total receipts were \$27 80 How many 25-cent cards were sold?

9. A grocer buys a box of oranges for \$4 He must discard one-fifth of them, and then sells the rest for 8 cents each How many oranges were there in the box if he made \$2 40 on the transaction?

10. One type of rug sells for \$7 50 per square yard and another sells for \$9 per square yard A man purchases two rugs totaling in area 42 square yards and costing him \$342 How many square yards in each rug?

Problems in per cent

In the arithmetic section of this book problems in per cent are included Since problems dealing with per cent are the most prevalent type found in business, and since they lend themselves admirably to algebraic representation, the principles of per cent are reviewed here as an application of algebra

The fundamental relationship in per cent problems is stated algebraically as $BR = P$ That is, B, the base, times R, the rate, equals P, the percentage This formula can readily be solved for R, since $R = \frac{P}{B}$, and for B, which is equal to $\frac{P}{R}$ In the earlier discussion, two other concepts were introduced A, the amount, was defined as the sum of the B and P, that is, $A = B + P$, and D, the difference, which was defined as the difference of B and P, that is, $D = B - P$

It would thus follow that, if $A = B + P$

and $P = BR$

by substitution $A = B + BR$

or $A = B(1 + R)$

or $B = \frac{A}{1 + R}$

Similarly it can be seen that, if	$D = B - P$
and	$P = BR$
by substitution	$D = B - BR$
or	$D = B(1 - R)$
or	$B = \dfrac{D}{1 - R}$

Such algebraic representations are more concise than the arithmetical, but the same answers are found no matter which method is used. It is best to know both, for it has been observed that one of the most serious shortcomings of students in schools of business is an inability to solve problems in percentage rapidly.

Illustrations:

a. What is 18% of \$420? What is the amount?

In this problem the base and the rate are given, and the percentage is the unknown. Thus the formulas are $P = BR$ and $A = B + P$. Since $B = \$420$ and $R = 18\%$, $P = \$420 \times 18\% = \$75.60$ and $A = \$420 + \$75.60 = \$495.60$.

b. \$360 is 25% more than how much?

Here the amount and the rate are given and the problem is to find the base. The formula would be $B = \dfrac{A}{1 + R}$. Since $A = \$360$ and $R = 25\%$, then $B = \dfrac{\$360}{1 + 25\%} = \dfrac{\$360}{1.25} = \$288$.

c. An article costs \$60 and sells for \$75. What is the per cent markup on *cost*?

Here the base and the amount are given. The formulas needed are $P = A - B$ and $R = \dfrac{P}{B}$. Since $A = \$75$ and $B = \$60$, then $P = \$75 - \$60 = \$15$ and $R = \dfrac{15}{60} = 25\%$.

d. An article costs \$60 and sells for \$75. What is the per cent markup on *selling price*?

Here the difference and the base are given. The formulas needed are $P = B - D$ and $R = \dfrac{P}{B}$. Since $B = \$75$ and $D = \$60$, then $P = \$75 - \$60 = \$15$ and $R = \dfrac{15}{75} = 20\%$.

EXERCISE 7.10

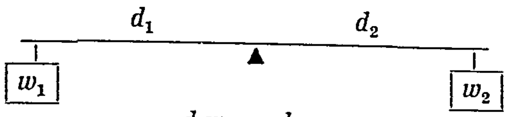
Show what is known, what is unknown, then write the formulas to be used, and solve each of the following problems

1. What is $12\frac{1}{2}\%$ of $1,200, and what is the amount?
2. $275 is 20% more than how much?
3. $320 is 20% less than how much?
4. An article costs $80 and sells for $125 What is the per cent markup on *cost*? What is the per cent markup on *selling price*?
5. What is the base if the amount is $600 and the rate is $12\frac{1}{2}\%$?
6. What is the base if the difference is $800 and the rate is $16\frac{2}{3}\%$?
7. What is the base if the percentage is $300 and the rate is $37\frac{1}{2}\%$?
8. What is the percentage if the amount is $8,100 and the rate is $12\frac{1}{2}\%$?
9. What is the percentage if the difference is $9,600 and the rate is $16\frac{2}{3}\%$?
10. What is the difference if the base is $480 and the rate is $4\frac{1}{2}\%$?
11. A merchant sold goods for $3,710 and made a gain of $32\frac{1}{2}\%$ of the cost What was the cost of the goods?
12. A baseball team with a standing of .725 has won 87 games How many games have been played?
13. In the community chest drive $117,725 was raised This was 97% of the quota Find the quota
14. The cost accountant found that for the past 12 months Department A has averaged 90,120 castings a month, with a monthly average of $6\frac{3}{4}\%$ defective included in the total Assuming that a similar per cent will be defective, what is the smallest number that should be made to assure 500 perfect castings for a job?
15. In a particular industry past records indicate that $3\frac{1}{2}\%$ of all charge sales are uncollectible If 50% of all sales are charge sales, what should be the anticipated loss on bad debts if sales last year were $278,112?

Lever problems

In the financial aspects of business it is unlikely that you will have any problems dealing directly with levers and their uses Yet there are a multitude of problems in finance which use the basic principle of the lever problem

If a rod or bar, called a *lever*, is balanced on a support, called the *fulcrum*, the length of the rod d_1 (read d sub one) on the left of the fulcrum, times the weight w_1 on the left must equal the product found by multiplying the length of the rod d_2 (read d sub two) on the right of the fulcrum, times the weight w_2 on the right



$$d_1w_1 = d_2w_2$$

It is readily apparent that if $d_1 = d_2$, then w_1 must equal w_2 if the lever is to balance. Once in balance, with $d_1 = d_2$, the balance will be maintained as long as equal weights are added or subtracted simultaneously from both sides. On the other hand, problems arise from the fact that d_1 and d_2 are not equal, shown in the algebraic sign language as $d_1 \neq d_2$.

Illustration: A 25-pound weight 6 feet from a fulcrum is balanced by a weight $7\frac{1}{2}$ feet from the fulcrum. How heavy is the weight?

Given the formula $d_1w_1 = d_2w_2$, it can be seen that $w_2 = \frac{d_1w_1}{d_2}$. Since $d_1 = 6$ feet, $w_1 = 25$ pounds, and $d_2 = 7\frac{1}{2}$ feet, $w_2 = \frac{6 \text{ feet} \times 25 \text{ pounds}}{7\frac{1}{2} \text{ feet}}$ $= 20$ pounds. That is, a weight of 20 pounds $7\frac{1}{2}$ feet from the fulcrum will balance a weight of 25 pounds 6 feet from the fulcrum.

The same principle involved in lever problems is found in what are sometimes referred to as work problems. If for example, 5 men complete a job in 4 days, there are in effect 20 man-days of work. If 10 men worked on the job, $10x = 20$, or $x = 2$. That is, it would require only 2 days. If only 2 men worked it would require 10 days. Problems in which the number of items or units remain unchanged are similar to lever problems.

Illustration: One machine can do a piece of work in 12 days. How many machines should be used to complete the job in 4 days?

There are in effect 12 machine-days involved in the job. If x machines are to do the job in 4 days, then $4x = 12$ or $x = 3$ machines. That is, 3 machines will be needed to complete the job in 4 days.

EXERCISE 7.11

Solve the following problems.

1. A weight of 2,000 pounds, 6 inches from a fulcrum, is balanced by what weight 2 feet from the fulcrum?

2. What weight $\frac{1}{4}$ inch from a fulcrum will balance a $2\frac{1}{2}$-pound weight 16 inches from the fulcrum?

3. If 30 men finish 1,500 units of a product in 10 days, how long should it take 40 men of the same efficiency to produce the same number of units?

4. If 30 men produce 1,800 units in 10 days, how many men should be employed to produce the same number of units of the same product in 6 days?

5. Eighteen men can do a job in 5 days. How long would it take 2 men?

6. It took 6 markers 5 days to put the selling price tags on a shipment of goods. How many markers would be needed to do the job in 2 days?

7. Two rectangles have the same area. One is 24 inches long and the other is 32 inches long. The longer rectangle is 2 inches narrower than the shorter rectangle. What are their widths?

8. A square and a rectangle have the same areas. The length of the rectangle is 8 inches longer than a side of the square, and its width is 6 inches shorter than a side of the square. What are the dimensions of each?

9. Two trains travel the same distance. One ran at the rate of 42 miles per hour, and took 8 hours for the trip. What is the speed of the other if it took 10 hours 30 minutes to make the trip?

10. Two boats make the same run. The faster boat travels at 30 knots and makes the trip in 9 hours. How much time is required by the slower boat if it travels at 24 knots?

Proportion

The equality of two ratios is called a *proportion*. A common example is a fraction (or ratio) reduced to its lowest terms.

$$\frac{12}{27} = \frac{4}{9}, \quad \frac{15}{24} = \frac{5}{8}$$

The 12, the 27, the 4, and the 9 are called the *terms* of the proportion.

A proportion, such as $\frac{12}{27} = \frac{4}{9}$, may be written as 12 : 27 :: 4 : 9, and is read, 12 is to 27 as 4 is to 9. In recent years this form of stating a proportion has not been much used since the principle of ratios is more easily understood and applied when the relationship is shown as one fraction to another.

Solution of proportion problems

The relationships among the terms of a proportion are important. If these relationships are understood and if three terms are known, the

fourth can be found readily. This fact is of fundamental importance in accounting and business as well as in engineering and surveying.

If it is known that 5 is to 8 as some unknown quantity is to 24, the unknown quantity can be found. If the letter x is used to represent the unknown, the proportion can be written $\frac{5}{8} = \frac{x}{24}$. Since the product of the means equals the product of the extremes in every proportion, $8 \times x = 5 \times 24$. Carrying out the multiplication, the result is

$$8x = 120; \qquad x = 15$$

Therefore

$$\frac{5}{8} = \frac{15}{24}$$

Thus if it known that 5 gallons of gasoline will run a motor for 8 hours, it is not difficult to determine that 15 gallons are necessary to run it for 24 hours, or 3 times as long.

Illustration: Find the missing term in the proportion $\frac{3.5}{8} = \frac{x}{24}$. Or, to state the problem in words, if it takes $3\frac{1}{2}$ minutes to fly 8 miles, how long should it take to fly a distance of 24 miles?

The product of the extremes (3.5×24) is equal to the product of the means ($x \times 8$). So

$$8x = 3.5 \times 24 = 84.0; \qquad x = 10.5$$

Thus if it takes $3\frac{1}{2}$ minutes to fly 8 miles, it will take $10\frac{1}{2}$ minutes to fly 24 miles.

In the solution of a problem in proportion, cancellation is often possible. In any problem of the type

$$\frac{a}{b} = \frac{x}{c}$$

cancellation can take place between a and b, and between b and c. That is, cancellation can take place either horizontally or vertically. Using cancellation, the preceding problem can be solved as follows:

$$\frac{3.5}{\cancel{8}} = \frac{x}{\cancel{24}} \quad \text{or} \quad \frac{3.5}{1} = \frac{x}{3}, \quad \text{so} \quad x = 3.5 \times 3 = 10.5$$

Thus, no matter what method is used, the solution shows that 3.5 is to 8 as 10.5 is to 24.

EXERCISE 7.12

Find the missing term in each proportion

1. $\frac{25}{15} = \frac{x}{9}$
2. $\frac{x}{12} = \frac{6}{9}$
3. $\frac{7}{8} = \frac{3\frac{1}{2}}{x}$
4. $\frac{120}{144} = \frac{x}{12}$
5. $\frac{35}{105} = \frac{x}{75}$
6. $\frac{2}{3} = \frac{x}{4}$
7. $\frac{5}{19} = \frac{4}{x}$
8. $\frac{25}{4.8} = \frac{60}{x}$
9. $\frac{18.75}{60} = \frac{x}{12}$
10. $\frac{4}{13} = \frac{200}{x}$
11. $\frac{3}{7} = \frac{x}{84}$
12. $\frac{5}{9} = \frac{x}{144}$
13. $\frac{9}{8} = \frac{x}{100}$
14. $\frac{3}{16} = \frac{x}{100}$
15. $\frac{7}{24} = \frac{x}{60}$
16. $\frac{3}{4} = \frac{x}{500}$
17. $\frac{11}{40} = \frac{x}{200}$
18. $\frac{15}{24} = \frac{50}{x}$
19. $\frac{20\%}{30\%} = \frac{x}{400}$
20. $\frac{0.075\%}{6\%} = \frac{x}{600}$

Applications of proportion

It is frequently necessary to find one or more numbers which bear the same relation to one another as the numbers in a known ratio bear to one another For example, cost accountants have many problems which take the form of a proportion Many of these problems can be solved by arithmetic, but the solution is usually much shorter if it can be worked as a proportion

Illustrations

a A plant has been operating at 60% of capacity In Department A, direct labor costs have been \$2,416.20 If the activity in Department A is increased and the direct labor cost increases proportionately, what will direct labor costs be when the factory is operated at 80% of capacity?

The present labor costs are to the new labor costs, x, as the present rate of operation is to the new rate of operation

$$\frac{\$2{,}416.20}{x} = \frac{60\%}{80\%}, \quad \text{or} \quad x = \$3{,}221.60$$

EXERCISE 7.13

Solve the following problems.

1. An article which costs the retailer 65 cents is priced to sell at 90 cents. If a proportionate increase is made in the price of an article costing $1.30, find the selling price.

2. An article which costs the retailer $4.50 is priced to sell at $7.50. If a proportionate increase is made in the price of an article costing $17.40, find the selling price.

3. The commission paid on the purchase of 300 shares of stock was $35. What is the proportionate commission on 800 shares?

4. The commission paid on the purchase of 900 shares of stock was $54. What is the proportionate commission on 1,600 shares?

5. If 1,680 square feet of floor space is required for 8 machines, how much more floor space should be added for 6 additional machines?

6. When a store moved to a new location the floor space of one department was increased from 1,800 square feet to 2,100 square feet. If the average daily sales in the old store were $850, what should be the proportionate figure in the new store?

7. If 12 men can produce 160 units in 1 hour, how many units can 25 men produce in 1 hour?

8. Nine men can dig 24 cubic yards of dirt out of a future basement in one day. How many men are needed to complete the operation in 2 days if 80 cubic yards are to be removed?

9. A map 3 feet high by 5 feet wide is to be enlarged until it is 8 feet wide. How high will it be?

10. A 3 by 5 picture has the same shape as a 14 by what picture?

11. If 8 pounds of raw material are needed to make 15 units, how much will be necessary to make 225 units?

12. Divide $35 between two men so that their shares shall be in the ratio of 3 to 4.

13. Divide $3,800 between two partners so that their shares shall be in the ratio of 7 to 12.

14. Last year's budget anticipated operation at 75% of capacity and provided for direct labor expenses of $24,000. The budget for next year provides for operation at 90% of capacity. If the basic wage rates have increased 10% above last year, what estimate should be made for direct labor costs?

15. The cost accountant found that for the past 12 months Department A has averaged 21,000 castings a month, with a monthly average of 1,000 defective castings included in the total. Assuming that a pro-

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portionate number of castings will be defective, what is the smallest number that should be made to assure 500 perfect castings for a job?

Continued proportion

When two or more persons engage in a business operation together, they often do not go through the formal procedure of forming a corporation, but instead operate as a partnership. They usually sign a partnership agreement in which they make some provision for the division of profits, and it is up to an accountant to divide the profits in the way indicated in the agreement. A common method of stating the division of profits is in the form of a certain fixed or determinable ratio. Since the relationship may be among more than two members, the method of distribution may not be stated in the form of an ordinary proportion or ratio but in the form of what might be termed a *continued proportion*, such as 2 : 3 : 5. For instance, suppose that partners X, Y, and Z agree to distribute the profits on the basis of 2, 3, and 5, respectively. If the profits were $21,000, it would be distributed in 2 parts to X, 3 parts to Y, and 5 parts to Z. Obviously, there must be 10 parts, since $2 + 3 + 5 = 10$.

To divide a sum in proportion to a given series of numbers:

1. Find the sum of the numbers in the series.
2. Divide this sum into the amount.
3. Multiply each number in the series by the quotient obtained.

Illustration. If three partners, A, B, and C, are to divide profits of $14,400 in the ratio of 5, 3, and 4, how much should each receive?

Step 1	$5 + 3 + 4 = 12$	
Step 2	$14,400 ÷ 12 = $1,200	
Step 3	A's share = 5 × $1,200 =	$6,000
	B's share = 3 × 1,200 =	3,600
	C's share = 4 × 1,200 =	4,800
		$14,400

The ratio in which profits are to be divided may not be stated as an arbitrary amount but may be based on the investment of each partner. If the amount of investment over the year has remained constant, the problem of dividing the profits does not differ greatly from that in the preceding illustration.

Illustration. Four partners, A, B, C, and D, have contributed $4,000, $6,000, $7,500, and $2,500, respectively, to form a partnership. Profits and losses are to be shared in proportion to the capital contributed.

a. Find the share borne by each partner if the partnership lost $4,000.

$$\$4,000 \div \$20,000 = 0.20$$

Partner	*Contribution*		*Loss*
A	$4,000	$4,000 \times 0.20$	$800
B	6,000	$6,000 \times 0.20$	1,200
C	7,500	$7,500 \times 0.20$	1,500
D	2,500	$2,500 \times 0.20$	500
	$20,000		$4,000

b. Assume that the next year the partnership made a profit of $12,000. How would this profit be divided if the investment made by each partner had changed as follows:

Partner		*Loss*	*Net Investment*		*Share in Profits*
A	$4,000 −	$800 =	$3,200	$3,200 × 0.75 =	$2,400
B	6,000 −	1,200 =	4,800	4,800 × 0.75 =	3,600
C	7,500 −	1,500 =	6,000	6,000 × 0.75 =	4,500
D	2,500 −	500 =	2,000	2,000 × 0.75 =	1,500
			$16,000		$12,000

$$\$12,000 \div \$16,000 = 0.75$$

It should be observed that the fact that the original contributions had been reduced did not change the relationship among the partners, since the reductions were all proportionate.

Equated time

If a partnership agreement states that profits are to be divided on the basis of average investment and if the amount invested by the partners has fluctuated during the year, the first problem is to find the average investment. In calculating the average amount of the partners' investments, *equated time* is used. Since the problem is to find the proportionate contribution of each partner to the capital of the business, it may be assumed that a dollar of investment left in the business for two months is twice as effective as a dollar left in for only one month. If the changes in investment have not always occurred at the beginning or end of a monthly period, the basis of comparison can just as well be the number of dollars contributed multiplied by the number of days rather than by the number of months.

To divide profits in a partnership on the basis of average investment when changes in investment have occurred during the year, it is necessary

to multiply the original investment by the number of months it remained unchanged during the year If the amount is changed, the net amount of each partner's investment after the change is multiplied by the number of months for which it remained unchanged This product can be called *month dollars* of capital, if days are used instead of months, the product can be called *day-dollars* The ratio of the sum of month-dollars for each partner to the sum of month-dollars for all partners gives the ratio of each partner's investment to the total investment in the partnership

Illustration The profits of A and B partnership are divided in the same ratio as the partners' contribution to the average investment of partnership If profits are $5,236 50, what share should be allocated to each if they made the following changes in accounts during the course of the year?

		A		B
January 1	Balance	$5,000		$8,000
March 1	Added	2,000	Withdrew	1,000
June 1	Withdrew	1,000	No Change	
July	No Change		Added	2,000
September 1	Added	1,000	Added	1,000
December 1	Withdrew	1,500	Withdrew	1,000
		$5,500		$9 000

PARTNER A

Date	*Investment*	*Months in Business*	*Month-Dollars*
January 1	$5,000	2	$10,000
March 1	7,000	3	21,000
June 1	6,000	3	18,000
September 1	7,000	3	21,000
December 1	5,500	1	5,500
	Sum of A's month-dollar investments		$75 500

PARTNER B

Date	*Investment*	*Months in Business*	*Month-Dollars*
January 1	$8,000	2	$ 16,000
March 1	7,000	4	28,000
July 1	9,000	2	18,000
September 1	10,000	3	30,000
December 1	9,000	1	9,000
	Sum of B's month-dollar investments		$101,000
	Sum of A's month-dollar investments		75,500
	Total month-dollar investments		$176,500

A's share of the profits would be

$$\frac{\$75{,}500}{\$176{,}500} \times \$5{,}236.50 = \$2{,}239.98$$

B's share of the profits would be

$$\frac{\$101{,}000}{\$176{,}500} \times \$5{,}236.50 = \$2{,}996.52$$

Cost accountants, too, often make use of continued proportion in distributing expenses among various departments. For example, heating or lighting expenses might be distributed among departments on the basis of floor space.

Illustration: The cost of lighting a department store for the month of December was \$1,872. The accounting department wants to distribute this on the basis of floor space. The areas of the various departments are

Department	*Area in Square Feet*
A	4,500
B	2,000
C	3,300
D	2,200
	12,000 square feet

Each department is charged for a part of the expense for lighting based on the ratio of its area to the total area.

$$\text{Department A, } \frac{4{,}500}{12{,}000} \times \$1{,}872 = \$702.00$$

$$\text{Department B, } \frac{2{,}000}{12{,}000} \times \$1{,}872 = \$312.00$$

$$\text{Department C, } \frac{3{,}300}{12{,}000} \times \$1{,}872 = \$514.80$$

$$\text{Department D, } \frac{2{,}200}{12{,}000} \times \$1{,}872 = \$343.20$$

The total expense, \$1,872, could be divided by 12,000 and the quotient multiplied by the floor space in each department. This would give the proportionate charge.

EXERCISE 7.14

Solve the following problems.

1. Four partners divide profits of \$27,850 in the ratio of 1, 2, 3, and 4. How much does each receive?

2 Three partners divide profits of $68,250 in the ratio of 3, 4, and 6 How much does each receive?

3 Three partners divide profits of $106,224 40 in the ratio of 2, 3, and 3 How much does each receive?

4 A cost accountant decides that the heating expense of an office should be apportioned among the departments on the basis of floor space If the total heating expense is $1,200, find the amount which should be borne by each of the following departments

Department	*Area in Square Feet*
A	600
B	1,000
C	1,800
D	2,400
E	1,600

5. In determining a budget for a department store, the rent factor (taxes, insurance, depreciation, etc) is estimated at $240 000 to be distributed among the departments in proportion to the floor space occupied, as follows

Department	*Area in Square Feet*
A	5,000
B	800
C	4,200
D	12,000
E	8,000

Find the amount allocated to each

6. Six partners, A, B, C, D, E, and F, agree to share profits and losses in the ratio of 1 2 3 3 5 6 If profits for one year total $24,000, how much should each receive?

7. The two partners in the Wilson and White Company contributed $20,000 and $25,000, respectively, to the partnership If they agree to share profits in proportion to their capital contributions and if profits amount to $25 875, how much does each receive?

8 A, B, and C as partners contribute $4,000, $6,000, and $5 000 respectively, to form a partnership in which they agree to share profits in proportion to their capital contributions If profits amount to $37,500, how much does each receive?

9. In the A and B partnership, profits are shared in the ratio of average investment. During the year the following changes were made in the capital account:

		A	B
Balance January 1		$6,000	$4,000
February 1	Added	None	1,000
March 1	Withdrew	500	500
July 1	Added	1,000	800
November 1	Withdrew	500	None
December 1	Withdrew	1,000	500

Profits for the year were $7,400. How much should be given to each partner?

10. A and B enter a partnership agreement to distribute profits on the ratio of average investment. A invested $15,000, but three months later withdrew $5,000. B invested $3,000 originally and added $1,000 at the beginning of each of the next 11 months. At the end of the year profits of $11,492 are to be distributed. How much should each receive?

REVIEW PROBLEMS

Chapters 6 and 7

Carry out the indicated operations

1. $(+7)+(+3)+(-5)-(+8)-(-4)+(+12)$
2. $(-18)-(-7)-(+6)+(+21)+(-34)-(-14)$
3. $18-23-26-(-45)+67-31-(+19)-12$
4. $(-6b)-(-4b)-(-8b)+(-12b)-(+7b)$
5. $-7wr+9wr-(-12wr)-15wr+8wr-3wr$
6. $(+1)(-6)(-1)(+4)(-7)(-1)(-1)$
7. $\dfrac{(-6)(-9)}{+4}$
8. $\dfrac{(+32)(+12)}{(-24)(-8)}$
9. $\dfrac{(-6)(+8)(-4)}{(+12)(-1)}$

Find the values of the following if $x=2$, $y=-1$, $z=3$, $w=0$

10. $3x-4y+8z-2w$
11. $7x^2-2y^3+\dfrac{z}{3}$
12. $\dfrac{4x+3y}{z-2w}$
13. $\dfrac{4y^2-z}{7x-2z}$
14. $\dfrac{x(2z-y)}{3z-4w}$
15. $x^2-3y^2+\dfrac{w^2}{8}+\dfrac{4x-y}{z}$

Find the value of the unknown, and check the solution

16. $2x+9-5x=21$
17. $8x-13=5x+2$
18. $2x+7=6x-1$
19. $36-5x=4x$
20. $6x-2+x=-3x+8$
21. $-3x+8=x+6$
22. $2x+5=5x-2$
23. $8x-6=3x+2$
24. $4x-15=10x$
25. $(3x-7)-(x+1)=0$
26. $5x-2(x-3)=15$
27. $3-2(5-x)=5$
28. $1+3(x-3)=2(1-x)$
29. $5(7-2x)=3(2x-1)$
30. $4(2x+5)-(x-1)=0$
31. $1-2(3-x)=5$
32. $-5(x-2)=7x-2$
33. $3(6x-7)=4(5x-4)$
34. $3(x-7)+6=5(x-2)$

35. $3(2x - 7) - 5 = 4(3x - 2)$
36. $3 + (2x - 5) = x + 2$
37. $7 - (x - 2) = 3x + 1$
38. $3x - 4 - (2 - x) = 6$
39. $-2x - (7 - x) = 3x + 1$
40. $2(x - 2) = 3(x + 2)$
41. $3(x + 4) = 2(x + 6) - 2$
42. $x + (3x - 16) = 3(x - 12) + 18$
43. $5x - 3(x + 7) + x = 24$
44. $4(x - 2) - 4 = 5x + 7$
45. $7 - 3(x - 4) = x - 13 + 2(x - 2)$
46. $(2x - 3) = 6 - 8x$
47. $2.4x = 3(2x - 6)$
48. $8(x + 2) = 5(x + 5)$
49. $3(x - 1) - 2(x + 1) = 4(x + 3) + 7$
50. $5(x + 2) - 4(x - 7) = 20$
51. $8 - 3(x - 2) - 4(3 - 2x) = 7x$
52. $47 - 4(5x - 7) = 5(6x - 5)$
53. $3(x + 2) - (x + 4) = 3x - 2(x + 6)$
54. $2(x + 3) - 5(x + 6) = 3(x - 2) + 12 - 8x$
55. $3 - 2(x - 4) = 7(x + 2) + 6$
56. $7(x - 3) = 9(2 - x) + 1$
57. $3.2(x - 5) - 13 = 1.4(x + 5)$
58. $0.4(x - 3) = 1.1(6 - x) - 0.3$
59. $3.2 + 0.2x = -2(4.7x + 0.8)$
60. $3(0.9x + 0.4) = 1.2(x - 4)$
61. $\dfrac{x + 3}{2} + \dfrac{2x - 5}{5} = 1 + \dfrac{3x - 2}{4}$
62. $\dfrac{7x + 1}{4} + \dfrac{4x + 9}{6} = 2x + 3$
63. $\dfrac{3}{4} + \dfrac{3x - 2}{2x} = \dfrac{8}{x}$
64. $\dfrac{x}{4} = \dfrac{x}{9} + 2 + \dfrac{x}{12}$
65. $\dfrac{7}{6} - \dfrac{x - 4}{3} = \dfrac{x + 3}{6}$
66. $\frac{2}{3}(x - 4) = 1 + \frac{1}{3}(x + 5)$
67. $15x - \dfrac{9x}{8} - \dfrac{3}{4} = \dfrac{3x}{4} + 12\frac{3}{8}$
68. $\dfrac{x}{3} - \dfrac{x - 4}{3} = \dfrac{3}{2} - \dfrac{2x - 5}{6}$

69. $\frac{x-3}{4} = 1 - \frac{5x}{8}$

70. $\frac{x+5}{4} - \frac{17}{4} = \frac{x-3}{2}$

71. $\frac{3x+1}{10} = \frac{9}{2} - \frac{6x+7}{5}$

72. $\frac{4x+1}{3} - (x+1) = \frac{x-5}{15} + \frac{2x+5}{9}$

73. $\frac{x-1}{6} - \frac{x}{9} + \frac{x}{4} = \frac{4}{9}$

74. $5x - \frac{x}{6} = 62 - \frac{x}{3}$

75. $\frac{5x+6}{2} - \frac{x+10}{6} = \frac{x-2}{3} + 18$

Solve for the unknown in the following proportion problems

76. $\frac{x}{24} = \frac{5}{6}$

77. $\frac{x}{45} = \frac{5}{9}$

78. $\frac{27}{x} = \frac{9}{7}$

79. $\frac{4}{x} = \frac{32}{8}$

80. $\frac{27}{9} = \frac{3}{x}$

81. $\frac{x}{72} = \frac{3}{8}$

82. $\frac{60}{x} = \frac{15}{90}$

83. $\frac{x}{450} = \frac{13}{90}$

84. $\frac{6}{50} = \frac{x}{120}$

85. $\frac{128}{32} = \frac{x}{12}$

86. $\frac{350}{x} = \frac{25}{16}$

87. $\frac{125}{40} = \frac{5}{x}$

88. $\frac{35}{800} = \frac{x}{160}$

89. $\frac{x+5}{x+2} = \frac{8}{5}$

90. $\frac{3}{5} = \frac{6}{x+2}$

Factor the following

91. $3x + 6y - 9z$
92. $4x^2 - 8x^3y + x^2y$
93. $4x^2 - 25y^2$
94. $3x^2 - 12y^2$
95. $25x^2y^2 - 16z^2$
96. $x^2 + 12xy + 36y^2$
97. $4x^2 - 12xy + 9y^2$
98. $x^2 + x - 12$
99. $x^2 - 3x - 10$
100. $x^2 - 11x + 24$
101. $x^2 + 2x - 15$
102. $x^2 - 7x + 12$
103. $x^2 - 7x + 10$
104. $x^2 - 8x - 9$
105. $x^2 + 2x - 8$
106. $x^2 - 5x - 6$
107. $2x^2 - 9x + 4$
108. $2x^2 + 7x - 15$
109. $2x^2 - 3x - 9$
110. $2x^2 + 5x - 12$
111. $3x^2 + 7x - 6$
112. $12x^2 + x - 20$
113. $6x^2 - 17x + 12$
114. $3x^2 + 21x + 36$
115. $5x^2 + 10x - 40$
116. $4x^2 + 5x - 6$
117. $6x^2 - 13x + 6$
118. $4x^2 - 12x + 5$
119. $8x^2 + 10x - 3$
120. $8x^2 - 26x + 15$

121. If a stenographer with an electric typewriter can bill 100 customers in 2 hours, and a second stenographer can bill 100 customers in 3 hours, how long will it take them together to bill 1,000 customers?

122. With an electric calculator A can perform a job in $\frac{2}{3}$ the time it takes B, using a hand-operated machine. Working together they can complete a job in 24 hours. How long would it take each working alone?

123. A syllabus containing 80 pages can be reduced to 64 by adding 150 words to each page. How many words in the syllabus?

124. A manufacturer mixes two chemicals, A and B. A costs $\frac{1}{3}$ more than B. The total cost of a mixture consisting of 4 units of A to 5 units of B is $1.55. What is the price of each per unit?

125. A real estate agent is to receive 5% on the sales price of a house and an additional bonus of 10% for what he gets above $12,000. If he sells the house for $15,000, how much does the owner receive?

126. An owner signs a contract with a real estate agent, agreeing to give the agent 5% of the total selling price and an additional bonus of 10% for anything above $15,000. If the owner receives $18,500 net, what commission did the real estate agent receive?

127. A public speaker pays his agent 10% of net income after taxes. The agent's fee is deducted in calculating the federal income tax, and the tax is deductible in calculating the agent's fee. If the income tax rate is 25%, what does the agent receive if the speaker's income was $18,000 before either the tax or the agent's fee was deducted?

128. In renewing the contract between the speaker and the agent in the preceding problem, the agent asks for either 10% of net income before taxes, or $12\frac{1}{2}$% of the speaker's income after taxes. Assuming that the total income and the tax rate remain unchanged, which contract is more favorable to the speaker?

129. A 20% solution of a certain type of drug is needed. Only a 12% and a 24% solution of the drug are available. What per cent of the final mixture is the 12% solution?

130. A subdivider decides to offer 2-bedroom homes and 3-bedroom homes. He has space for 50 homes that will sell at an average price of $15,000. If the 2-bedroom homes will sell for $13,500 and the 3-bedroom homes will sell for $16,000, how many of each will be build?

131. If 8 spark plugs at 90 cents each will increase the mileage of a car from 13 to 15 miles per gallon, and if gasoline costs 28 cents per gallon, how many miles must a car be driven in order that the saving in gasoline be as much as the cost of the spark plugs?

132. In acquiring 640 acres in a new irrigation district, a buyer paid a total of \$172,000 Some of the land was bought for \$325 an acre and the remainder at \$250 an acre How much was bought at each price?

133. Austin Dixon bought some stock at \$25 a share Later he bought twice as many shares at \$20 a share He sold all the shares at \$23 and made a profit of \$400 How many shares did he buy at \$25?

134 If one stenographer can type 100 form letters in 3 hours and a second stenographer can do 100 in 4 hours, how many minutes will it take them working together to do 100 letters?

135. A hardware dealer bought some bolt cutters at \$3 20 each He marked them for sale at a price such that he could sell them for 10% less than the market price and still make 25% over cost What was the market price?

136 One machine can do a piece of work in 12 hours After a second machine is installed, the two can complete the same amount of work in 3 hours How long should it take the second machine to do the job alone?

137. One car starts from Chicago, going to Champaign, and at the same time another car starts from Champaign, going to Chicago If the cities are 132 miles apart, and the cars travel at 40 miles and 50 miles per hour, respectively, in how many minutes will they meet?

138. A tank can be filled by one pipe in 10 hours and by another in 2 hours How long will it take to fill the tank if both pipes operate?

139. One salesman sells $\frac{2}{3}$ as much as another Their combined daily sales average \$1 800 What are the average daily sales of each?

140. If it costs \$9,000 to provide underground electric service for a subdivision of 48 lots, how much should it cost for a subdivision of 36 lots?

141. A buyer of a motel receives a net return of \$4,800 on an investment of \$24,000 If he receives the same rate of return, how much should he expect on a larger unit costing \$37,500?

142. When a small fabricator of aluminium invested \$39 600 in additional equipment, his net profits were increased by \$4,250 a year If the expenditure of another \$26,400 will increase his profits at $1\frac{1}{3}$ times the rate of the previous expenditure, what increase in profit should he expect?

143. Find the two numbers whose sum is 81 which are in the ratio of 2 7

144. Find two numbers whose sum is 88 which are in the ratio of 4 7

145. The dividend received on a stock decreased 25% to \$720 What was the previous income? What per cent increase would be necessary to bring dividends back to the previous level?

146. Jim Willis bought 2 pieces of property in the desert for $600. He sold one at a 15% profit and the other at a 5% loss. His net gain was $50. Find the cost of each.

147. An investor bought stock at $40 a share. When the price of the stock declined to $24 he bought an equal number of shares. How high must the price rise before he can sell with no gain or loss?

148. An investor bought stock at $40 a share. When the price of stock declined to $24 a share, he invested an equal amount of money. How high must the price of the stock rise before he can sell with no gain or loss?

149. Assuming that the value of a used car is inversely proportional to its age, find the value of a car, when it is 5 years old, which was valued at $900 when it was 2 years old.

150. Assuming that the value of a used car is inversely proportional to its age, find the original cost of a car that was worth $650 when it was 3 years old.

151. One corner of a lot 90 feet wide is $4\frac{1}{2}$ feet higher than the other. The slope of the lot is uniform. If a house 60 feet wide is to be built, how much higher must one foundation be than the other to assure a level floor?

152. A man buys 3 acres of land for $25,000 which he expects to subdivide into 10 lots. He borrows $18,000 to pay for the land with the understanding that he will pay the principal and the interest by giving the lender $2,000 when each lot is sold. He borrows another $10,000 from the bank, which he spends on improvements, and in addition he owes a contractor $8,000 for other work on the project. He offers lots for sale at $7,000 each. From each sale he pays a withholding tax equal to 25% of the profit. How much money would he retain from the sale of 5 lots, assuming he paid the bank and the contractor as soon as possible?

153. If electronic equipment can be rented for $25,000 a year which will do the work now done by 10 clerks and require only $\frac{1}{6}$ the floor area, the amount of rent will be reduced by $1,200 per year. If fringe benefits now paid to the employees average $10 per month for each of the workers, what is the highest amount that the company can economically afford to pay each of the 10 clerks?

154. The cost of bringing a crane to a building site to pour concrete is $75. The rental of the crane is $4.25 an hour. The operator is paid $2.75 per hour, and his helper $2.00 per hour. The two men with the crane can do in one hour what it takes 12 men at $2.00 per hour to accomplish. Find the point in time and cost at which it is just as economical to use the two men and the crane as it is to use hand labor.

155. Divide \$1,000 into 3 amounts so that the first is 4 times the second, and the third is 3 times the second

156. The Grant, Sheridan, and Dewey partnership was formed with investments of \$7,000, \$8,000, and \$10,000, respectively It was agreed that on dissolution of the partnership each partner should receive a share of the net assets proportionate to his investment. At the dissolution date the partnership had net assets of \$37,500 How much should each partner receive?

157. Phelps and Phillips are partners with investments of \$40,000 and \$50,000, respectively On June 1, Phelps withdrew \$5,000, on August 1 he reinvested \$15,000 Phillips withdrew \$5,000 on March 1, and \$5,000 on September 1 How should a profit of \$16,000 be divided on the basis of their average investment for the year?

158. Allen, Backon, and Cronk as partners have contributed capital of \$10,000, \$15,000, and \$25,000, respectively They agree to share profits and losses in the ratio of their contribution Net profit for the year was \$6,250 What was each partner's share in the net profit and what was each partner's capital after he was credited with his share of the gain?

159. The Terry White Sports Store had total sales last year of \$346,400 Credit sales averaged \$12,000 a month What was the ratio between cash and credit sales?

160. A bank with deposits of \$84,000,000 has total capital funds of \$7,000,000 What is the ratio of deposits to capital funds?

161. In settling an estate of \$250,000 the executor's fees amount to \$4,330 What is the fee when stated as a per cent of the estate?

162. The cooling system of a truck holds 4 gallons of water. After 1 gallon of antifreeze has been added to 3 gallons of water, it was found that the owner wanted a 40% solution of antifreeze How much of the solution must be withdrawn and replaced by pure antifreeze to obtain a 40% solution?

163. During a 10-year period the cost of living increased 1 82 times How much income was required after the increase to provide the same standard of living that \$400 had provided earlier?

164. The cost of building 9 minimum-sized apartments of 550 square feet each was estimated by a contractor at \$8 25 a square foot The value of the lot on which the 9 units were to be placed was \$4,200. If the owner assumes that it will take $\frac{1}{2}$ of his income to pay taxes and to provide for the other necessary expenses of owning and operating the building, how much monthly rental must he charge on each of the nine units if he expects to earn 6% on the money he has invested?

165. An owner of a vacant lot builds a building to be rented as a branch bank. The annual rental is to be $\frac{1}{10}$ of 1% of deposits. How much in deposits must the branch have in order to assure an income of \$500 a month to the owner?

166. An investor buys 3 acres of land at the edge of the city. He leases it for 10 years to an amusement park catering to children, at 15% on all annual sales over \$12,000. If it is assumed that 75% of the sales will be made during 50 days of the year, and that each child attending will spend on the average of 50 cents, how many children must he anticipate will attend on each of the 50 days if he is to receive \$100 a month in rentals?

167. In the Jay Stanton Manufacturing Company 12% of the employees in the production department are women, 40% of the office employees are women, and 20% of all the employees in the two departments are women. What is the ratio of the number of employees in the production department to the number in the office?

168. An investor's last dollar of income is taxed at $37\frac{1}{2}$%. On some securities he will receive income that is not subject to the federal income tax. Find the rate of taxable income he must earn which is equivalent to nontaxable income of 4%.

Problems regarding stock rights

Stockholders in corporations are commonly given the right to subscribe to new stock issued by the corporation. To determine the value of the *right attaching to each old share*, called a New York Right (V_y), the following formula has been developed:

$$V_y = \frac{M - S}{R + 1}$$

where M = market price; S = subscription price; and R = ratio of old stock holding to new issue. Find the value of a New York Right in the following.

	M	S	R		M	S	R
169.	120	100	3	**174.**	185	160	4
170.	75	60	4	**175.**	$8\frac{3}{4}$	$7\frac{1}{2}$	4
171.	$7\frac{1}{2}$	4	$2\frac{1}{2}$	**176.**	20	10	1
172.	19	10	8	**177.**	35	32	5
173.	160	140	8	**178.**	225	100	20

The value of a right to subscribe to *one new share of stock* is called a Philadelphia Right to differentiate from a New York Right. The value of

a Philadelphia Right (V_p) is equal to $\frac{P \times R}{R+1}$ P is equal to $M - S$ Find the value of a Philadelphia Right in the following

	M	S	R		M	S	R
179.	120	100	4	**184.**	140	130	6
180.	72	60	3	**185.**	225	100	20
181.	185	150	8	**186.**	20	10	1
182.	19	10	5	**187.**	80	75	1
183.	36	30	3	**188.**	$8\frac{3}{4}$	$7\frac{1}{2}$	4

Problems regarding preferred stock

To test the investment position of preferred stock in a public utility, some security analysts apply the following formulas

$$\frac{\frac{2}{3}N - Br}{i} = x_1$$

$$75\% \times T - (C + B) = x_2$$

$$\frac{N - 15\% \times G - Br}{i} = x_3$$

Find x_1, x_2, and x_3, given

189. $T = \$160{,}000{,}000$, $B = \$75{,}000{,}000$, $C = \$25{,}000{,}000$, $G = \$25{,}000{,}000$, $N = \$8{,}000{,}000$, $i = 6\%$, $r = 4\%$

190. $T = \$36{,}000{,}000$, $B = \$18{,}000{,}000$, $C = \$2{,}000{,}000$, $G = \$8{,}000{,}000$, $N = \$3{,}200{,}000$, $i = 5\%$, $r = 4\%$

191. $T = \$110{,}000{,}000$, $B = \$50{,}000{,}000$, $C = \$8{,}000{,}000$, $G = \$18{,}000{,}000$, $N = \$4{,}600{,}000$, $i = 6\%$, $r = 3\frac{1}{2}\%$

8

Linear Systems and Quadratic Equations

Introduction

Frequently in problems in business there are two unknowns, the value of one being dependent on the value of the other. In determining taxes, for example, the tax paid to the state may be deductible in determining the federal tax, while the amount paid in federal taxes may be deductible in determining the tax paid to the state. It appears, therefore, that one cannot be determined without knowing the other.

Simultaneous equations

Solutions of problems involving two or more unknowns traditionally are a part of the basic course in algebra. If there is only one equation with two unknowns, the value of one unknown can be stated only in terms of the other. Thus in the equation $3x + 2y = 12$, for every value of x there is a corresponding value of y and vice versa. Some of these may be paired off as follows:

For x:	-1	0	1	2	3	4	5
For y:	$7\frac{1}{2}$	6	$4\frac{1}{2}$	3	$1\frac{1}{2}$	0	$-1\frac{1}{2}$

When two equations can be developed which represent the conditions included in the problem, each equation serves to regulate the relationships of the unknowns in the other. Two or more linear equations satisfied by the same set of values of the unknown quantities are called *simultaneous linear equations* because they must be considered at the same time (simultaneously) in order to get a solution. For example, if the equation $3x + 2y = 12$ is paired with the equation $2x - 5y = -11$, there is only one set of values which will satisfy both. The integer values between -1 and 5 for x which give corresponding values for y in the first equation —determined earlier—give the following corresponding values for the second equation.

For x:	-1	0	1	2	3	4	5
For y:	$1\frac{4}{5}$	$2\frac{1}{5}$	$2\frac{3}{5}$	3	$3\frac{2}{5}$	$3\frac{4}{5}$	$4\frac{1}{5}$

a Philadelphia Right (V_p) is equal to $\frac{P \times R}{R+1}$ P is equal to $M - S$ Find the value of a Philadelphia Right in the following

	M	S	R		M	S	R
179.	120	100	4	**184.**	140	130	6
180.	72	60	3	**185.**	225	100	20
181.	185	150	8	**186.**	20	10	1
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$$\frac{N - 15\% \times G - Br}{i} = x_3$$

Find x_1, x_2, and x_3, given

189. $T = \$160{,}000{,}000$, $B = \$75{,}000{,}000$, $C = \$25{,}000{,}000$, $G = \$25{,}000{,}000$, $N = \$8{,}000{,}000$, $i = 6\%$, $r = 4\%$

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8

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When two equations can be developed which represent the conditions included in the problem, each equation serves to regulate the relationships of the unknowns in the other. Two or more linear equations satisfied by the same set of values of the unknown quantities are called *simultaneous linear equations* because they must be considered at the same time (simultaneously) in order to get a solution. For example, if the equation $3x + 2y = 12$ is paired with the equation $2x - 5y = -11$, there is only one set of values which will satisfy both. The integer values between -1 and 5 for x which give corresponding values for y in the first equation —determined earlier—give the following corresponding values for the second equation.

For x:	-1	0	1	2	3	4	5
For y:	$1\frac{4}{5}$	$2\frac{1}{5}$	$2\frac{3}{5}$	3	$3\frac{2}{5}$	$3\frac{4}{5}$	$4\frac{1}{5}$

These series of possible values for each equation show that the only pair of values which satisfies both equations is $x = 2$ and $y = 3$ This common pair of values ($x - 2$, $y = 3$) is called the common solution of the two equations since it satisfies both equations

Solution of simultaneous linear equations

One method of solving a system of linear equations is to eliminate one of the unknowns and to find the value of the remaining unknown Once this value is found, it can be substituted in either of the original equations and the value of the other unknown determined

If the numerical coefficients of one unknown are the same and have like signs that unknown can be eliminated by subtracting one equation from the other If the numerical coefficients of the unknowns are the same but with unlike signs, that unknown can be eliminated by adding the two equations

In solving such equations, decide first which unknown is to be eliminated if the numerical coefficients of the unknown to be eliminated are not the same Then multiply the first equation by the coefficient of the unknown to be eliminated in the second equation, and multiply the second equation by the coefficient of the unknown to be eliminated in the first

Illustrations

a Solve the following pair of equations

$$3x + 2y = 12$$
$$2x - 5y = -11$$

To eliminate y, use the coefficients 2 and 5 as multipliers

$$3x + 2y = 12 \text{ (multiply by 5)} = 15x + 10y = 60$$
$$2x - 5y = -11 \text{ (multiply by 2)} = \underline{4x - 10y = -22}$$

Adding the new equation, $19x = 38$

$x = 2$

Substituting the value of x in the first equation,

$$6 + 2y = 12, \quad 2y = 6, \quad y = 3$$

b Solve the following set of equations

$$2x - y = 3$$
$$3x - 2y = 2$$

To eliminate x, use the coefficients 2 and 3 as multipliers

$$2x - y = 3 \text{ (multiply by} > 3) = 6x - 3y = 9$$
$$3x - 2y = 2 \text{ (multiply by} > 2) = \underline{6x - 4y = 4}$$

Subtracting, $y = 5$

Substituting the value of y in the first equation,

$$2x - 5 = 3; \quad 2x = 8; \quad x = 4$$

When the coefficients are larger numbers—as they often are in business problems—the work can be simplified by reducing the size of the multiplier by removing the common factors.

Illustration: Solve the following set of equations.

$$\begin{aligned} 23x - 32y &= 24 \\ 13x - 24y &= -16 \end{aligned}$$

To eliminate y, the multipliers 24 and 32 could be used. When 8 as a common factor is used as a divisor, these are reduced to 3 and 4, respectively.

$$\begin{aligned} 69x - 96y &= 72 \\ 52x - 96y &= -64 \end{aligned}$$

Subtracting: $17x = 136$; and $x = 8$

Substituting in the second equation: $104 - 24y = -16$

$$-24y = -120; \quad y = 5$$

EXERCISE 8.1

Solve the following pairs of equations and check your answers.

1. $x + y = 9$
   $x - y = 5$
2. $3x - 2y = 14$
   $4x + y = 4$
3. $2x + 3y = 12$
   $5x - 3y = 9$
4. $3x + 2y = 8$
   $2x - 5y = 18$
5. $2x - 3y = 2$
   $3x + 2y = 16$
6. $2x - y = 6$
   $x - 2y = -3$
7. $3x - y = -2$
   $x + 3y = 6$
8. $5x - 2y = 3$
   $x + y = 2$
9. $2x + 3y = 7$
   $4x - 6y = 5$
10. $4x - 7y = 1$
    $9x - 10y = 28$

A system of linear equations can be solved also by first solving for one unknown in terms of the other unknown in one equation; and second, substituting the value found for the same unknown in the other equation. This gives rise to an equation in one unknown, which can then be readily solved.

Illustration: Solve

$$\begin{aligned} 3x + 2y &= 12 \\ 2x - 5y &= -11 \end{aligned}$$

Since $3x + 2y = 12$ then $y = \frac{12 - 3x}{2}$ This equation is called the *equation of substitution*, since one letter has been solved for in terms of the other letter Substitute the value of y determined from the first equation for the value of y in the second equation

$$2x - 5\left(\frac{12 - 3x}{2}\right) = -11$$

or

$$4x - 60 + 15x = -22$$
$$19x = 38, \quad x = 2$$

Substitute the value of x in one of the original equations and solve for y Or, substitute 2 for x in the equation of substitution The corresponding value of y is $\frac{12 - 6}{2} = 3$ If the second method is used to find y, it is necessary to check both original equations

$$6 + 6 = 12 \quad \text{and} \quad 4 - 15 = -11$$
$$12 = 12 \qquad\qquad -11 = -11$$

EXERCISE 82

Solve the following pairs of equations by substitution Check your answers

1. $x + 2y = 5$
   $2x - y = 5$
2. $2x + 7y = 1$
   $5x + 8y = 12$
3. $2x + y = 0$
   $3x - 4y = 11$
4. $2x - 7y = 8$
   $3x - 8y = 7$
5. $x + 4y = 10$
   $2x - 3y = -13$
6. $8x - 6y = -1$
   $6x + 4y = -5$
7. $5x + 3y = 17$
   $x - 5y = -5$
8. $3x - 4y = 19$
   $4x - 3y = 16$
9. $3x - 4y = 7$
   $x + 6y = 6$
10. $9x - 3y = 12$
    $2x + y = -1$

Stated problems in more than one unknown

Time may be saved and work simplified by using simultaneous equations for problems which can be solved with only one unknown Simultaneous equations can be used often in business problems Frequently their use is avoided simply because the person who could use them advantageously lacks sufficient confidence in his own ability, or is able to accomplish the same results by a long system of trial and error

The types of problems traditionally included in algebra texts provide excellent training in setting up the necessary equations, and in developing a sense of thinking in terms of conditional equations. Many of the problems have limited practical application. They should be viewed, however, as training devices for stating relationships in algebraic terms. Such problems include number problems, rate and distance problems, problems of areas, motion problems, and so-called investment problems.

The following sequence may be used advantageously in solving such problems.

1. Read the problem carefully to see what is given, and what is to be found.

2. Find the independent conditions which will furnish equations.

3. Let one letter, such as x, represent one unknown, and establish that value for each equation.

4. Let another letter, such as y, represent the other unknown, and establish that value for each equation.

5. Set up the equations and solve them.

6. Check the results by testing whether they satisfy the conditions of the original problem.

Number problems

Statements expressing the relationships between numbers give rise to the so-called number problems. Although presented in many variations, such problems usually involve the relationship of the sum and difference of two numbers, or the magnitude of one relative to the magnitude of the other, before and after the additions, subtractions, multiplications, or divisions of the two have taken place.

Illustrations:

a. The sum of two numbers is 86, their difference is 28. Find the numbers.

Let $x =$ the larger number and $y =$ the smaller number. Then

$$\begin{array}{ll} x + y = 86 & \\ x - y = 28 & \\ \hline 2x = 114; & x = 57 \\ 57 + y = 86; & y = 29 \end{array}$$

Check: Since the sum of 57 and 29 is 86 and the difference of 57 and 29 is 28, the problem checks.

b Four tımes the larger number ıncreased by 3 tımes the smaller number ıs equal to 57, and, 3 tımes the larger number decreased by twıce the smaller number ıs equal to 30 What are the numbers?

Let $x =$ the larger number and $y =$ the smaller number Then

$$\begin{array}{lcl} 4x + 3y = 57 & \text{or} & 8x + 6y = 114 \\ 3x - 2y = 30 & & 9x - 6y = 90 \\ \hline & & 17x \qquad = 204, \quad x = 12 \\ & & 48 + 3y = 57 \\ & & \qquad 3y = 9, \quad y = 3 \end{array}$$

Check Since 4 tımes 12 plus 3 tımes 3 ıs equal to 57 $(48 + 9 = 57)$, and 3 tımes 12 less twıce 3 ıs equal to 30 $(36 - 6 = 30)$, the problem checks

EXERCISE 83

Solve the followıng

1. The sum of two numbers ıs 44 and theır dıfference ıs 4 Fınd the numbers

2. Three tımes the larger number decreased by 4 tımes the smaller number ıs equal to 5 Fıve tımes the smaller number decreased by 3 tımes the larger number ıs equal to 5 Fınd the numbers

3. Half the sum of two numbers ıs 32 Twıce the dıfference of the two numbers ıs 32 What are the numbers?

4. Three tımes the larger number ıncreased by half of the smaller number ıs 13, and half of the larger number ıncreased by 3 tımes the smaller number ıs 8 What are the numbers?

5. Twıce the larger number less 5 tımes the smaller number ıs 6 Twıce the larger number ıncreased by 5 tımes the smaller number ıs -4 Fınd the numbers

6. The sum of two numbers ıs 5 Twıce the larger number equals 3 tımes the smaller number Fınd the numbers

7. If twıce one number ıs taken from 3 more than twıce another number, the dıfference ıs 9 If the fırst number ıs added to 4 less than 3 tımes the second number, the sum ıs 9 What are the numbers?

8. The sum of two numbers ıs 143 The larger number ıs 27 more than the smaller number What are the numbers?

9 The dıfference of two numbers ıs 3 Fıve tenths of the larger number less $\frac{4}{10}$ the smaller number ıs 2 Fınd the numbers?

10 One hundred twenty-five tımes the larger number decreased by 63 tımes the smaller number ıs 16, and 50 tımes the larger number decreased by 21 tımes the smaller number ıs 7 Fınd the numbers

Time, rate, and distance problems

Interesting problems sometimes arise because of movements of ships or planes, aided in one direction by the currents of water or air, and hindered to an equal degree in movements against the current. In navigation such problems have some practical significance, and are invariably found in examinations for the selection of personnel in the armed forces. Sometimes the problems are presented in such a way that only one unknown need be used, although some problems can be solved to better advantage by the use of simultaneous equations.

Illustrations:

a. A ship required 6 hours for a trip of 48 miles against the current. It took 2 hours to travel 32 miles of the return journey with the current. Find the rate of the current and the speed of the ship.

Let x = rate of the ship in still water and y = rate of the current. Then

	Rate of ship	*Time of Travel*	*Distance*
Against current	$x - y$	6	48
With current	$x + y$	2	32

Thus, since rate times time equals distance,

$$\begin{aligned} 6(x-y) &= 48 \\ 2(x+y) &= 32 \end{aligned} \quad \text{or} \quad \begin{aligned} 6x - 6y &= 48 \\ 2x + 2y &= 32 \end{aligned} \quad \text{or} \quad \begin{aligned} 6x - 6y &= 48 \\ 6x + 6y &= 96 \\ \hline 12x \qquad &= 144; \quad x = 12 \\ 24 + 2y &= 32 \\ 2y &= 8; \quad y = 4 \end{aligned}$$

Therefore the rate of the ship in still water is 12 miles per hour, and the rate of the current is 4 miles per hour.

Check: Since 6 times the rate of the ship against the current, namely 6 (12 − 4), equals 48; and 2 times the rate of the ship with the current, namely 2 (12 + 4), equals 32, the problem checks.

b. Two automobiles leave at the same time from two cities 420 miles apart and travel toward each other. If one travels at an average rate which is 5 miles per hour greater than the average rate of the other, what is the rate of each if they pass each other 6 hours after starting?

Let x = the rate of the slower automobile and y = the rate of the faster automobile. Then, since y is 5 greater than x,

$$y = x + 5$$

And, since the distance traveled by the first automobile plus the distance

traveled by the second automobile is equal to the total distance between the cities,

$$6x + 6y = 420 \quad \text{or} \quad x + y = 70$$

Substitute the value of y in the first equation for the value of y in the second equation

$$x + x + 5 = 70$$
$$2x = 65$$
$$x = 32\tfrac{1}{2} \text{ miles per hour}$$
$$y = 37\tfrac{1}{2} \text{ miles per hour}$$

Check 195 miles + 225 miles = 420 miles

EXERCISE 8.4

Solve the following

1. An airplane flies between two cities 1,260 miles apart If it took 7 hours against the wind and 5 hours with the wind, what was the speed of the airplane in still air and what was the velocity of the prevailing wind?

2. A man can walk 60 miles in 17 hours If he walks at the rate of 3 miles an hour uphill and 4 miles an hour downhill, how much of the distance is uphill, and how much is downhill?

3. A ship required 7 hours for a trip of 91 miles against the current If it takes the ship 3 hours for a trip of 69 miles with the current, find the current and the speed of the ship in still water

4. A college student finds that by driving at 30 miles per hour he is 6 minutes late for his first class, by driving at 45 miles per hour he is 2 minutes early How far does he drive? What should be his average speed to be neither early or late?

5. An airplane flying east with a tail wind made a trip of 630 miles *in 1 hour 45 minutes Had it been flying west at the same rate against* the wind, it would have taken 2 hours 15 minutes Find the speed of the plane in still air and the velocity of the wind

6. Two trains leave the same station going in the same direction If the faster train, whose rate is 48 miles per hour, leaves the station 1 hour later than a slower train, whose rate is 40 miles per hour, how far from the station will the faster train pass the slower train?

7. An airplane went 840 miles with the wind in 3 hours, and went 640 miles against the wind in 4 hours Determine the speed of the airplane in still air and the velocity of the wind

8. A steamer goes 5 miles downstream in the same time that it would take to go upstream 3 miles. If its rate each way is diminished 4 miles per hour, its rate downstream will be twice its rate upstream. How fast does the steamer go in each direction?

9. A man walks a certain distance at an average rate of 4 miles per hour. He then rides back in an automobile at an average rate of 40 miles per hour. If he is gone from home 5 hours and 30 minutes, how far did he walk?

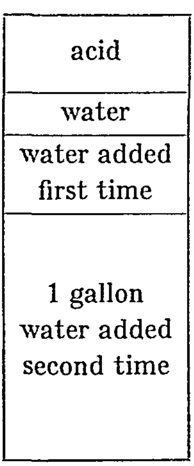
Mixture problems

Mixture problems arise when either the relative quantities of two or more materials already mixed are to be changed by the addition of one of the materials in a pure state, or when the amount of each of several simple ingredients, whose prices or quantities are known, are combined to form a mixture of any required price or quantity. In all cases a problem in simultaneous equations arises. Problems that involve combinations of ingredients may be encountered on many occasions.

Illustrations:

a. A 75% acid solution is diluted with water to make a 50% solution. When 1 gallon of water is added to dilute it again, the solution becomes 25% acid. How much water was added to the solution the first time?

Let x = amount of water in the original solution. Then $3x$ = amount of acid in the original solution, since the total solution is $4x$ and 75% of $4x$ is $3x$. Let y equal the amount of water added the first time. When the amount of water, y, is added, there is a solution of 50% acid and 50% water. Then the amount of acid, $3x$, is equal to the original amount of water, x, plus the amount of water added, y. That is: $3x = x + y$, or $y = 2x$.



When 1 gallon of water is added, the solution is changed to 75% water and 25% acid. The amount of acid is still $3x$. Since the total amount of water is now 3 times as great as the amount of acid, it must be $9x$. Thus $9x - 1 = 3x$, or $6x = 1$. So $x = \frac{1}{6}$ gallon. Therefore $y = \frac{1}{3}$ gallon, since $y = 2x$.

Check: In the original solution there is $\frac{1}{6}$ gallon of water and $\frac{1}{2}$ gallon (i.e., $\frac{3}{6}$ gallon) of acid, making $\frac{2}{3}$ gallon of solution. By adding $\frac{1}{3}$ gallon of water to the solution, there will be 1 gallon of solution, half of which is water and half of which is acid. Now add 1 gallon of water, making a total solution of 2 gallons of mixture. Since

the $\frac{1}{2}$ gallon of acid is 25% of the total solution of 2 gallons of mixture, the problem checks

b A subdivision has for sale some homes with two bedrooms and some with three bedrooms If 3 of the smaller homes are sold for \$9 000 each to create interest in the tract, how many homes of each size must be sold to average \$11,500, knowing that the larger homes will sell for \$12,500 each and the smaller homes will sell for \$10,520 each? There are 60 homes in the tract

Let x be the number of small homes, and y be the number of large homes planned Then

$$x + y = 60 \qquad \text{(Equation 1)}$$

Since 3 of the small homes are sold for \$9,000 each, there are $(x - 3)$ homes that will sell for \$10,520 each, therefore

$$\$10{,}520(x - 3) + \$12{,}500y + 3 \times \$9\,000 = 60 \times \$11{,}500$$

or $\quad \$10{,}520x + \$12{,}500y = \$694{,}560 \qquad$ (Equation 2)

and $\quad \$10{,}520x + \$10{,}520y = \$631{,}200 \quad$ (Eq 1 $\times$ \$10,520)

Subtracting $\quad \$\ 1{,}980y = \$\ 63\,360$

Then $\quad y = 32, \quad x = 28$

EXERCISE 8 5

Solve the following

1. A chemist desires to make 800 cubic centimeters of a 15% solution He has a 40% solution How much water and how much 40% solution must he combine to make the desired solution?

2. A subdivider has 48 homes, some of which sell for \$12,000 each and the rest for \$15,000 each If he desires an average return of \$12,500 per home, how many should he build to sell at each price?

3. The total sales for 800 tickets were \$860 Some sold for \$1 and the rest sold for \$1 25 How many of each type were sold?

4. There are 4 gallons of a fluid that is 60% alcohol and 10% water in a 10-gallon tank One gallon of pure alcohol is added Then sufficient pure alcohol and pure water are added to fill the tank with a fluid that is 50% alcohol and 50% water How much of each is added?

5. A gardner has two solutions, one consisting of 7 gallons of water and 2 of insecticide, the other consisting of 6 gallons of water and 5 of insecticide He needs 6 gallons of solution with 25% insecticide How much of each solution should he use?

6. A subdivider had 18 acres of land which he divided into 54 lots. The lots were sold at \$6,000 and \$7,200 each. Had he sold the lower-priced lots at the higher price and the higher-priced lots at the lower price, he would have received \$16,800 less. How many did he sell at each price?

7. If a buyer is offered 5 Fords and 8 Mercuries for \$36,000 or 4 Mercuries and 6 Fords for \$26,400, what is the price of each?

8. A hotel has 200 rooms. Those with bath rent for \$8 a day, and those without bath rent for \$6 a day. On a certain day 80% of the rooms with bath were rented and 90% of the rooms without bath were rented. If the gross receipts for rent were \$1,140, how many rooms of each type were rented?

9. In a paper and magazine drive 24 tons are collected. If the paper is worth \$6 a ton and the magazines \$4 a ton, how many tons of each were collected for a welfare organization if their gross receipts were \$132?

Investment problems

The managers of institutions which invest large amounts of money in securities, such as insurance companies, investment companies, and eleemosynary institutions predicate their own operational policies on the assumption of certain amounts of income. With changes in interest rates, changes in the amount of funds to be invested, and with alternative choices of outlets for funds, it is essential that the managements be able to ascertain what shifts should be made in securities to gain stated objectives. The solutions to some of the problems that arise are facilitated by the use of simultaneous equations.

Illustration: Walter Larsh receives an annual income of \$4,020 on a total investment of \$75,000. Part of this money is invested in mortgages at 6% interest, and the balance in preferred stock which pays $4\frac{1}{2}$%. How much is invested at each rate?

Let $x =$ the amount invested at $4\frac{1}{2}$% and $y =$ the amount invested at 6%; then $x + y$ must equal \$75,000, the total amount. That is,

$$x + y = \$75{,}000$$

The income from investing x at $4\frac{1}{2}$% is $0.045x$; and the income from investing y at 6% is $0.06y$. Then $0.045x + 0.06y$ must equal \$4,020, the total income. That is,

$$0.045x + 0.06y = \$4{,}020$$

Since each equation satisfies a condition of the problem, the solution of such a system, if possible, will determine the amount invested at each rate.

$$\begin{aligned} x + y &= \$75,000 \\ 0.045x + 0.06y &= 4,020 \end{aligned} \quad \text{or} \quad \begin{aligned} 45x + 45y &= \$3,375,000 \\ 45x + 60y &= 4,020,000 \\ \hline 15y &= \$645,000 \\ y &= \$\ 43,000 \\ x &= \$\ 32,000 \end{aligned}$$

EXERCISE 8.6

Solve the following

1. The Merchants National Bank last year had commercial loans of \$4,500,000 and installment loans of \$1,500 000 The rate on the installment loans is $4\frac{1}{2}\%$ above the rate on the commercial loans If the total income was \$337,500, what was the rate on each type?

2. The Teachers Credit Union lends at the annual rate of 10% on unsecured loans and 9% on secured loans If the total amount of loans was \$800,000 and the union earned \$75 000, how much was lent at each rate?

3. George Mapes has two investments which total \$25,000 From one of these which yields 4% he receives \$100 more than from the other, which yields 5% Find the amount of each investment

4. C W Gimby desires his funds to yield $3\frac{1}{4}\%$ Three investments are purchased with his funds with yields of $2\frac{3}{4}\%$, 3%, and $3\frac{3}{4}\%$ If \$5 000 is invested at the $2\frac{3}{4}\%$ rate, how much is invested at the other rates in order to get his desired yield if his total investment is \$19,000?

5 Charles Wiseman has \$40,000 invested in funds that yield at $3\frac{1}{2}\%$ $3\frac{3}{4}\%$, and 4% If he has $\frac{1}{4}$ of the fund invested at the $3\frac{1}{2}\%$ rate, how much does he have at $3\frac{3}{4}\%$, and at 4% if the total yield is \$1,494 a year?

6. F G Essig desires his funds to yield $3\frac{1}{2}\%$ Three investments are purchased with funds totaling \$28,000 that yield 3%, $3\frac{1}{4}\%$, and 4% If \$8,000 is invested at the 3% rate, how much must be invested at the other rates to get the desired yield?

7. Louis Martin makes two investments totaling \$9,600 On one investment he made a 5% profit, but on the other he takes a 12% loss If his net profit is \$123, how much was in each investment?

8. Diane Hawkins makes two investments totaling \$12,000 On one investment she made an 8% profit, but on the other she takes a 20% loss If her net loss is \$160, how much was in each investment?

9. A mortgage banker has \$240,000 outstanding in loans Some is lent at $4\frac{1}{4}\%$ and the balance is at 5% His income is \$11,100 He seeks an average return of 5% How much of the money now out at $4\frac{1}{4}\%$ must be called in and reinvested at 6% to get the desired average return?

10. Two amounts of money are invested. If the first amount is invested at 3% and the second amount invested at 2%, the annual interest return is $780. If the first amount is invested at 2% and the second amount is invested at 3%, the annual interest return is $720. How large is each amount?

Tax and bonus problems

With the growth of state income taxes, and the use of bonuses in corporate salaries, the business application of simultaneous equations has increased many fold. Such problems usually arise because the bonus is to be paid after the deduction of income taxes, but the amount of the bonus is a deductible item in computing the income tax. Similarly the amount paid in state income taxes may be deductible in computing the federal tax, and vice versa. Tax laws change frequently, rates change, and the methods of computing them change. The purpose of this discussion is to teach the application of principles under given conditions. The purpose is not to furnish definitive information on the present tax laws.

Illustration: The sales manager of a company is to receive a flat salary plus a bonus of 10% of the net profits after taxes. Net profits for the year are $100,000, and the tax rate on that sum is 40%. Since the bonus is deductible in arriving at the tax, and since the tax is deductible in arriving at the bonus, what is the tax and what is the bonus?

Let b represent the bonus and t the tax. Then

(Equation 1) $$b = 0.10(100{,}000 - t)$$
(Equation 2) $$t = 0.40(100{,}000 - b)$$

Substituting for t in Equation 1 gives a linear equation in b.

$$b = 0.10[100{,}000 - 0.40(100{,}000 - b)]$$

Solving for b,

$$b = \$6{,}250$$

Substituting the value of b in Equation 2,

$$t = 0.40(100{,}000 - 6{,}250)$$
$$t = \$37{,}500$$

EXERCISE 8 7

Solve the following

1. Besides a flat salary, the manager of the ABC Company is to receive a bonus of 15% of the net profits after taxes The company earns $252,000 before paying the tax or the bonus The tax rate is 50% If the bonus is deductible in arriving at the tax, and if the tax is deductible in arriving at the bonus, what is the tax and what is the bonus?

2. A taxpayer has a net income of $9,900 before federal and state income taxes Assume that the federal income tax rate is 20%, and the state income tax rate is 5% If the federal tax is allowed as a deduction in computing the state tax, and the state tax is allowed as a deduction in computing the federal tax, find the federal tax and the state tax

3 Nancy Stanton s net income is $13 200 before federal and state income taxes Assume that the federal income tax rate is 25%, and the state income tax rate is 4% If each tax is allowed as a deduction in computing the other tax, what are her federal tax and her state tax?

4. The construction foreman of the Hasty Home Corporation receives a bonus of 10% of profits after payment of federal income taxes The corporation pays taxes at 45% of income after deducting the foreman's bonus If the earnings of the corporation last year were $189,000 before taxes or bonus, how much did the foreman receive and what were the taxes?

Equations in three unknowns

The principles involved in solving equations in more than two unknowns are exactly the same as those involved in solving simultaneous equations in two unknowns When there are 3 equations in 3 unknowns, or 4 equations in 4 unknowns, the process of solution is to eliminate 1 of the unknowns in favor of the others, and continue the procedure until there are only 2 equations in 2 unknowns The value for the third or the fourth unknown is found by substituting the other values found in the original equations The process may be long but it is not complex

Illustration Solve
$$\begin{aligned} 2x + y + z &= 7 \\ x + 2y + z &= 3 \\ x + y + 2z &= 6 \end{aligned}$$

Any one of the unknowns can be eliminated first To eliminate z, subtract the second equation from the first The result is $x - y = 4$ A second equation in x and y is needed One possibility is to multiply the second equation by 2 and subtract the third equation The result is $x + 3y = 0$

Solving the pair of equations $x - y = 4$ and $x + 3y = 0$, the results are $x = 3$, $y = -1$. Substituting these values in any one of the given equations, the value of z is 2. Thus the values are: $x = 3$, $y = -1$, $z = 2$.

EXERCISE 8.8

Solve the following systems.

1. $x - 3y + 2z = -1$
   $2x + y - 3z = -2$
   $x + 2y - z = 3$

2. $3x + 2y - z = -1$
   $x - 4y + 3z = 11$
   $2x + 2y + z = 2$

3. $x + y - z = 11$
   $2x - y + z = 2$
   $-x + 2y + z = 1$

4. $2x - y + 4z = 7$
   $3x + 2y - 2z = 17$
   $5x - 4y + 6z = 11$

5. $2a + 5b - 4c = 3$
   $5a + 2b + c = -4$
   $4a - 3b + 2c = -1$

6. $4a - 3b + 2c = 2$
   $3a + 4b - 3c = 7$
   $a - 3b + c = 3$

7. $x = 2y + z - 5$
   $y = x - 4z + 2$
   $z = x + y$

8. $x + 2y + z - w = -2$
   $3x - 2y - z + 2w = 9$
   $2x - y - z + w = 4$
   $x - 3y - 2z - w = -3$

9. $x + 2y = 1$
   $2x + 3z = 14$
   $y - 2z = -10$
   $3x - w = 11$

10. $x + 3y - 2z = -1$
    $2x - y - 3w = 11$
    $y - 3z + 2w = -7$
    $x - 2z + w = 0$

Business problems in more than two unknowns

Many persons engaged in business and related occupations, who possess no knowledge of simultaneous equations and their solution, would benefit immeasurably by the application of these simple principles. More and more management consultants and financiers are seeing and applying more complex algebraic methods to the solution of problems which otherwise would be solved only by much more time-consuming and costly methods.

Suppose, for example, that a company manufactures several products, and that it has inadequate knowledge of the specific costs of each item. From the accounting records, however, it can be found what monthly sales have been of each item, and the amount of profit. Regardless of the number of items, a system of simultaneous linear equations can be set up, and the profitability of each item determined.

Illustration Officials of the local tool company have never before tried to determine the cost of each item they manufacture A new manager feels that some items are unprofitable From an analysis of the sales records, he gets total sales, from the accounting records he finds the net profits From the figures given, find the average net profit or loss on each item

Year	*Number of Items Made and Sold*			*Net Profit*
	Hammers	*Mallets*	*Axes*	
1	10,000	5,000	2,000	\$8,400
2	8,000	6,000	3,000	\$7,900
7	9,000	4,000	4,000	\$6,500

If x = profit per hammer, y = profit per mallet, and z = profit per ax, we have the three following formulas

$$
\begin{aligned}
&(1) \qquad 10{,}000x + 5{,}000y + 2{,}000z = \$8{,}400 \\
&(2) \qquad 8{,}000x + 6{,}000y + 3\ 000z = \$7{,}900 \\
&(3) \qquad 9{,}000x + 4{,}000y + 4{,}000z = \$6{,}500
\end{aligned}
$$

These equations can all be reduced by dropping 2 zeros from each number Eliminate z by multiplying Equation (1) by 2, and subtracting Equation (3) from the product

$$
\begin{array}{lrl}
 & 200x + 100y + 40z & = \$168 \\
 & 90x + 40y + 40z & = \$\ 65 \\ \hline
(4) & 110x + 60y & = \$103
\end{array}
$$

Multiply Equation (2) by 4, and deduct Equation (3) times 3

$$
\begin{array}{lrll}
 & 320x + 240y + 120z & = \$316 & \\
 & 270x + 120y + 120z & = \$195 & \\ \hline
(5) & 50x + 120y & = \$121 & \\
 & 220x + 120y & = \$206 & \text{Equation (4) times 2} \\ \hline
 & 170x & = \$\ 85 & \\
 & x & = 50 \text{ cents} &
\end{array}
$$

Substituting in Equation (4),

$$\$55 + 60y = \$103$$
$$60y = \$\ 48, \quad y = 80 \text{ cents}$$

Substituting in Equation (1),

$$\$5{,}000 + \$4{,}000 + 2{,}000z = \$8{,}400$$
$$2{,}000z = -\ \$600, \quad z = -\ 30 \text{ cents}$$

Thus the profit is 50 cents on each hammer, 80 cents on each mallet, and a loss of 30 cents on each ax

A somewhat similar problem arises in trying to determine the most economical combination of employees. The following illustration shows how this may be solved.

Illustration: Sales in a given department may be influenced by the season of the year, the day, or the week, the type of weather, etc. In one department four people are employed, and each person is allowed one day off per week. By rotating the days which each person had off weekly over a period of one year, and checking total sales volumes on the separate dates, the manager of the store obtained the following information.

Salesman on Duty	*Average Total Daily Sales*
A, B, and C	\$3,750
A, B, and D	4,500
A, C, and D	4,250
B, C, and D	3,250

Obviously from the data it can be determined that the salesmen had different sales records. The sales were lowest when A was absent, second lowest when D was absent, third lowest when B was absent, and highest when C was absent. Hence with no algebra involved, the salesmen ranked from highest to lowest would be A, D, C, and B.

To determine average daily sales, however, set up the four equations.

$$\begin{array}{lllr}
(1) & & A + B + C & = \$3{,}750 \\
(2) & & A + B + D & = \$4{,}500 \\
(3) & & A + C + D & = \$4{,}250 \\
(4) & & B + C + D & = \$3{,}250 \\
(5) & \text{(Equation 2 less 1)} & -C + D & = \$750 \\
(6) & \text{(Equation 2 less 3)} & B - C & = \$250 \\
(7) & \text{(Equation 3 less 4)} & A - B & = \$1{,}000 \\
(8) & \text{(Equation 6 plus 7)} & A - C & = \$1{,}250 \\
(9) & \text{(Equation 3 plus 5)} & A + 2D & = \$5{,}000 \\
(10) & \text{(Equation 1 less 4)} & A - D & = \$500 \\
 & & A & = \$500 + D
\end{array}$$

Substituting the value $A = \$500 + D$ in Equation (9):

$$\$500 + D + 2D = \$5{,}000$$
$$3D = \$4{,}500; \quad D = \$1{,}500$$

Substituting the value $D = \$1{,}500$ in Equation (10):

$$A - \$1{,}500 = \$500; \quad A = \$2{,}000$$

Substituting the value A = \$2,000 in Equation (8),

$$\$2{,}000 - C = \$1{,}250, \quad C = \$750$$

Substituting the value of A in Equation (7),

$$A - B = \$1{,}000$$
$$\$2000 - B = \$1{,}000, \quad B = \$1{,}000$$

Thus the average daily sales were

$$A = \$2{,}000, \quad B = \$1{,}000, \quad C = \$750, \quad D = \$1{,}500$$

If one salesman is to be replaced it should be C It may be, however, that from this preliminary investigation the problem of management is *to determine why the differences in average sales exist*

EXERCISE 8.9

Solve the following

1. A company which manufactures punches, chisels, and screwdrivers has no adequate cost system Sales for the last 3 quarters have been totaled, and the profits for those periods found Find the average net profit or loss for each 100 items

Quarter	*Number of Items Sold (in 100 s)*			*Net Profit for Quarter*
	Screwdrivers	*Punches*	*Chisels*	
1	8,000	4,000	3,200	\$10,320
2	7,200	6,000	4,800	\$15,528
3	10,000	5,000	3,840	\$12,580

2. Three employees, denoted by A, B, and C, in a store rotate their days off each week Although no individual record of sales was maintained, their employer wanted to compare the sales records of each employee By checking total sales volumes on the separate dates involved over a period of a year, the employer obtained the following information

Salesmen on Duty	*Average Total Daily Sales*
A and B	\$1,000
A and C	\$860
B and C	\$900

What was the average daily sales volume of each?

3. The president of the Heavy Metals Corporation was anxious to find out which of three salesmen, X, Y, and Z, would be the best sales manager. Two were assigned to outside sales while the third stayed at the plant, studying the organization of the company. Such a plan was carried out for 6 months and the average weekly outside sales of these men were recorded as follows:

	Average Weekly Sales
X and Y outside, Z inside	\$39,900
X and Z outside, Y inside	38,100
Y and Z outside, X inside	41,000

If the salesman with the highest total weekly sales was selected as sales manager, who was selected?

4. The Economy Blueprint Company has a contract with the general manager which provides that he will receive 10% of the profits before federal income taxes have been paid and a contribution has been made to the pension fund. The contribution to the pension fund is 10% of the profits after the deduction of the general manager's bonus has been made and the income taxes have been paid. Earnings before these allocations were made were \$240,000. What amount should be allocated to: (a) the general manager's salary; (b) the pension fund; (c) federal income taxes if the rate is 50% and if the payments to the general manager and to the pension fund are both deductible in computing the tax.

5. A taxpayer has a net income of \$10,000 before federal and state income taxes. The federal tax rate is 20%. Sixty per cent of the income was earned in State A, where a 5% tax rate prevails after allowing deductions for federal taxes and *other* state taxes. The remainder was earned in State B where the tax rate is 2% and deductions are allowed for federal taxes and other state taxes *including* its own. Both states levy taxes only on income derived within the state. Find the federal tax, the state tax in A, and the state tax in B.

6. A taxpayer who earned \$50,000 before providing for federal and state taxes has agreed to pay his manager a commission of 10% of net income after federal and state taxes. The federal rate is 40% and the state rate is 2%. The commission paid to the manager is a deductible expense in arriving at federal and state taxes. The state tax may be deducted in arriving at the federal taxes, and the state allows deduction for both federal and state taxes. Find the amount of commission, state tax, and federal tax.

Quadratic equations

If the highest power of the unknown in an equation written in its simplest form is a square, the equation is called an *equation of the second degree* or a *quadratic equation* The equation $x^2 + 4x + 4 = 0$ is a complete quadratic, since it contains both the square and the first power of the unknown The expression $9x^2 - 81$ is called an incomplete quadratic, or a pure quadratic, since it contains only the square of the unknown

Solution of incomplete quadratic equations

An incomplete quadratic equation can be solved in the following way (1) carry out the indicated operations and simplify the equation so that it can be stated in the form of $ax^2 - b = 0$, (2) isolate the unknown term on one side of the equation and then extract the square root of both members

Illustration Solve $5x^2 - 6 = 75 - 4x^2$

Step 1	Transpose and collect terms	$9x^2 = 81$
Step 2	Divide both sides by 9	$x^2 = 9$
Step 3	Take the square roots	$x = \pm 3$

The sign $\pm$ is read ' plus or minus," signifying that the root of the equation is either $x = +3$ or $x = -3$

Check If $x = +3$,

$$\begin{aligned} 5 \times (+3)^2 - 6 &= 75 - 4 \times (+3)^2 \\ 5 \times 9 - 6 &= 75 - 4 \times 9 \\ 45 - 6 &= 75 - 36 \\ 39 &= 39 \end{aligned}$$

If $x = -3$,

$$\begin{aligned} 5 \times (-3)^2 - 6 &= 75 - 4 \times (-3)^2 \\ 5 \times 9 - 6 &= 75 - 4 \times 9 \\ 45 - 6 &= 75 - 36 \\ 39 &= 39 \end{aligned}$$

EXERCISE 8.10

Solve and check the following

1. $x^2 = 4$
2. $4x^2 - 36 = 0$
3. $x^2 + 9 = 3x^2 - 15$
4. $16x = \dfrac{64}{x}$
5. $x + 3 = \dfrac{72}{x - 3}$
6. $5x^2 - 21 = 101$
7. $\dfrac{8}{5x^2 - 2} = \dfrac{4}{2x^2 + 7}$
8. $8x^2 - 50 = x^2 - 1$
9. $2(x^2 - 43) = -4(3x^2 - 10)$
10. $\dfrac{x}{3} + \dfrac{15}{x} = 2x$

Solution of complete quadratic equations by factoring

In the study of multiplication it is learned that if the product of two numbers is zero, one of the numbers must be 0. Thus if $AB = 0$, it is known that either $A = 0$, $B = 0$, or $A = 0$ and $B = 0$.

This fact provides the basis for solving quadratic equations by the method of factoring. If the right-hand member of the equation is 0, and if the left-hand member can be factored, the roots of the equation can be found by setting each factor equal to 0 and solving the resulting linear equations.

To solve a complete quadratic equation by factoring, the following steps are necessary:

1. Move all the terms of the equation to the left-hand side, and set their value equal to zero.
2. Factor the left-hand side.
3. Set each factor containing the unknown equal to zero.
4. Solve each resulting equation.

Illustrations:

a. Solve and check $x^2 = 7x - 10$.

Step 1. $x^2 - 7x + 10 = 0$

Step 2. $(x - 5)(x - 2) = 0$

Step 3. $x - 5 = 0$ and $x - 2 = 0$

Step 4. $x = 5$ and $x = 2$

Check: When $x = 5$,
$$5^2 = 7 \times 5 - 10$$
$$25 = 35 - 10$$
$$25 = 25$$

When $x = 2$,
$$2^2 = 7 \times 2 - 10$$
$$4 = 14 - 10$$
$$4 = 4$$

b. Solve and check $x(5x + 7) = -2$.

Step 1. $5x^2 + 7x + 2 = 0$

Step 2. $(5x + 2)(x + 1) = 0$

Step 3. $5x + 2 = 0$ and $x + 1 = 0$

Step 4. $x = -\frac{2}{5}$ and $x = -1$

Check: When $x = -\frac{2}{5}$,
$$-\tfrac{2}{5}[5(-\tfrac{2}{5}) + 7] = -2$$
$$-\tfrac{2}{5}(-2 + 7) = -2$$
$$-2 = -2$$

When $x = -1$,
$$-1[5(-1) + 7] = -2$$
$$-1(-5 + 7) = -2$$
$$-2 = -2$$

c Solve and check $-12x^2 + 42x + 24 = 0$

Step 1 Since the coefficient of the squared term is not positive, and since 6 is a common factor of each term, change the signs and reduce the coefficients by dividing by -6 Thus

$$2x^2 - 7x - 4 = 0$$

Step 2 $(2x + 1)(x - 4) = 0$

Step 3 $2x + 1 = 0$ and $x - 4 = 0$

Step 4 $x = -\frac{1}{2}$ and $x = 4$

Check When $x = -\frac{1}{2}$,

$$-12(-\tfrac{1}{2})^2 + 42(-\tfrac{1}{2}) + 24 = 0$$
$$-3 - 21 + 24 = 0$$
$$0 = 0$$

When $x = 4$,

$$-12(4)^2 + 42 \times 4 + 24 = 0$$
$$-192 + 168 + 24 = 0$$
$$0 = 0$$

EXERCISE 8.11

Solve the following equations and check the roots

1. $x^2 - 7x = 0$
2. $4x^2 + 5x = 0$
3. $3x^2 = 7x$
4. $x^2 - 6x + 9 = 0$
5. $x^2 = 10x - 25$
6. $4x = x^2 + 4$
7. $x^2 + x = 12$
8. $x^2 = 10 - 3x$
9. $x^2 + 10 = -7x$
10. $x^2 + x = 20$
11. $x^2 = 2(x + 12)$
12. $x^2 + 11x = -28$
13. $2x^2 - 3x = 2$
14. $4x(x + 2) = -3$
15. $12x^2 - 5x = 3$
16. $12x^2 = 14x + 6$
17. $18x^2 + 39x = -18$
18. $4x^2 - 2x = 3x^2 + 8$
19. $4x^2 - 2 = 3x + 2x^2$
20. $5x + 2(x^2 - 1) = 1$

The quadratic formula

If the general quadratic formula $ax^2 + bx + c = 0$ is solved in terms of general numbers, then to solve any quadratic equation it is necessary only to reduce it to the form of the general quadratic formula and substitute the values in the formula

Given the general quadratic formula	$ax^2 + bx + c = 0$
Subtract c from both sides	$ax^2 + bx = -c$
Divide each side by a	$x^2 + \frac{b}{a}x = \frac{-c}{a}$

The left-hand member would be a perfect square if it had a third term equal to the square of $\frac{1}{2}$ the coefficient of x. Hence the next step is to add to both sides of the equation the square of $\frac{1}{2}$ the coefficient of x. The coefficient of x is $\frac{b}{a}$, so half of this is $\frac{b}{2a}$, and the square of this is $\frac{b^2}{4a^2}$.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

If we rewrite the left member as the square of a binomial and then collect the terms of the right member,

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Extract the square root of both sides, using the double sign on the right side.

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{2a}}$$

Solve for the value of the unknown.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To solve any quadratic equation by the use of this formula, adopt the following procedure:

1. Reduce the equation to the form $ax^2 + bx + c = 0$.
2. List the values of a, b, and c, where $a =$ the coefficient of the quadratic term, $b =$ the coefficient of the linear term, $c =$ the term not containing the unknown.
3. Substitute these values in the quadratic formula. Solve for the unknown and check.

Illustrations:

a. Solve by formula and check $2x^2 + 5x = 18$.

Put the equation in the form $2x^2 + 5x - 18 = 0$. Then

$$a = 2, \quad b = 5, \quad c = -18$$

Therefore

$$x = \frac{-5 \pm \sqrt{25 - 4 \times 2 \times (-18)}}{4} = \frac{-5 \pm \sqrt{25 + 144}}{4} = \frac{-5 \pm \sqrt{169}}{4}$$

So, $$x = \frac{-5 + 13}{4} = 2 \quad \text{or} \quad x = \frac{-5 - 13}{4} = -\frac{9}{2}$$

Check
$$2 \times 2^2 + 5 \times 2 = 18$$
$$8 + 10 = 18$$
$$18 = 18$$
$$2 \times (-\tfrac{9}{2})^2 + 5 \times (-\tfrac{9}{2}) = 18$$
$$\tfrac{81}{2} - \tfrac{45}{2} = 18$$
$$18 = 18$$

b Solve by formula and check $x^2 + 5x = 9$

Put the equation in the form $x^2 + 5x - 9 = 0$ Then
$$a = 1, \quad b = 5, \quad c = -9$$

Therefore
$$x = \frac{-5 \pm \sqrt{25 - 4 \times 1 \times (-9)}}{2} = \frac{-5 \pm \sqrt{25 + 36}}{2} = \frac{-5 \pm \sqrt{61}}{2}$$

So
$$x = \frac{-5 + \sqrt{61}}{2} \quad \text{or} \quad x = \frac{-5 - \sqrt{61}}{2}$$

Check
$$\left(\frac{-5 + \sqrt{61}}{2}\right)^2 + 5\left(\frac{-5 + \sqrt{61}}{2}\right) = 9$$
$$\frac{25 - 10\sqrt{61} + 61}{4} + \frac{-25 + 5\sqrt{61}}{2} = 9$$
$$\frac{43}{2} - \frac{5\sqrt{61}}{2} - \frac{25}{2} + \frac{5\sqrt{61}}{2} = 9$$
$$9 = 9$$
$$\left(\frac{-5 - \sqrt{61}}{2}\right)^2 + 5\left(\frac{-5 - \sqrt{61}}{2}\right) = 9$$
$$\frac{25 + 10\sqrt{61} + 61}{4} + \frac{-25 - 5\sqrt{61}}{2} = 9$$
$$\frac{43}{2} + \frac{5\sqrt{61}}{2} - \frac{25}{2} - \frac{5\sqrt{61}}{2} = 9$$
$$9 = 9$$

EXERCISE 8.12

Solve the following by quadratic formula and check

1. $4x^2 + 3 = 7x$
2. $5x^2 = 2x + 72$
3. $4x^2 + 8x + 1 = 0$
4. $9x^2 = 32x - 15$
5. $9x^2 + 15x + 4 = 0$
6. $4x^2 + 11x + 7 = 0$
7. $7x - 3 = 2x^2$
8. $3(x + 1) - x(x - 1) = 4x$
9. $9x^2 + 12x = 1$
10. $(3x + 2)(2x + 3) = 2(x - 3)(x - 2)$

Quadratic-form word problems

Quadratic equations can often be used in solving stated problems.

Illustrations:

a. Divide 18 into two parts so that the sum of their squares shall be 170.

Let $x =$ one number; then $(18 - x)$ is the other number; and thus $(18 - x)^2 + x^2 = 170$. Therefore $324 - 36x + x^2 + x^2 = 170$, or $x^2 - 18x + 77 = 0$, or $(x - 7)(x - 11) = 0$. So $x = 7$ or $x = 11$. Thus one number is 7, and the other number is $18 - 7 = 11$; or, one number is 11, and the other number is $18 - 11 = 7$.

Check: Since 11^2 is 121 and 7^2 is 49 and 121 plus 49 equals 170, the problem checks.

b. The power poles along a certain highway are equally spaced. If it was decided to place 2 more poles in each mile, the space between each pole would be decreased by 20 feet. Find the number of poles in a mile.

Let $x =$ the number of poles in a mile. Then

$$\frac{5{,}280}{x} - \frac{5{,}280}{x+2} = 20; \quad \text{or} \quad x^2 + 2x - 528 = 0$$

or

$$(x - 22)(x + 24) = 0 \quad \text{and} \quad x = 22$$

Therefore, at present, there are 22 poles per mile. If the distance apart of the poles is decreased by 20 feet, there would be 24 poles per mile. Note that the solution of $x + 24 = 0$ has no meaning here since the answer is negative.

Check: When there are 22 poles per mile they are 240 feet apart; when there are 24 poles per mile they are 220 feet apart; and since 240 less 220 is 20 feet, the problem checks.

EXERCISE 8.13

Solve and check the following word problems.

1. Divide 18 into two parts so that the sum of their squares is 234.
2. Divide 20 into two parts so that one is the square of the other.
3. The difference of two numbers is 7. If their sum multiplied by the greater is 400, what are the numbers?
4. The area of a rectangular yard is 3,600 square feet, and the perimeter is 260 feet. Find the dimensions.
5. A certain plot of ground is in the shape of a rectangle whose length is 3 times its width. If the length is increased 20 feet and the width increased 8 feet, the plot of ground would be trebled. What were the original dimensions?

6. Two men can do a job together in 6 days Working alone one would require 5 more days to do the job than the other How many days would each require working alone?

7. A man traveled 1,530 miles If his average speed had been 4 more miles an hour, he could have made the trip in 6 hours less time What was his average speed?

8. A jeweler has sold all his watches for a gross income of $875 If he had sold them for $5 less per watch, he could have sold 20 more for the same gross income How many watches did he sell and what was the selling price of each watch?

9. A farmer bought a number of sheep for $720 If there had been 8 more they would have cost him $3 apiece less What was the cost of a sheep and how many did he purchase?

10 A man dies, leaving children and a sum of $46,800 By his will it is to be divided equally among them, but it happens that immediately after the death of the father, two of the children also die In consequence of this, each remaining child receives $1,950 more than he or she was entitled to by the will How many children were there and how much does each of the survivors receive?

9

Exponents, Logarithms, and the Slide Rule

Introduction

In carrying out any type of economic endeavor, costs must be considered and the prices of goods or services set sufficiently high to recover all the costs. Otherwise sooner or late the uneconomic operation will be replaced. One of the principal costs is the payment for the time involved—payment for your time or payment to others for the labor they contribute. In either case, if time is wasted either by you or by someone else, there is an added cost, and other persons offering a similar commodity or service may be able to sell their product at a lower cost because they do not have the extra cost arising from wasted time.

The time required to make simple mathematical operations can often be reduced by the application of the principles of higher mathematics. Often such principles are not used simply because the person responsible for supervising the work is not competent to see how the time can be saved. One method of saving many hours of labor is to use a knowledge of exponents, logarithms, and the slide rule. It is intended that this chapter shall furnish such knowledge.

Laws of exponents

It has previously been pointed out that if any number, such as a, is to be multiplied by itself, the operation can be indicated by the use of exponents. Thus $a \times a$ can be indicated by the symbol a^2. The number, in this case a, is referred to as the *base*, and the small numeral written at the right and slightly above the number (in this case, 2) is referred to as the *exponent*. In Chapter 6, it was also shown that if a^2 is multiplied by a^3, the product is a^5. That is, the exponent of a product equals the sum of the exponents of its factors.

Operations with exponents follow fixed relationships which are generally referred to as the *Laws of Exponents* The *Law of Multiplication* previously mentioned, is that *the exponent of a product equals the sum of the exponents of its factors* Thus $a^x \times a^y = a^{x+y}$, and $2^3 \times 2^4 = 2^{3+4} = 2^7$

Division is the second fundamental operation which can be carried on using exponents when the bases are the same The *Law of Division* states that *the exponent of a quotient is equal to the difference between the exponents of the dividend and the divisor* This relationship is generally illustrated by writing the two in fractional form Thus $a^m \div a^n$ is written $\frac{a^m}{a^n}$ Suppose that $m = 5$ and $n = 3$, then $a^5 \div a^3$ can be written $\frac{aaaaa}{aaa} = a^2$ Thus $a^m \div a^n = a^{m-n}$

Suppose, however, that the value of n is greater than the value of m, such as, $m = 3$ and $n = 5$ Then $a^3 \div a^5$ can be written $\frac{aaa}{aaaaa} = \frac{1}{a^2}$ But according to the Law of Exponents the exponent of a quotient is equal to the difference between the exponent of the dividend and the exponent of the divisor Thus $a^3 \div a^5$ might be thought of as $a^{3-5} = a^{-2}$ Indeed a^{-2} is defined as being equal to $\frac{1}{a^2}$

Negative exponents appear frequently, they represent fractional values Thus a^{-n} is the same as $\frac{1}{a^n}$

If $m = n$, then $a^m \div a^n = a^n \div a^n = a^{n-n} = a^0$ Anything divided by itself equals 1 By definition any number to the zero power is equal to 1 For example, $258^0 = 1$

A third law of exponents states that the exponent of a number with an exponent raised to a power is the product of the exponent and the power This can be illustrated as follows If the quantity a^2 is cubed, it is expressed as $(a^2)^3$ It is not hard to see that according to the Law of Multiplication, this is equivalent to $a^2 \times a^2 \times a^2 = a^{2+2+2} = a^6$ Frequently the law is illustrated as $(a^m)^n = a^{mn}$

A fourth law of exponents states that the root of a number with an exponent can be extracted by dividing the exponent by the root Since the number a is presumed to have the exponent of 1, the square root of a could be indicated as $a^{1/2}$ If this is true, then $a^{1/2} \times a^{1/2} = a^{1/2+1/2} = a^1 = a$ From this it follows that $a^{1/2}$ has the same meaning as $\sqrt{a}$ Thus $a^{1/n}$ has the same meaning as $\sqrt[n]{a}$, and the expression $a^{1/n}$ is defined as the nth root of a, and $a^{m/n}$ is defined as the nth root of a raised to the mth power, or $\sqrt[n]{a^m}$, while $a^{2/3} = \sqrt[3]{a^2}$

EXERCISE 9.1

Solve the following:

1. $a^3 \times a^4$	7. $a^2 \div a^{-3}$	13. $(1+i)^9 \div (1+i)^3$
2. $a^2 \times a^6$	8. $a^3 \div a^{-1}$	14. $(1+i)^{12} \div (1+i)^{-2}$
3. $a^6 \times a^{-4}$	9. $(a^2)^4$	15. $[(1+i)^6]^4$
4. $a^5 \times a^{-7}$	10. $(a^3)^3$	16. $[(1+i)^{-6}]^{-4}$
5. $a^7 \div a^4$	11. $(a^{-2})^{-3}$	17. $(a+b)^4\,(a+b)^3$
6. $a^5 \div a$	12. $(a^{-3})^4$	18. $28^3 \div 7^3$

The powers of 10

Since numbers cannot be multiplied by adding the exponents unless the base is the same, the practice has developed of changing numbers to the base of 10.

If 10 is raised to the first power, it is $10^1 = 10$.
If 10 is raised to the second power, it is $10^2 = 100$.
If 10 is raised to the third power, it is $10^3 = 1{,}000$.
If 10 is raised to the fourth power, it is $10^4 = 10{,}000$.

In each of these four statements, it can be seen that the number of zeros after the 1 following the equal sign is the same as the power to which 10 is raised. The same relationship exists for all powers of 10, in fact even for the 0 power, since 10^0 (or any number raised to the zero power) is equal to 1.

As pointed out the discussion of the Laws of Exponents, a negative exponent indicates that a quantity with the same positive exponent is to be divided into 1—that is, a negative exponent indicates the reciprocal of that power. Written as a decimal it can be seen that:

If 10 is raised to the -1 power, it is $10^{-1} = 0.1$.
If 10 is raised to the -2 power, it is $10^{-2} = 0.01$.
If 10 is raised to the -3 power, it is $10^{-3} = 0.001$.
If 10 is raised to the -4 power, it is $10^{-4} = 0.0001$.

In each of the four preceding statements, it can be seen that if the number is a decimal less than 1, the number of zeros immediately following the decimal point is one *less* than the negative exponent to which 10 is raised. Thus in the decimal 0.1 there is no zero following the decimal point. Since one more than 0 is 1, the decimal 0.1 represents 10^{-1}. In the decimal 0.01, one zero follows the decimal point. Since 1 more than 1 is two, the decimal 0.01 indicates 10^{-2}.

Logarithms

In dealing with exponents, any base can be selected, but for practical purposes of computation 10 is almost universally used as the base *The exponent to which the number 10 must be raised to express a number is called the logarithm of that number*

The power of 10 just discussed is in effect the logarithm, or as it is most commonly abbreviated, the log, of the number The log of 10 is 1, the log of 100 is 2, the log of 1,000 is 3

When it is recognized that the numbers which represent the ascending powers of 10 increase by adding a zero before the decimal point as the values rise—that is, $10^1 = 10$, $10^2 = 100$, $10^3 = 1{,}000$, and so on—and that numbers which represent the negative powers of 10 (such as $10^{-1} = 0\ 1$, $10^{-2} = 0\ 01$, $10^{-3} = 0\ 001$, $10^{-4} = 0\ 0001$, and so on) decrease by adding a zero after the decimal point as the negative power gets higher, the method of writing numbers in scientific or standard notation can be understood

A number is said to be in standard notation when it is written as the product of a number between 1 and 10 and a power of 10

Since any number written in ordinary decimal notation can be stated as a product of a number between 1 and 10 and a power of 10, any number written in ordinary decimal notation can be rewritten in standard notation

A number expressed in standard notation must have the decimal point immediately following the first digit on the left This is referred to as the standard position To express any number in standard notation

1 Shift the decimal point to the standard position

2 Multiply the resulting number by a power of 10

3 The exponent of 10 will be *positive* if the decimal point has been shifted to the *left*, it will be *negative* if the decimal point has been shifted to the *right*

4 The exponent of 10 will be equal to the number of places the decimal point has been shifted

Illustrations

(a) Express 1,234 5 in standard notation

1 Move the decimal point to the position following the 1 Thus 1 2345

2 Show as a product 1 2345 and 10 raised to a power

3 Since the decimal point was shifted 3 places to the *left*, the exponent is 3 and is positive Thus $1{,}234\ 5 = 1\ 2345 \times 10^3$

(b) Express 12 345 in standard notation

This can be written as $1\ 2345 \times 10^1$ Since the shift was only one place

to the left, the exponent is positive and is 1. An exponent of 1 is not usually written. Thus 12.345 would be written as 1.2345×10.

(c) Express 0.0012345 in standard notation.

This would be written as 1.2345×10^{-3}. The decimal point must be moved to the *right* 3 places to be in standard position. Thus the number would be written as 1.2345×10^{-3}.

EXERCISE 9.2

Write the following in standard notation.

1. 29,487
2. 294.87
3. 0.29487
4. 45,987
5. 4.5987
6. 0.0045987
7. 0.45987
8. 459.87
9. 67,200
10. 0.00672
11. 0.0000672
12. 672

In the preceding illustrations showing numbers written in standard notation, it was seen that numbers made up of the same series of digits—namely, 12345—when written in standard notation differ only in the power of 10 used.

Thus in standard notation, the following numbers differ only in the exponent applied to 10:

$$\begin{aligned} 4{,}985 &= 4.985 \times 10^3 \\ 49.85 &= 4.985 \times 10 \\ 0.4985 &= 4.985 \times 10^{-1} \\ 0.004985 &= 4.985 \times 10^{-3} \end{aligned}$$

It has already been stated that numbers raised to the same base can be multiplied by adding their exponents. *A log is defined as an exponent.* Therefore the logarithm of 4,985 can be written as the sum of the log of 4.985, and 1,000 ($10^3 = 1{,}000$) as follows:

$$\begin{aligned} \log 4{,}985 &= \log (4.985 \times 10^3) = \log 4.985 + \log 10^3 \\ &= \log 4.985 + 3 \log 10 = \log 4.985 + 3 \end{aligned}$$

In the same manner

$$\begin{aligned} \log 49.85 &= \log 4.985 + 1 \\ \log 0.4985 &= \log 4.985 - 1 \\ \log 0.004985 &= \log 4.895 - 3 \end{aligned}$$

It is seen that when numbers are written in standard notation, the logarithm of the power of 10 is known. All that is needed is the logarithm of the number less than 10. Since the logarithm of 10 is 1, it is logical to

assume that the logarithm of the number less than 10 is a decimal Since the logarithm of a number less than 10 is a decimal, the logarithm of any number written in standard notation can be written as the sum of (first) the decimal less than 1 and (second) the exponent applied to 10 The whole number is called the *characteristic*, the decimal is called the *mantissa*

The characteristic depends only on the position of the decimal point in the number To determine the characteristic of the logarithm, write the number in standard notation The characteristic is the exponent of 10 when the number is written in standard notation

Illustration Find the characteristics of the logarithms of the following numbers (a) 3,456, (b) 37, (c) 10,009 43, (d) 0 7937, (e) 0 0015

(a) 3,456 written in standard notation is $3\ 456 \times 10^3$ Therefore the characteristic is 3

(b) 37 written in standard notation is $3\ 7 \times 10$ Therefore the characteristic is 1

(c) 10,009 43 written in standard notation is $1\ 000943 \times 10^4$ Therefore the characteristic is 4

(d) 0 7937 written in standard notation is $7\ 937 \times 10^{-1}$ Therefore the characteristic is -1

(e) 0 0015 written in standard notation is $1\ 5 \times 10^{-3}$ Therefore the characteristic is -3

EXERCISE 9 3

What is the characteristic of each following?

1. 48
2. 481
3. 4 81
4. 1,548
5. 154 81
6. 837
7. 83,481
8. 8 00481
9. 481 481
10. 48 1
11. 0 481
12. 0 0481
13. 4 00481
14. 481,000
15. 4
16. 0 4
17. 0 00000481
18. 0 000481
19. 49
20. 9 000481

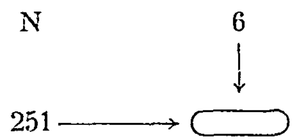
Table of logarithms

The mantissa is exactly the same for all numbers made up of the same series of digits, no matter where the decimal point may be among the digits Since the same mantissa applies to a whole series of numbers

made up of the same digits, the values of the mantissas have been calculated and can be found by referring to printed tables called Tables of Logarithms. One page of a 6-place table is reproduced here. The table is called a 6-place table because the mantissas are stated as decimal fractions with 6 places to the right of the decimal point. The decimal points are not shown; assume them to be before each set of digits.

To find the logarithm of a number: first, determine the characteristic in the way shown; second, find the mantissa of the series of digits from the table.

In using the table presented here, if a number consists of only 3 digits, the mantissa is found in the left-hand column headed 0, on a horizontal line with the number. Thus the mantissa for the number 252 is 401401. If the number consists of 4 digits, the first 3 are found in the column under N, and the fourth is found to the right of the letter N. The method of finding the mantissa of such a 4-digit number is indicated in the accompanying diagram. For example, the logarithm of 251.6 is determined somewhat as follows:



Since 251.6 has three places to the left of the decimal point, the characteristic is 2. To find the mantissa, first find the number 251 under N. Then on the line horizontal to 251, find the value in the vertical line below 6. The value 400711 is shown in the table. The logarithm of 251.6 (generally written merely as log) is 2.400711 and is written log 251.6 = 2.400711.

Although the log of 251.6 is to the base 10, the 10 does not appear. Written as an exponent, 251.6 to the base 10 is

$$251.6 = 10^{2.400711}$$

The 6-place table of logarithms reproduced here is a copy of one page of a complete table of logarithms included in the appendix of this book. The table has two features which facilitate its use. In the first place, if the mantissa is shown as 6 digits, copy the 6 digits shown. For instance, the mantissa for the digits 2821 is 450403. In progressing from one number to another, the first 2 digits of the mantissa do not change for each succeeding number. The first 2 digits of every mantissa are not reproduced. If the mantissa of the desired number is shown in the table with only 4 digits, the first 2 digits are found by looking up the same

Six-Place Logarithms of Numbers 250–300

N	0	1	2	3	4	5	6	7	8	9	D
250	39 7940	39 8114	39 8287	39 8461	39 8634	39 8808	39 8981	39 9154	39 9328	39 9501	173
251	9674	9847	40 0020	40 0192	40 0365	40 0538	40 0711	40 0883	40 1056	40 1228	173
252	40 1401	40 1573	1745	1917	2089	2261	2433	2605	2777	2949	172
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
257	9933	41 0102	41 0271	41 0440	41 0609	41 0777	41 0946	41 1114	41 1283	41 1451	169
258	41 1620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	4973	5140	5307	5474	5641	5808	5974	6141	6308	6474	167
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
263	9956	42 0121	42 0286	42 0451	42 0616	42 0781	42 0945	42 1110	42 1275	42 1439	165
264	42 1604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
269	9752	9914	43 0075	43 0236	43 0398	43 0559	43 0720	43 0881	43 1042	43 1203	161
270	43 1364	43 1525	1685	1846	2007	2167	2328	2488	2649	2809	161
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
273	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	159
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
275	9333	9491	9648	9806	9964	44 0122	44 0279	44 0437	44 0594	44 0752	158
276	44 0909	44 1066	44 1224	44 1381	44 1538	1695	1852	2009	2166	2323	157
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
280	7158	7313	7468	7623	7778	7933	8088	8242	8397	8552	155
281	8706	8861	9015	9170	9324	9478	9633	9787	9941	45 0095	154
282	45 0249	45 0403	45 0557	45 0711	45 0865	45 1018	45 1172	45 1326	45 1479	1633	154
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
288	9392	9543	9694	9845	9995	46 0146	46 0296	46 0447	46 0597	46 0748	151
289	46 0898	46 1048	46 1198	46 1348	46 1499	1649	1799	1948	2098	2248	150
290	2398	2548	2697	2847	2997	3146	3296	3445	3594	3744	150
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
295	9822	9969	47 0116	47 0263	47 0410	47 0557	47 0704	47 0851	47 0998	47 1145	147
296	47 1292	47 1438	1585	1732	1878	2025	2171	2318	2464	2610	146
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145
300	7121	7266	7411	7555	7700	7844	7989	8133	8278	8422	145

column until a mantissa with 6 digits appears. Thus the mantissa for 2895 shows only 4 digits (1649) in the column headed 5, across from 289 in the N column. In the same column under the number 5 above the mantissa 1649, the first 6-digit mantissa shown has as the first 2 digits 46. Thus the mantissa for 2895 is 461649.

Logarithms of numbers less than 1

Since the mantissa of a series of digits is the same regardless of the location of the decimal point, the mantissa is a positive number which is added to the characteristic to give the logarithm for all numbers greater than 1. Suppose, however, that the logarithm of the number 0.02895 is desired. In the discussion in the preceding paragraph it was shown that the mantissa for this series of digits is 461649. Under the rules for determining characteristics, however, the characteristic of the number would be -2. If the characteristic and the mantissa are combined it would indicate the sum of a negative number (the characteristic) and a positive number (the mantissa). Hence the two are not combined algebraically.

It is customary to write a negative characteristic as a positive value, and to show after the mantissa a negative 10, which, when added to the positive characteristic, gives a negative sum equal to the negative exponent or characteristic desired. For example, a logarithm with a negative characteristic can be indicated as follows:

Characteristic

$$9.\ (\text{mantissa}) - 10 = -1$$
$$8.\ (\text{mantissa}) - 10 = -2$$
$$7.\ (\text{mantissa}) - 10 = -3, \text{ etc.}$$

Thus the characteristic for the number 0.02895 is -2, and the logarithm is written

$$\log 0.02895 = 8.461649 - 10$$

EXERCISE 9.4

Using the table on page 226, give the logarithm of each of the following:

1. 252.0
2. 2,571
3. 2,655
4. 3.000
5. 30.09
6. 27.00
7. 270.5
8. 275.5
9. 0.2897
10. 0.02976

11. 2,601
12. 2 666
13. 290 5
14. 28 37
15. 2,999
16. 266 7
17. 261 4
18. 0 02794
19. 0 0003056
20. 30,010

Interpolation to find the mantissa

If a number consists of 5 or more figures, the mantissa cannot be read directly from the table However, by a process called *interpolation*, an approximate value of the mantissa can be found Interpolation is based on the principle of proportion For example, in interpolation it is assumed that if the mantissa for 2,716 is 433930, and the mantissa for 2,717 is 434090, then the mantissa of the logarithm for 2,716 3 is the same as the mantissa for 2,716, plus 0 3 of the difference between the mantissa for 2 716 and the mantissa for 2,717 Thus $433930 + 0\ 3\ (434090 - 433930) = 433930 + 0\ 3 \times 160 = 433930 + 48 = 433978$ Therefore log 2 716.3 $= 3\ 433978$

So far as the mantissa is concerned, 2,716 and 2 717 can be considered as 27,160 and 27,170 Then perhaps it can be more readily seen that the mantissa of the number 27,163 may be considered as $\frac{3}{10}$ of the way between the two known mantissas This can be shown as follows

Numerical difference	Number	Mantissa	Tabular difference
	27,160	433930	
3	27,163	?	x
10	27,170	434090	160

As a proportion, this can be written

$$\frac{\text{Smaller numerical difference (3)}}{\text{Larger numerical difference (10)}} = \frac{\text{Smaller tabular difference } (x)}{\text{Larger tabular difference (160)}}$$

$$\frac{3}{10} = \frac{x}{160}, \quad 10x = 480, \quad x = 48$$

The mantissa for 27,163 is $433930 + 48 = 433978$

Interpolation from the table in the appendix is greatly facilitated by using the table of proportional parts published as part of the logarithm

table. The table shows, under the column headed D, the tabular difference between one mantissa and the mantissa for the number just higher.

By looking across the table, it is seen that the D column shows 160 on the same horizontal line as 271 in the N column. Actually the difference between one mantissa and the next higher one is not 160 for all in the line, since there is a difference of 161 between the first two shown on the line. For practical purposes, however, the number shown in the difference column is of much help. If it is desired to find a proportional part, namely 0.3 of 160, look at the right-hand side of the page on which the logarithm is found for the column headed 160. Reading across on a horizontal line from the 3 shown in the left-hand column of the proportional parts table to the vertical column headed 160, we see that 48 is the proportional part. This, added to the mantissa of the 2,716, gives the mantissa of 27,163.

In the lower numbers, the tabular differences between the 10 succeeding mantissas in one horizontal line vary greatly. Because of lack of space, however, the actual differences are not shown under the heading D, but only the median difference for the mantissas in the one horizontal line. Since space does not permit every possible difference to be shown in the proportionate parts table, the proportionate parts for the larger amounts vary in units of 5. Thus the proportionate parts for 432 do not appear, although the proportionate parts for both 435 and 430 do.

For differences of less than 100, the proportionate parts tables show the decimal parts. The difference between the mantissa for 8,615 and the mantissa for 8,616 is 51. In the column headed D, 50 appears on the same horizontal line as 861 in the column headed N. The proportionate parts for 51 show that $\frac{1}{10}$ of 51 is 5.1. The $\frac{1}{10}$ appears under n\d simply as 1, although actually it signifies 0.1. The proportionate parts for a tabular difference of 51 appear as follows:

n\d	51
1	5.1
2	10.2
3	15.3
4	20.4
5	25.5
6	30.6
7	35.7
8	40.8
9	45.9

The proportionate parts table is helpful in determining logarithms of numbers not shown in the table. If the mantissa of 86,154 is desired, it

can readily be ascertained from the table that the difference between the mantissas for 86,150 and 86,160 is 51 The number 86,154 lies $\frac{4}{10}$ of the way between these two From the table of proportionate parts it can readily be seen that $\frac{4}{10}$ of the 51 is 20 4 The mantissa for 86,150 is 935255

$$\begin{array}{ccc} n\backslash d & & 51 \\ & & \downarrow \\ 4 & \longrightarrow & (20\ 4) \end{array}$$

If the proportional part of 20 4 is added to this, the mantissa would be 9352754 This must be rounded to 935275, however, by dropping the 4 since one rule of logarithms is that the number whose logarithm is found must be rounded to the number of places in the logarithm tables That is, the mantissa may contain no more places than the number of places shown in the table Thus, if a 6-place table is used, the mantissa is rounded to 6 places

EXERCISE 9.5

Using the logarithm table in the appendix, find the logarithm of each of the following

1. 4 9927
2. 26 623
3. 6 1776
4. 122 43
5. 8,768 2
6. 4 7738
7. 2 7926
8. 817 22
9. 66,287
10. 0 058632
11. 0 0041824
12. 4 1824
13. 2177 9
14. 43 376
15. 8 7123
16. 83,403
17. 8 0034
18. 26,111
19. 0 00080003
20. 0 000066666

Antilogarithms

Certain properties of exponents or logarithms make them very useful for computation It has previously been indicated that the logarithm of a product equals the sum of the logarithms of the factors Applying this principle, multiply 240 $\times$ 325 From the table it can be determined that

$$\begin{aligned} \log 240 &= 2\ 380211 \\ \log 325 &= \underline{2\ 511883} \\ \log (240 \times 325) &= 4\ 892094 \end{aligned}$$

What is desired, however, is not the logarithm of the product, but the actual product. The number corresponding to a given logarithm is called the *antilogarithm*. The product of 240×325 is the antilogarithm of 4.892094. To find the value corresponding to a given logarithm: first, find in the table the number corresponding to a given logarithm; second, determine the location of the decimal point from the value of the characteristic. In this problem it is found from the table that the mantissa 892095 corresponds to a series of digits 7800; from the characteristic of 4, it is known that the answer must contain 5 places; therefore one zero must be added, giving 78,000 as the product.

If the exact mantissa is not found in the table, a close approximation of the antilogarithm can be found by interpolation. The antilogarithm, however, may have no more significant places than the number of digits on which the table is based. Even though the mantissa contains 6 places, the antilogarithm can contain only 5 significant figures.

If the logarithm of a product is found to be 1.553472, the product can be determined as follows:

From the table, it is found that the nearest mantissa less than 553472 is 553398, and that the nearest greater than 553472 is 553519. The difference between the two mantissas ($553519 - 553398 = 121$) is called the *tabular difference*, and is shown in the D column of the table. The mantissas in the table represent the series of digits 3,576 and 3,577, respectively. Since the characteristic of the logarithm 1.553472 is 1, it is known that the antilogarithm is more than 35.76 but less than 35.77. The difference between the mantissa of 3576 (553519) and the given mantissa (553472) is 74. The tabular difference is 121. It is logical therefore to assume that the number 35.766 ($\frac{74}{121} \times 0.10 = 0.06$) is the antilogarithm of 1.553472.

The table of proportional parts may also be helpful in determining antilogarithms.

Illustration: Determine the number whose mantissa is 553472. The mantissa 553472 lies between the mantissas 553398 and 553519, which represent the series of digits 35760 and 35770, respectively. The tabular difference, D, is 121. The difference between the mantissa 553472 and the next smaller mantissa (553398) is 74. Under 121 in the proportional parts table, find the nearest value to 74, here 73. Thus 74 is about $\frac{6}{10}$ of 121. The desired number is 35766.

$$\begin{array}{cc} n\backslash d & 121 \\ & \downarrow \\ 6 & \longleftarrow 73 \end{array}$$

EXERCISE 9 6

Find the antilogarithm of each of the following

1. 2 103119
2. 0 311754
3. 1 502973
4. 9 564903 − 10
5. 0 622835
6. 8 715167 − 10
7. 3 789933
8. 1 864096
9. 7 920436 − 10
10. 0 966892
11. 1 094978
12. 9 334917 − 10
13. 2 512332
14. 8 574512 − 10
15. 0 632700
16. 3 726451
17. 9 797467 − 10
18. 7 867725 − 10
19. 1 926072
20. 4 970194

Multiplication by logarithms

Since a logarithm is an exponent, multiplication problems can be solved by adding logarithms, just as they can be solved by adding exponents The chief advantage of using logarithms rather than exponents is that the problem can then be laid out in orderly form before consulting the tables

Illustrations

a Using logarithms, find the product of 2 784 × 31 62

Layout		*Layout filled in*	
log 2 784 = 0		log 2 784 = 0 444669	
log 31 62 = 1	(+)	log 31 62 = 1 499962	(+)
log product =		log product = 1 944631	
product =		product = 88 030	

In the layout, show the plus (+) sign indicating that addition is to take place

b Using logarithms, find the product of 0 008527 × 62 34

Layout		*Layout filled in*	
log 0 008527 = 7 − 10		log 0 008527 = 7 930796 − 10	
log 62 34 = 1	(+)	log 62 34 = 1 794767	(+)
log product =		log product = 9 725563 − 10	
product =		product = 0 53157	

EXERCISE 9.7

Using logarithms, find the product of the following:

1. 28.32×4.726
2. 5.623×81.75
3. 127.8×61.68
4. 44.94×53.86
5. 38.27×8.716
6. 27.82×0.3324
7. 4.875×0.006168
8. 0.07269×0.6383
9. $1{,}874 \times 0.008226$
10. 38.27×0.05186
11. 4.9927×103.74
12. 26.623×61.776
13. 122.43×87.682
14. 2.7926×0.006362
15. 81.722×0.00055734
16. $8.382 \times 4.773 \times 27.93$
17. $3.822 \times 7.270 \times 21.38 \times 8.282$
18. $48.28 \times 32.27 \times 0.06336$
19. $283.45 \times 6.117 \times 0.007173$
20. $41.824 \times 0.058632 \times 0.0066287$

Division by logarithms

A second law of logarithms is that the logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.

Illustrations:

a. Using logarithms, find the quotient of 83.62 ÷ 6.286.

Layout		*Layout filled in*	
log 83.62 = 1.		log 83.62 = 1.922310	
log 6.286 = 0.	(−)	log 6.286 = 0.798374	(−)
log quotient =		log quotient = 1.123936	
quotient =		quotient = 13.303	

In the layout, show the minus (—) sign indicating that subtraction is to take place.

b. Using logarithms, find the quotient of 33.84 ÷ 147.6.

The log of 33.84 is 1.529430; the log of 147.6 is 2.169086. To avoid the necessity of subtracting a larger number (2.169086) from a smaller number (1.529430), the number 10 (or any number, for that matter) may be added and deducted at the same time from the characteristic of the dividend. The log of 33.84 is written not *1.529430* but *11.529430 — 10.* Then the subtraction becomes relatively simple. Nothing deducted from the — 10 results in the — 10 appearing as part of the quotient. In other words, the characteristic of the quotient is negative. The problem thus takes the form:

Layout		*Layout filled in*		
log 33 84 = 1		log 33 84 = 1 529430	= 11 529130 − 10	
log 147 6 = 2	(−)	log 147 6 = 2 169086	= 2 169086	(−)
log quotient =		log quotient	= 9 360344 − 10	
quotient =		quotient	= 0 22927	

From the logarithm table it is found that the mantissa 360344 represents the digits 22927 Since the characteristic is − 1 (represented in the log quotient as 9 − 10), the decimal point must precede the first digit

EXERCISE 9 8

Using logarithms, find the quotient of the following

1. 43 83 − 7 269	11. 37 258 − 84 811
2. 583 6 − 87 64	12. 148 36 − 72 504
3. 2,723 − 432 2	13. 97 562 − 8 3456
4. 3 826 − 12 72	14. 32 118 − 64 554
5. 41 62 − 587 2	15. 2 3269 − 0 81712
6. 2 781 − 138 6	16. 21 801 − 0 52816
7. 4 112 − 0 3472	17. 0 63628 − 0 0051378
8. 0 06528 − 0 8867	18. 0 0013271 − 0 62683
9. 0 08931 − 0 0009845	19. 27 365 − 0 0087629
10. 300 4 − 7,006	20. 5,687 3 − 8,856 4

Raising a number to a power

The process of raising a number to a power is called *involution* It is not much of a trick to raise a number such as 2 to the sixth power $2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$ For larger numbers, the long laborious process can be simplified by the use of logarithms Since the multiplication of numbers is carried out by adding logarithms, the logarithm of a number raised to a power is the product of the logarithm of the number and the exponent of the number Thus if the logarithm of 42 is 1 623249, the logarithm of 42^2 would be twice as much, and of 42^6 just 6 times as much Orderly presentation is equally important in this type of problem

Illustrations

a Using logarithms, find 42^3

Layout	*Layout filled in*
log 42 = 1	log 42 = 1 623249
log 42^3 = 3 × log 42 =	log 42^3 = 3 × log 42 = 4 869747
answer =	answer = 74,088

b. Using logarithms, find 3.824^6.

Layout	*Layout filled in*
$\log 3.824 = 0.$	$\log 3.824 = 0.582518$
$\log 3.824^6 = 6 \times \log 3.824 =$	$\log 3.824^6 = 6 \times \log 3.824 = 3.495108$
answer $=$	answer $= 3{,}126.9$

c. Using logarithms, find $(0.05158)^4$.

Layout	*Layout filled in*
$\log 0.05158 = 8. \quad -10$	$\log 0.05158 = 8.712481 - 10$
$\log (0.05158)^4$	$\text{Log } (0.05158)^4$
$= 4 \times \log 0.05158 =$	$= 4 \times \log 0.05158 = 34.849924 - 40$
answer $=$	$= 4.849924 - 10$
	(since $34 - 40 = -6 = 4 - 10$)
	answer $= 0.0000070782$

EXERCISE 9.9

Using logarithms, find the following:

1. 2.272^5
2. 3.184^4
3. 2.837^3
4. 7.744^2
5. 187.8^2
6. $(0.4388)^4$
7. $(0.06172)^3$
8. $(0.7727)^5$
9. $(0.33826)^4$
10. $(0.023258)^2$

Extracting the root of a number

The inverse of raising a number to a power is extracting the root of a number. The process of extracting the root of a number is called *evolution*. Through the use of logarithms, it is a relatively simple process, since to extract the root of a number it is necessary only to divide the logarithm of the number by the index of the root and then find the antilogarithm of the quotient.

Illustration: Using logarithms, find $\sqrt[3]{729}$.

Layout	*Layout filled in*
$\log 729 = 2.$	$\log 729 = 2.862728$
$\log \sqrt[3]{729} = \dfrac{\log 729}{3} =$	$\log \sqrt[3]{729} = \dfrac{\log 729}{3} = 0.954243$
answer $=$	answer $= 9.0000$

It is desirable to have the characteristic of a logarithm either a positive integer or a positive integer minus 10. Hence when the logarithm of a

decimal is to be divided, the characteristic may be changed so that the quotient will be a positive integer minus 10 For example, the logarithm for the number 0 007483, with a characteristic of -3, can be written in any of the following ways

7 874076 -10	27 874076 -30
17 874076 -20	37 874076 -40

In extracting the root of a number with a negative characteristic, change the minus 10 of the characteristic by multiplying it by the index of the root to be taken, and make the corresponding change in the positive integer of the characteristic Thus, if the square root of 0 007483 is taken the logarithm is written 17 874076 $-$ 20, since 2 (the index) $\times$ 10 = 20 if the cube root is taken, it is written 27 874076 $-$ 30, since 3 (the index) $\times$ 10 = 30

Illustration Using logarithms, find $\sqrt[4]{0\ 007483}$

Layout

$$\log 0\ 007483 = 7 \qquad -10$$

$$\log \sqrt[4]{0\ 007483} = \frac{\log 0\ 007483}{4}$$

$$=$$

$$=$$

answer =

Layout filled in

$$\log 0\ 007483 = 7\ 874076 - 10$$

$$37\ 874076 - 40$$

$$\log \sqrt[4]{0\ 007483} = \frac{\log 0\ 007483}{4}$$

$$= \frac{37\ 874076 - 40}{4}$$

$$= 9\ 468519 - 10$$

answer = 0 29112

EXERCISE 9.10

Using logarithms, find the root of the following

1. $\sqrt[3]{438\ 4}$
2. $\sqrt[4]{327\ 6}$
3. $\sqrt{2,387}$
4. $\sqrt[5]{38\ 628}$
5. $\sqrt[10]{1,826\ 6}$
6. $\sqrt{0\ 6682}$
7. $\sqrt[4]{0\ 3327}$
8. $\sqrt[3]{0\ 05342}$
9. $\sqrt[12]{0\ 006687}$
10. $\sqrt[20]{0\ 00055224}$

Using logarithms to find an exponent

Logarithms are particularly useful in working problems in compound interest Often, for example, the value of a number raised to a power is known without the power being known By the use of logarithms it is

possible to find the exponent of a number. In solving for the exponent, logarithms are treated as logarithms part of the time and as actual numbers part of the time.

(a) If it is known that $(1.05)^n = 1.3401$, find the value of n.

To find the log of a number raised to a power, multiply the log of the number by the power. Thus it is known that

$$n \log 1.05 = \log 1.3401$$

$$n = \frac{\log 1.3401}{\log 1.05}$$

Substituting the values of the logs,

$$n = \frac{0.127134}{0.021189} = 6$$

Particular attention should be given to this example, since here the logarithm of one number is divided into the logarithm of another to find the exponent.

(b) If 1.02 raised to the nth power is equal to 2.208, what is n?

$$n \log 1.02 = \log 2.208$$

$$n = \frac{\log 2.208}{\log 1.02} = \frac{0.343999}{0.008600} = 40$$

Another type of problem often encountered which can be solved most readily by the use of logarithms is finding the rate of interest when money is invested at a compound rate. Compound interest is discussed in Chapter 12, but the algebra used in some of the solutions is first introduced at this point.

Illustration: Find i in the equation $\$1,250.23 = \$1,000\ (1 + i)^{15}$.

Transposing and dividing

$$(1 + i)^{15} = \frac{\$1,250.23}{\$1,000}$$

Stating the relationships in terms of logarithms

$$15 \log (1 + i) = \log 1,250.23 - \log 1,000$$

$$\log (1 + i) = \frac{\log 1,250.23 - \log 1,000}{15}$$

Up to this point no log need actually to be written. Filling in the value of the logs on the left side,

$$\begin{array}{lr} \log 1,250.23 = & 3.096990 \\ \log 1,000 \quad = & \underline{3.000000}\ (-) \\ & 0.096990 \end{array}$$

That is,

$$\log(1+i) = \frac{0\ 096990}{15} = 0\ 006466$$

If the log $(1 + i) = 0\ 006466$, then the antilog of 0 006466 must equal $(1 + i)$ Thus $1 + i = 1\ 015$, or $i = 1\ 5\%$ or $1\frac{1}{2}\%$

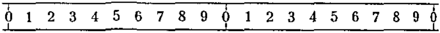
EXERCISE 9.11

Solve the following

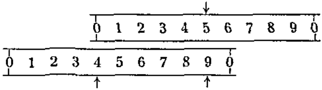
1. Find n, given $(1\ 035)^n = 2\ 6202$
2. Find n, given $(1\ 05)^n = 1\ 407$
3. Find i, given $(1 + i)^8 = 1\ 9206$
4. Find i, given $(1 + i)^{12} = 2\ 2522$
5. Find n, if $(1\ 09)^n = 3\ 970$
6. Find i, if $(1 + i)^{14} = 2\ 2609$

The theory of the slide rule

The theory of the slide rule is easily illustrated by the use of two ordinary rulers If one ruler 10 inches long is placed end to end with

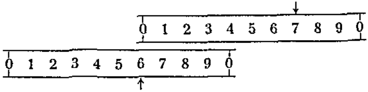


another ruler 10 inches long, the total distance covered is 20 inches Knowing this, it is easy to illustrate addition by the use of the two 10-inch rulers If 4 and 5 are to be added, the two rulers are placed in this position

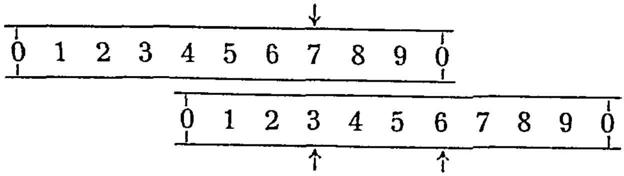


That is, to add 4 and 5, set the left end 0 of the upper ruler over the 4 of the lower ruler Below the 5 on the upper ruler read the answer, 9 on the lower ruler

If 6 and 7 are to be added, this procedure cannot be followed since the 7 would be beyond the end of the lower ruler This difficulty can be



avoided by putting the right end 0 (really 10) of the upper ruler over the 6 on the lower ruler. The sum of the two numbers, 13, can be read on



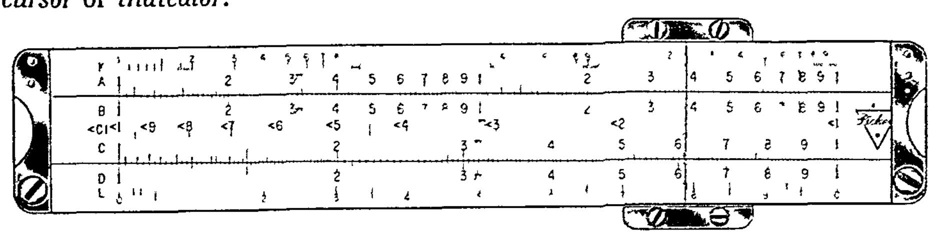
the lower ruler if it is assumed that both rulers run from 0 to 10 (right end 0), but that in going over them the second time the left 0 becomes 10, the 1 becomes 11, the 2 becomes 12, and so on. Thus the 3 on the lower ruler is really 13.

It is also possible to carry out subtraction by using the two rulers. To find the difference between 9 and 5, set the 5 on the upper ruler over the 9 on the lower ruler. The left 0 on the upper ruler will be over the difference, 4, on the lower ruler. The setting looks exactly as it did when 5 was added to 4. This is because subtraction is the inverse of addition.

A standard slide rule uses exactly the same procedure just shown for two ordinary rulers. Addition on ordinary rulers is multiplication on a standard slide rule; subtraction on ordinary rulers is division on a standard slide rule, since the scale shown on slide rules represents logarithms.

The slide rule

A slide rule is a mechanical device which employs the principle of logarithms. It is used to simplify computation. It consists essentially of three parts. The fixed part is called the *stock*; the movable center section is called the *slide*; and the movable glass with the fine line is called the *cursor* or *indicator.*



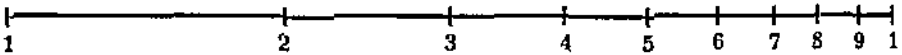
Along the top of the stock is the A scale. Immediately below it on the slide is an identical scale marked B. In the middle of the slide is a scale marked CI. At the bottom of the slide is a scale marked C; and immediately below it on the stock is an identical scale marked D. The C and D scales are used in multiplication and division.

The use of the slide rule hinges on the principles of addition and subtraction illustrated with two ordinary rulers To make it possible however, to carry on multiplication, division, involution, and evolution the scales are based on logarithms

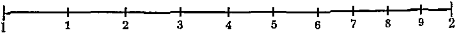
The following three sets of figures show the logarithms of numbers to three decimal places

log 1 = 0 0	log 10 = 1	log 100 = 2
2 = 0 301	20 = 1 301	200 = 2 301
3 = 0 477	30 = 1 477	300 = 2 477
4 = 0 602	40 = 1 602	400 = 2 602
5 = 0 699	50 = 1 699	500 = 2 699
6 = 0 778	60 = 1 778	600 = 2 778
7 = 0 845	70 = 1 845	700 = 2 845
8 = 0 903	80 = 1 903	800 = 2 903
9 = 0 954	90 = 1 954	900 = 2 954
10 = 1 000	100 = 2 000	1,000 = 3 000

If a ruler were selected which had 1,000 subdivisions the numbers 1 to 10 being represented by distances corresponding to their logarithms the end of the ruler marked 1 called the *left index*, would represent the log of 1 There is no 0 on a logarithm scale The number 2 would appear at the 301 subdivision, or about one third of the length of the ruler 3 would be at the 477 point, or about half the length, and so on until 10 or 1 appeared at the right end of the ruler The right end of the ruler called the *right index*, represents the log of 10 These relationships are shown in the following figure



In the discussion of logarithms, it was emphasized that the mantissa of a series of digits depends on the series of digits, not on the decimal point Thus the left index 1 can represent log 1 or log 10 or log 100, etc , and correspondingly, the right index can represent log 10 or log 100 or



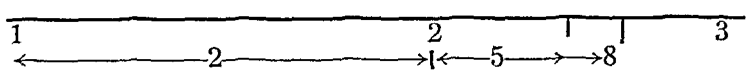
log 1,000, etc Because this is true, secondary divisions are introduced into the scale The secondary divisions between 1 and 2 are shown in the following figure If the left index is considered as log 10, then the 2 represents log 20, and the intermediate numbered divisions represent log 11, log 12, log 13, log 14, etc

Slide rules are commonly available in lengths of 5 inches (really 125 millimeters) and 10 inches (really 250 millimeters) The longer rule has

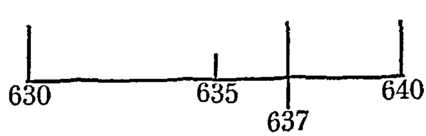
more subdivisions than the shorter one. The following discussion pertains to the 10-inch rule on which the secondary divisions are usually numbered between 1 and 2. Further subdivisions, known as *tertiary* divisions, are also shown. Between 1 and 2, the secondary divisions are each subdivided into 10 tertiary parts. They, too, represent logarithms and consequently are not equidistant.

The various positions, known as prime, secondary, and tertiary, indicate the power or order of digits represented on the index. Thus if the left index is considered as 1 and the right as 10, the numbers represented are all of the *first* order (1, 2, 3, 4, 5, 6, 7, 8, 9, 10). If the left index is considered as 10 and the right as 100, then the numbers from 10 to 100 (that is, numbers of the *second* order) are represented.

When any number with three significant digits is to be located on the C or D scale, the left index represents 100, and the right index represents 1,000. A number such as 258 (or 25.8 or 2,580 or 0.0258, etc.) lies between the *big* 2 and the *big* 3. Each digit in turn from left to right represents a movement to the right from the left index. The left digit of 258, known as the *prime* digit, is the 2 (of 258) and is represented by the distance from the left index to the *big* 2 on the scale. The middle digit, known as the *secondary* digit, is the 5 (of 258) and is the distance between the *big* 2 and the fifth long line between the *big* 2 and the *big* 3 on the scale. The right digit, known as the *tertiary* digit, is the 8 (of 258) and is the distance between the fifth long line just referred to and the fourth short line between this fifth long line and the next long line.



There should be no difficulty in picking out the position of the primary and secondary digit lines, but when one moves from left to right, many tertiary digital positions are not shown as distances get shorter. Between 1 and 2, where the secondary spaces are long, all third-digit positions are recorded; between 2 and 4, the *even* third-digit positions are marked, and the odd third digits must be approximated as about halfway between the adjacent third-digit positions. Between 4 and the right index, there are only 20 divisions in each of the prime spaces. Each division is equal to 5, and any other third-digit figures must be approximated. Thus 637 is about two-fifths of the way from 635 to 640.



To use the indicator to locate 637 on the D scale (which is exactly the same as the C scale), move the indicator over to 600 (the *big* 6), then move to the right to 630 (the third *long* line between the *big* 6 and the *big* 7), then move to the right to 635 (the only *short* line between the third and fourth *long* lines), and finally approximate 637 as about two-fifths the way between 635 and 640 The more skilled one is at approximating a position on the slide rule, the more accurate his work

EXERCISE 9.12

Locate the following numbers on the C or D scale

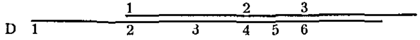
1. 4, 6, 8, 7, 5
2. 10, 12, 14, 25, 68
3. 127, 225, 350, 775, 835
4. 437, 482, 523, 617, 808
5. 501, 682, 403, 998, 821
6. 400, 420, 425, 427, 428
7. 700, 750, 752, 754, 757
8. 300, 302, 303, 307, 309
9. 100, 102, 120, 107, 170
10. 900, 960, 965, 967, 968

Multiplication, using the slide rule

The C and D scales, which are alike, are used together to find the product of one number times another For example, find 2×3 By logarithms,

$$\begin{aligned} \log 2 &= 0\ 301030 \\ \log 3 &= 0\ 477121 \quad (+) \\ \log \text{product} &= 0\ 778151 \\ \text{product} &= 6 \end{aligned}$$

That is, $\log 2 + \log 3 = \log 6$ On the slide rule, this addition is done by adding the distance from 1 to 3 (which represents log 3) to the distance from 1 to 2 (which represents log 2) Since the prime division mark 3 on



the C scale represents log 3 and the prime division mark 2 on the D scale represents log 2, put the left index on the C scale over the *big* 2 on the D scale Then under the *big* 3 on the C scale, read 6 (the *big* 6) on the D scale

As a matter of fact, when the left index of the C scale is over the *big* 2 on the D scale, each number on the C scale is directly over the product of itself and 2 on the D scale Thus 2 on C is over 4 on D, 3 on C is over 6 on D, 4 on C is over 8 on D

To find the product of 25 and some number, place the left index over the fifth long line between the *big* 2 and the *big* 3 on the D scale which represents 25. Again every number on the C scale is directly over the product of itself and 25. For example, the second long line between the *big* 1 and the *big* 2, representing 12, is directly over the *big* 3. At this point the big numbers on the D scale are considered to be of the third order and represent values from 100 to 1,000. The *big* 3 represents 300. That is, $25 \times 12 = 300$. The product of 25×25 is 625 since the 25 on the C scale is directly over 625.

The product of two three-digit numbers is found in a similar manner.

Illustration: Find 1.82×3.24.

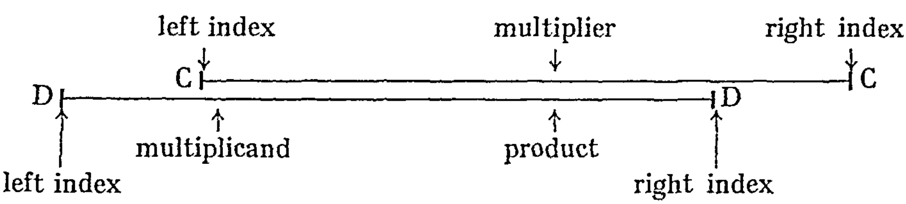
Put the left index of the C scale over 182 on the D scale; under 324 on the C scale read the product 590 on the D scale. That is, $1.82 \times 3.24 = 5.90$.

Frequently when the left index of the C scale is put over the multiplicand, the multiplier, which is on the C scale, is beyond the range of the D scale; that is to say, the D scale is not long enough. When this occurs, put the right index of the C scale over the multiplicand on the D scale and read the product on the D scale under the multiplier on the C scale.

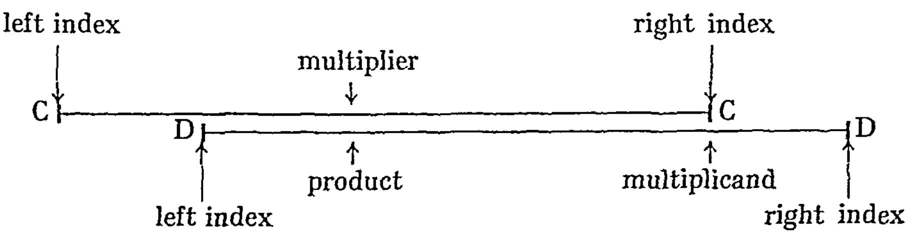
Illustration: Using a standard slide rule, find 8.35×5.32.

Put the right index of the C scale over 835 on the D scale; under 532 on the C scale, read the product 445 on the D scale. That is, $8.35 \times 5.32 = 44.5$.

The following diagrams show how to use the standard slide rule for multiplication.



Or



To find the product of two numbers, such as 38.4 and 52.6, with the standard slide rule, determine the location of the decimal point in the product by estimating the product ($40 \times 50 = 2{,}000$, the estimated product), and proceed as if the problem were 384×526

EXERCISE 9.13

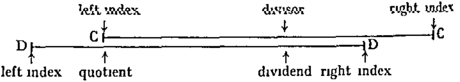
Find the following products with a standard slide rule Determine the position of the decimal point in the answer by finding each estimated product

1. 2×4
2. 8×12
3. 6×9
4. 25×40
5. 52×48
6. 17×120
7. 48.3×7.27
8. 58.2×10.7
9. 63.6×9.85
10. 122×6.03
11. 487×51.2
12. 33.8×0.552
13. 437×0.0406
14. $5{,}680 \times 0.00156$
15. 0.731×0.682
16. 0.0902×0.0552
17. $23{,}800 \times 0.000753$
18. 456×827
19. 0.00824×0.0915
20. 17.8×0.0000506

Division, using the slide rule

Division is the inverse of multiplication In multiplication, the multiplicand times the multiplier equals the product In division, the quotient times the divisor equals the dividend That is, $6 \div 3 = 2$, and $2 \times 3 = 6$ Therefore, using the indicator, put the divisor on the C scale over the dividend on the D scale, read the quotient on the D scale under whichever index (left or right) is within the range of the D scale

In other words, if the dividend and the divisor are set opposite each other, the quotient appears opposite the index on the same scale as the dividend This is shown on the following diagram



Or

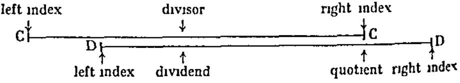


Illustration: Find the quotient of two three-digit numbers, $8.35 \div 5.28$.

Using the indicator, set 528 on the C scale over 835 on the D scale; under the left index on the C scale, read the quotient 158 on the D scale. That is, $8.35 \div 5.28 = 1.58$.

To find the quotient of two numbers such as 38.4 and 526 with the standard slide rule, determine the location of the decimal point by estimating the quotient as discussed in chapter 4 ($40 \div 500 = 0.40 \div 5 = 0.08$, the estimated quotient), and proceed as if the problem were $384 \div 526$.

EXERCISE 9.14

Find the following quotients with a standard slide rule. Determine the position of the decimal point in the answer by finding each estimated quotient.

1. $18 \div 3$
2. $12 \div 4$
3. $52 \div 13$
4. $44 \div 8$
5. $84 \div 12$
6. $14 \div 20$
7. $39 \div 52$
8. $43 \div 12$
9. $62 \div 22$
10. $35 \div 15$
11. $8.35 \div 3.27$
12. $12.8 \div 7.24$
13. $51.6 \div 21.3$
14. $44.4 \div 6.66$
15. $83.2 \div 5.16$
16. $14.9 \div 20.8$
17. $38.6 \div 52.9$
18. $43.3 \div 127$
19. $62.2 \div 227$
20. $3.28 \div 50.6$
21. $0.827 \div 5.28$
22. $0.0337 \div 2.08$
23. $61.6 \div 0.582$
24. $8.28 \div 0.0617$
25. $0.432 \div 0.781$
26. $0.00568 \div 0.0145$
27. $0.0428 \div 0.00689$
28. $3.68 \div 0.00854$
29. $34.8 \div 67{,}800$
30. $268{,}000 \div 4{,}260$

Proportion, using the slide rule

Any standard slide rule is a proportion rule. For example, when 1 (the left index) on the C scale is put over 2 on the D scale, then 2 on the C scale is over 4 on the D scale, and 3 on the C scale is over 6 on the D scale, etc., since $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$, etc. That is, all the values directly opposite each other on the C and D scales are in proportion to the original pair of values.

A simple way to solve a problem in proportion by the use of a standard slide rule is to set the numerators on the C scale over their respective denominators on the D scale.

Illustrations:

a. $\dfrac{x}{25} = \dfrac{48}{60}$; $x = 20$

By the use of the indicator, put 48 on the C scale over 60 on the D scale; read the answer 20 on the C scale over 25 on the D scale.

16. $\frac{14.9}{38.3} = \frac{x}{27.7}$

17. $\frac{8.38}{5.82} = \frac{29.3}{x}$

18. $\frac{x}{3.84} = \frac{87.3}{257}$

19. $\frac{128}{x} = \frac{48.3}{132}$

20. $\frac{88.3}{x} = \frac{2.27}{18.3}$

21. $\frac{18.6}{x} = \frac{487}{28.4}$

22. $\frac{x}{3.18} = \frac{18.4}{427}$

23. $\frac{x}{0.227} = \frac{11.4}{7.82}$

24. $\frac{0.0297}{x} = \frac{41.6}{384}$

25. $\frac{27.8}{3.42} = \frac{5{,}830}{x}$

26. $\frac{7.84}{0.0528} = \frac{x}{5.84}$

27. $\frac{0.0563}{0.00728} = \frac{x}{0.518}$

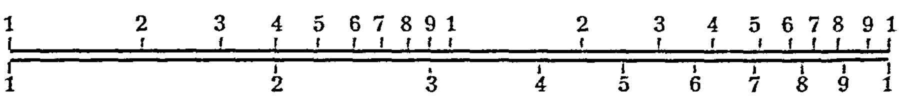
28. $\frac{22.7}{843} = \frac{0.684}{x}$

29. $\frac{3.03}{18.0} = \frac{x}{428}$

30. $\frac{22{,}800}{7{,}430} = \frac{x}{41.2}$

Finding squares and square roots with a slide rule

If the slide is completely removed from a standard slide rule, the relationship between the A and D scale can be readily seen. The A scale repeats the D scale, each time in half the space. Thus the distance representing the logarithm of a number is twice as great on the D scale as on the A scale. Or, twice the logarithm of any number on the D scale is the same as the logarithm of the corresponding number on the A scale (see accompanying diagram).



If the indicator is put over 2 on the D scale, the indicator is also over 4 on the A scale; if the indicator is put over 4 on the D scale, the indicator is also over 16 on the A scale. That is, since $2^2 = 4$ and $4^2 = 16$, any value on the A scale is the square of the corresponding value on the D scale. In like manner, any value on the B scale is the square of the corresponding value on the C scale. Care must be taken in getting the proper answer from the A scale, since its divisions are only half as long as those on the D scale, and since it has fewer tertiary lines. Note that all the tertiary lines between 6 and 1 (the middle index) and between 6 and 1 (the right index) are missing.

Illustration: Find 5.28^2 by using a standard slide rule. Put the indicator over 528 on the D scale; read 280 on the A scale under the indicator. That is, $5.28^2 = 27.8$.

Any value on the D scale is the square root of the corresponding value on the A scale. Since the two halves of the A scale look exactly alike, however, it is necessary to know the proper position to be used on the A scale. This can be determined as follows:

b $\frac{47.8}{x} = \frac{81.6}{63.4}$, $x = 37.1$

By the use of the indicator, put 816 on the C scale over 634 on the D scale, read the answer on the D scale under 478 on the C scale

c $\frac{x}{87.2} = \frac{67.5}{258}$, $x = 22.8$

By the use of the indicator, put 675 on the C scale over 258 on the D scale Since the C scale is not over 872 on the D scale, interchange the indexes of the C scale by setting the indicator over the right index of the C scale, and moving the slide to the right until the left index of the C scale is under the indicator Then read the answer 228 on the C scale over 872 on the D scale To summarize this last illustration

Step 1 First set 675 on the C scale over 258 on the D scale

Step 2 Move the indicator to the right index of the C scale

Step 3 Move the slide until the left index of the C scale is under the indicator

Step 4 Read the answer 228 on the C scale directly over 872 on the D scale

It is important that the decimal point in the answer be correctly located In proportion it is not possible to estimate the size of the answer as is done in multiplication and division Instead, it is necessary to compare the size of the numbers in the ratio where both parts are known, and make a similar comparison in the other ratio In Illustration b, since 63 4 is slightly less than 81 6, x must be slightly less than 47 8 In Illustration c, since 67 5 is about one-fourth of 258, x is about one fourth of 87 2

EXERCISE 9.15

Find the unknown in each of the following proportions using a standard slide rule

1. $\frac{2}{4} = \frac{x}{6}$
2. $\frac{7}{8} = \frac{x}{25}$
3. $\frac{x}{5} = \frac{3}{8}$
4. $\frac{7}{x} = \frac{5}{12}$
5. $\frac{21}{32} = \frac{9}{x}$
6. $\frac{82}{56} = \frac{25}{x}$
7. $\frac{x}{38} = \frac{84}{48}$
8. $\frac{13}{x} = \frac{42}{135}$
9. $\frac{563}{728} = \frac{x}{518}$
10. $\frac{178}{x} = \frac{252}{347}$
11. $\frac{38.6}{47.4} = \frac{x}{61.8}$
12. $\frac{42.2}{89.3} = \frac{31.8}{x}$
13. $\frac{27.9}{x} = \frac{8.28}{9.93}$
14. $\frac{47.2}{61.3} = \frac{x}{18.8}$
15. $\frac{x}{21.7} = \frac{4.07}{7.29}$

16. $\dfrac{14.9}{38.3} = \dfrac{x}{27.7}$

17. $\dfrac{8.38}{5.82} = \dfrac{29.3}{x}$

18. $\dfrac{x}{3.84} = \dfrac{87.3}{257}$

19. $\dfrac{128}{x} = \dfrac{48.3}{132}$

20. $\dfrac{88.3}{x} = \dfrac{2.27}{18.3}$

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22. $\dfrac{x}{3.18} = \dfrac{18.4}{427}$

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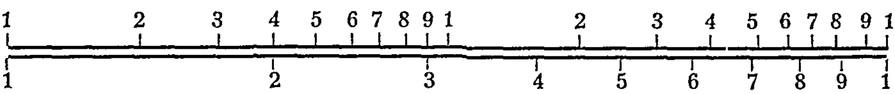

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Illustration: Find 5.28^2. by using a standard slide rule. Put the indicator over 528 on the D scale; read 280 on the A scale under the indicator. That is, $5.28^2 = 27.8$.

Any value on the D scale is the square root of the corresponding value on the A scale. Since the two halves of the A scale look exactly alike, however, it is necessary to know the proper position to be used on the A scale. This can be determined as follows:

1 Separate the number into groups of two figures each, beginning at the decimal point and moving in both directions

2 If the first group farthest to the left has *one* significant number, set the indicator over the number on the left half of the A scale and read the root on the D scale If the first group has two significant numbers, set the indicator over the number on the right half of the A scale and read the root on the D scale

3 Place the decimal point in the root by following the rule that the square root of a number has as many places as there are groups

Illustrations

a Find the square root of 625, using a standard slide rule

From the decimal point, mark off the digits in groups of two, as follows 6 25 Since there is only *one* digit in the group farthest to the left, use the left half (*first* half) of the A scale, and read the root on the D scale The result is $\sqrt{6\ 25} = 2\ 5$, or more clearly, $\sqrt{625} = 25$ That is, there are as many digits to the left of the decimal point in the answer as there are groups to the left of the decimal point in the number whose root is being taken

b Find the square root of 0 00837, using a standard slide rule

From the decimal point mark off the digits in groups of two until a group is reached containing one or two significant digits, as follows 0 00 83 7 The first group containing at least one significant digit has *two* of them Therefore use the right half (the *second* half) of the A scale and read the root on the D scale The result is $\sqrt{0\ 00\ 83\ 7} = 0\ 0\ 9\ 15$ or more clearly, $\sqrt{0\ 00837} = 0\ 0915$ That is, in getting the square root of a decimal, the number of zeros between the decimal point and the first significant digit must be determined In this illustration, one group contains zeros only, so there is one zero between the decimal point and the first significant digit

EXERCISE 9.16

Using a standard slide rule, square the following

1. 15, 22, 35, 43, 8 5, 3 25, 7, 9
2. 2 5, 5 8, 5 25, 5 28, 8 37, 9 40
3. 208, 337, 548, 382, 917, 1,450
4. 2 78, 3 84, 5 83, 8 73, 9 19
5. 0 0525, 0 0874, 0 236, 0 378, 0 889

Using a standard slide rule, find the square root of the following:

6. 144; 169; 289; 529; 1,024; 1,600
7. 3.87; 6.72; 9.28; 21.6; 34.8; 61.6
8. 4,780; 26,800; 245,000; 769; 6,620
9. 0.625; 0.772; 0.529; 0.807; 0.952; 0.0653
10. 0.0125; 0.0542; 0.0000529; 0.0000428; 0.00000729

REVIEW PROBLEMS

Chapters 8 and 9

Find the value of the unknowns, and check the solution.

1. $3x - y = 3$
   $x + y = 5$
2. $x - 3y = 1$
   $2x + y = 9$
3. $2x + 3y = -1$
   $3x - y = -7$
4. $2x - 5y = 16$
   $3x - 2y = 13$
5. $x + 4y = 14$
   $4x + y = 11$
6. $2x + 3y = 4$
   $2x - 3y = 16$
7. $3x - 2y = 3$
   $4x - 3y = 3$
8. $3x - 8y = 2$
   $5x + 3y = -13$
9. $3x + 2y = 8$
   $3x + y = 10$
10. $y = 3x$
    $5x - 2y + 2 = 0$
11. $2x = 5y - 1$
    $3x + y = 7$
12. $4x = 7y - 6$
    $x + 3y = 8$
13. $3x = 7 + 8y$
    $x = 8 - 3y$
14. $7x = y + 17$
    $x = 8 + 2y$
15. $x + 2y - z = -1$
    $2x + y + 2z = 9$
    $x - y + z = 6$
16. $2x + 3y - z = 2$
    $3x + y - 2z = -5$
    $x - 2y + 2z = -1$
17. $2x - 5y + 3z = 8$
    $3x - 2y + z = 7$
    $x - 4y + 2z = 4$
18. $x = 3y - 2z$
    $y = 2x - 3z + 2$
    $z = x + y - 1$
19. $x + z = 4$
    $y + z = 5$
    $x + y = 5$
20. $x + 2y - 2z = -2$
    $2x - 3z = 7$
    $3y - z = -8$

Find the values of the unknowns, and check the solution.

21. $x^2 + x = 12$
22. $x^2 + 2x = 15$
23. $x^2 = 7x - 12$
24. $x^2 = 7x - 10$
25. $2x^2 = 9x - 4$
26. $3x^2 + x = 2$
27. $3x^2 = 2x + 8$
28. $4x^2 + 5x = 6$
29. $2x^2 + 7x = 15$
30. $3x^2 = 11x + 4$
31. $6x^2 + 6 = 13x$
32. $4x^2 = 12x - 5$
33. $8x^2 + 10x = 3$
34. $6x^2 = 1 - x$
35. $8x^2 = 26x - 15$
36. $x^2 = 5x + 9$
37. $x^2 + 3x = 8$
38. $x^2 = 5x + 2$
39. $2x^2 - 3x = 8$
40. $2x^2 = 12 - 5x$

41. The sum of two numbers is 24, their difference is 4. Find the numbers.

42. Twice the larger number exceeds 3 times the smaller number by 1, but 5 times the smaller number exceeds 3 times the larger number by 1. Find the numbers.

43. The sum of two numbers is 154. The larger number is 24 more than the smaller number. What are the numbers?

44. The difference of two numbers is 35. One of the numbers exceeds twice the other by 5. What are the numbers?

45. In 5 years Jeffrey will be twice as old as Ellen is now. Thirteen years from now he will be 3 times as old as she is now. How old is each?

46. Harry is 3 times as old as Richard was 22 years ago. In 4 years Harry will be twice as old as Richard was 7 years ago. How old is each?

47. An expedition started at the rate of 20 miles per hour. A messenger with instructions to overtake the expedition started 5 hours later from the same point and traveled at the rate of 40 miles per hour. In how many hours will he overtake it?

48. A flight of bombers started on a flight at the rate of 450 miles per hour. An escorting group of fighters took to the air 20 minutes later at the rate of 620 miles per hour. How long will the bombers be in the air without protection of the fighters?

49. If 5 is added to both the numerator and the denominator of a certain fraction, the value of the fraction becomes $\frac{4}{7}$. If 5 is subtracted from both the numerator and the denominator, the value of the fraction becomes $\frac{1}{2}$. Find the fraction.

50. If 4 is subtracted from both the numerator and the denominator of a fraction, the new fraction is $\frac{1}{2}$. If 2 is added to both the numerator and the denominator, the new fraction is $\frac{3}{4}$. Find the fraction.

51. Item A sells for 50 cents and item B for 80 cents. If 1,000 items are sold for \$710, how many of each are sold?

52. A wholesale druggist receives an order for 100 gallons of 80% alcohol. He has two kinds in stock; one is 100% and the other is 75%. How many gallons of each should he mix to fill the order?

53. The Friends of Music, a nonprofit corporation, sell concert tickets to students for \$1.00; to others for \$1.50. The figures for the last concert show that attendance was 900, and receipts were \$1,050.00. How many tickets of each type were sold?

54. If 70 feet of V-type ditch and 100 feet of 8 inch pipe cost \$585, and 50 feet of V-type ditch and 80 feet of 8 inch pipe cost \$459, what is the cost per foot of each?

55. On the sale of 5 lots for $10,000 each, and 4 lots for $5,000 each, the seller's escrow fee was $214 00 On the sale of 3 lots for $10,000 each and 2 lots for $5,000 each, he paid total escrow fees of $120 How much was the seller s escrow fee on each $5,000 and on each $10,000 lot?

56. In one month the escrow department of a bank had a total income of $700 for drawing 200 deeds and 100 mortgages The next month the combined income was $1,100 for drawing 250 deeds and 200 mortgages What charge was made for drawing each deed and each mortgage?

57. An automobile supply dealer buys tires for $284 Some cost him $12 each, the balance cost him $14 each When he sold them all at $20 apiece, he cleared $156 How many of each price did he buy?

58. A motorist drove 560 miles at a certain rate If he had driven 5 miles per hour faster, his time would have been 2 hours less Find the rate

59. A person traveled 500 miles in his automobile If his average speed had been 10 miles per hour slower, his time for the trip would have been $2\frac{1}{2}$ hours longer What was his average speed?

60. An airplane made a trip of 600 miles against a head wind in 2 hours 30 minutes It returned with the wind in 1 hour 40 minutes Find the speed of the plane, and the velocity of the wind?

61. The combined weight of the luggage of two airline passengers was 100 pounds One paid $2 50 for excess weight, the other paid $7 50 Had the luggage belonged to one man he would have had to pay $30 00 How many pounds of luggage was each passenger permitted to carry without charge?

62. A broker sold two pieces of real estate for $21,000 His commission on the first sale was 4% and on the second 8% What was the price of each if his total commission was $1,080?

63 A merchant sold two television sets for $410 His profit on the first set was 25% and on the second was 30% Find the selling price of each set if his total profit was $114

64. Last week the bulldozer operator was paid $149 50 for 48 hours, this week he was paid $132 25 for 44 hours How much is he paid for a 40-hour week and how much per hour overtime?

65. A part of $15,000 is invested at 4%, a part at 6%, and the balance at 5% The annual income is $760 If the amount at 4% were invested at 5%, and if the amount now at 5% were invested at 4%, the total income would be reduced by $20 Find the amount invested at each rate

66. A company has three branch plants When plants I and II are in operation their production is 55% of the total When plants II and III are in operation their production is 75% of the total When plants I and III are in operation their production is 70% of the total What fractional part of total capacity does each plant have?

67. An investor bought some stock which he subsequently sold at a loss. He immediately reinvested the money in other stocks which increased in price, and on their sale he recouped his loss. His per cent of gain on the second sale was 5% more than his per cent of loss on the first one. Find his per cent of loss on the first sale.

68. Find three consecutive integers, the sum of whose squares equals 110.

69. An investor bought a number of shares of stock for $1,200. If the stock had cost $5 a share more, he would have obtained 10 less shares for the same money. How many shares did he buy?

70. If a job is allocated to Department A it can be done in 7 days less than if it is given to Department B. If the job is divided between A and B it can be completed 9 days earlier than if A does it alone. How long will the job require if it is assigned to both departments?

71. An investor bought some shares of stock for $20,000. By selling all but 100 shares for the same amount he made $10 a share on the stock sold. How many shares did he buy?

72. A wholesaler adds a certain percentage to the manufacturer's price when he sells to the retailer. The retailer adds 5 times this percentage to the wholesaler's price when he sells to the consumer. If the price to the consumer is 65% more than the manufacturer's price, what percentage did the wholesaler add?

73. An investor used $12,500 to buy some shares of stock at $90, on which annual dividends of $4 are paid, and some $1,000 bonds which pay interest at 5%. The annual income on the fund is $600. How many shares of stock and how many bonds did he buy?

74. Three quarts of paint thinner are mixed with 10 quarts of logwood oil. How much paint thinner must be added to get a mixture that is $\frac{2}{3}$ logwood oil?

75. In one government bureau there are 10 employees. Those classified as A are paid $12,000 a year; those classified as B are paid $8,000; and those classified as C are paid $6,000. How many may the Director hire in each classification if he is allocated $78,000 for salaries and there will be twice as many classified as B than classified as C?

76. Desert Clay Pipe Company makes pipe in three sizes: 4 inch, 6 inch, and 8 inch. The company has no cost accounting system. Their inventories are stable. Hence their manufacturing and sales are just about the same. From their records for the past three months determine the average profit or loss on each type of pipe.

Month	*Number of Feet Sold*			*Net Profit*
	4 inch	6-inch	8-inch	*for Month*
First	8 000	22,000	20,000	$1,840
Second	20,000	25,000	15,000	2,150
Third	12,000	28,000	10,000	2,060

77. The Oil Well Tool Company has a contract with the general manager which provides that he will receive 5% of the profits after federal income taxes have been paid, plus a contribution to the pension fund The contribution to the pension fund is 5% of the profits after the general manager s bonus has been deducted and after the income taxes have been paid Earnings before these allocations were made were $740,000 What amount should be allocated to (a) the general manager s salary, (b) the pension fund, (c) federal income taxes if the rate was 50%?

78. *An actress has a net income before taxes of $46,000 The agent's* fee of 10% and the state income tax of 4% are deductible in computing the federal tax The federal tax and the agent's fee are deductible in computing the state tax, and both are deductible in computing the *agent s fee If the federal tax rate is 40%, find the agent's fee*

79 The Easter Aircraft Manufacturing Company has a contract with the general manager and the sales manager under which the general manager receives a salary equivalent to 20% of profits and the sales manager a bonus of 5% of the profits after deductions of the general manager s salary and the contribution to the pension fund have been made The contribution to the pension fund is equal to 15% of the profits after deduction of the general manager's salary and the bonus to the sales manager What amount should be allocated to the pension fund, the general manager s salary, and the sales manager's bonus if profits before allocations amount to $150,000? Make no allocation for income taxes

80. J A Thompson and Son, Inc have three departments which work on contract jobs, special jobs, and regular jobs Equipment is rented and used in each division and billed as part of the job The men who work for the company are shifted from job to job The only accurate records the company has show the man hours allocated to each type of job and the profit by months Find the average net profit or loss on each man-hour for each type of job

	Man-Hours Devoted to			*Net Profit*
Month	*Contract Jobs*	*Special Jobs*	*Regular Jobs*	*per Month*
First	600	400	1,000	$1,050
Second	300	500	1,200	1,175
Third	100	600	1,500	1,375

81. A factory building contains 105,000 square feet of floor space. The length is 50 feet more than the width. What is the width?

Give the following logarithms to 6 significant figures and antilogarithms to 5 significant figures.

82. log 5,042
83. log 0.0475293
84. log 1.00087
85. log 65,427.9
86. log 0.888601
87. log 269,931
88. antilog 1.965367
89. antilog 7.716090
90. antilog 7.860960 − 10
91. antilog 6.740101 − 10

Solve on a slide rule. Carry answers beginning with "one" to four significant figures.

92. 31.8×7.29
93. 5.24×13.67
94. 749×0.324
95. $0.00870 \times 1{,}153$
96. 0.0206×0.03108
97. 1.523×482
98. 0.00243×402
99. $0.0684 \times 14{,}500$
100. $2{,}560 \times 0.0307$
101. $3.1416 \times 6{,}542$
102. $427 \div 534$
103. $0.0438 \div 4.86$
104. $2.43 \div 1{,}118$
105. $98.3 \div 576$
106. $34.6 \div 0.629$
107. $3.1416 \div 0.762$
108. $0.00460 \div 0.0542$
109. $1{,}244 \div 0.0237$
110. $1.030 \div 1.003$
111. $9{,}456 \div 0.04735$
112. $\dfrac{8.84 \times 462}{0.0868}$
113. $\dfrac{37.8 \times 3.1416}{0.1666}$
114. $\dfrac{0.000272 \times 174.3}{534}$
115. $4.80 : 0.626 :: 68.7 : x$
Find x.

Using the slide rule, find the following square roots.

116. $\sqrt{3.48}$
117. $\sqrt{151}$
118. $\sqrt{84{,}900{,}000}$
119. $\sqrt{0.00627}$
120. $\sqrt{0.0649}$
121. $\sqrt{72.0}$
122. $\sqrt{368}$
123. $\sqrt{9.72}$
124. $\sqrt{86.4}$
125. $\sqrt{0.000748}$

10

Simple Interest and Discount

Introduction

Usually when money is lent, when goods are delivered or when services are performed, an obligation arises on the part of the recipient to pay a fixed sum to the one who lends the money, supplies the goods or performs the service The one to whom final payment must be made is called the *creditor*, and the one who must make the payment is called the *debtor* In normal business relations between commercial concerns, debts are constantly being created and paid with no documentary evidence of debt signed by the debtor For example a wholesaler buys goods from a manufacturer as the need arises and makes payments within a reasonable period after he has received the goods Such a transaction is called an *open account*

In many transactions however, a credit instrument is signed by the debtor as legal evidence of the debt For example, the person who borrows money from a bank signs a *note* under the terms of which he agrees to repay the debt on or before a specified date, and corporations which borrow money for long periods of time furnish the lenders, as evidence of the debt, certificates known as *bonds* Bonds and notes differ primarily in the length of time before the debt must be repaid

The payment of a debt is often postponed for a definite or indefinite period of time by agreement between the debtor and the creditor Since a creditor must forego the use of his funds until the debt is paid he is entitled to some payment for allowing another to use his money He usually expects not only a repayment of the debt but also an additional payment to compensate him for the use of his funds The payment for the use of money is called *interest* The sum of money which is borrowed, *lent, or invested is called the principal* The per cent charged for the use of the principal for one year is called the *rate*, or the *rate of interest* The period for which the interest is paid is called the *time*, or the *term*, of the

loan. The principal plus the interest is called the *amount*. The use of these terms follows closely the pattern already discussed in the chapter on percentage. The interest is the percentage, the principal is the base, the rate of interest times the time or term is the rate per cent, and the amount is the amount.

Simple interest is paid on the principal only. Usually it is charged on loans which extend for only a short period of time, or on the balance of accounts which are soon to be paid. Thus the time or term is usually a fractional part of a year. Simple interest is the product of the principal multiplied by the rate and time.

Illustration: If the rate paid on United States Postal Savings accounts is 2% simple interest, how much interest does the depositor receive on $750 deposited for 1 year?

By paying 2% simple interest, the government is in effect paying the depositor for the use of his money. The interest on $750 for 1 year at the rate of 2% per year is $\$750 \times 0.02 = \$15$. If the deposit, plus the interest, were withdrawn at the end of the year, the sum received, known as the amount, would be $765.

The simple interest formula

All problems in simple interest can be solved by arithmetic, but a knowledge of equations and their applications makes it possible to save much time in solving such problems. If general formulas are developed, it is necessary only to substitute the numerical values for the letters of the formula, and solve. Such formulas for simple interest are more or less standard. The symbols most commonly used are:

P = the principal, or sum of money invested, lent or borrowed;
r = the annual rate of interest charged, stated as a per cent or as the cents paid for the use of $1 for one year;
t = the time, or the term, of the loan, expressed in years, or fractional part of a year;
I = the total interest in dollars and cents;
S = the amount, or the sum, of the principal and interest.

It has already been pointed out that the total interest I is equal to the product of the principal P, times the annual rate of interest r, times the time or term t. That is,

$$I = Prt$$

It will be readily seen that the computation of simple interest is nothing more or less than the application of the principles studied in the chapter

on percentage The problems that occur most frequently in simple interest are those of finding a percentage as the product of a base called P, the principal, and a rate per cent called r, the rate of interest The only modification that occurs is that an additional factor, t, for time, is included

By definition, S, the amount, is equal to the sum of the principal P, and the total interest I That is,

$$S = P + I$$

From the first equation it is seen that I is equal to Prt If Prt is substituted for the value of I in the second equation, the result is

$$S = P + Prt$$

In other words, the amount S, is equal to the principal P, plus the interest I, which is the product of the principal P, times the rate r, times the time t

Since P is common to both terms of the right-hand side of the formula $S = P + Prt$, it can be written

$$S = P(1 + rt)$$

The primary use of this formula is when S, r, and t are known and P is the unknown To state the formula in its most useful form, divide both sides by the coefficient of P, namely, $1 + rt$ Therefore

$$P = \frac{S}{1 + rt}$$

With these basic formulas, any type of simple interest problem can be solved

Illustrations

a Find the simple interest and the amount of \$3,000 for 1 year at 4%

The numerical values are $P = \$3{,}000$, $r = 4\%$, or 0 04, and $t = 1$ The first formula needed is $I = Prt$ Substituting $I = \$3{,}000 \times 0\ 04 \times 1 = \$120$, the total interest To find the amount, the formula needed is $S = P + I$, or $S = \$3{,}000 + \$120 = \$3{,}120$, the amount

b How long must \$800 be invested at 5% simple interest to earn \$120 interest?

If $I = Prt$, then $t = \frac{I}{Pr}$ Here $P = \$800$, $r = 5\%$ or 0 05, and $I = \$120$

Substituting $t = \frac{\$120}{\$800 \times 0\ 05} = 3$ (years)

c. At what rate must \$700 be invested to earn \$42 in 2 years?

If $I = Prt$ then $r = \frac{I}{Pt}$. Here $P = \$700$; $I = \$42$; $t = 2$. Therefore $r = \frac{\$42}{\$700 \times 2} = 0.03 = 3\%$, the rate of interest.

d. What is the value today of \$784 due 2 years hence at 6% simple interest?

Since P is unknown use $P = \frac{S}{1 + rt}$. Here $S = \$784$; $r = 6\% = 0.06$; $t = 2$. Therefore

$$P = \frac{\$784}{1 + 0.06 \times 2} = \frac{\$784}{1.12} = \$700$$

That is, if \$700 is invested at 6% simple interest for 2 years, the amount is \$784.

The length of the interest period

The computation of simple interest is not an involved process when the charge is made for either a full year or a multiple of years as in the last four illustrations. Simple interest is not always collected for such periods; in fact, more often than not it is collected for a period shorter than a year. Since the formula is based on a rate r per year, and the time t is stated in years, it is necessary that the period, if less than a year, be stated as a fractional part of a year.

The question then arises, What is a year? We know that an ordinary year is 365 days. For most sums of money the computation of interest for fractional parts of a year will not vary by more than a few cents regardless of whether it is computed on the basis of a year of 12 months, 360 days, or 365 days. Under the influence of custom and law, certain traditional methods of computing interest have developed. Practices, now well established, have developed which are usually favorable to the one who actually draws up the loan contract.

It is readily apparent that if the numerators of two fractions are equal, the one with the larger denominator is the smaller number. Thus one-half of an amount is greater than one-third of the same amount. Similarly, if the year is considered to have 365 days, the interest per day is less than the interest per day if the year is considered to have only 360 days, since $\frac{1}{365}$ is less than $\frac{1}{360}$ of any amount.

If the lender draws the contract it is to his advantage, in computing interest, to assume that the year has 360 days. Since it is customary in many circumstances for the lender to draw the contract, the procedure has developed of referring to interest figured on the 360-day year as *ordinary interest.* When the contract is drawn in terms of months, each

month is considered to be 30 days or $\frac{1}{12}$ of a year in computing ordinary interest

When the borrower draws the contract the year is customarily considered to have 365 or 366 days as the case may be Since the computation is made on the basis of the exact number of days, it is referred to as *exact* or *accurate interest* The interest per day on an exact basis is always less than the interest at the same rate on an ordinary basis

Time between dates

It is customary in this country in computing times between dates to count either the first or the last day, but not both The time between January 1 and January 31 is 30 days (31 − 1 = 30) When the days of the year are numbered from 1 to 365, the number of the days between any two dates is found by subtracting the number of the first date from the number of the last date The difference is said to be the exact number of days Using this method the exact number of days between any two dates can readily be found by consulting Table 1 For example, to find the exact number of days between March 15 and September 15, look up the two dates in the table and subtract the larger from the smaller

September	15 is	258th day
March	15 is	74th day
Difference	is	184 days

The exact time between the two dates is 184 days It can be computed without a table in the following way

Days remaining in March (31 − 15)	16
Days in April	30
Days in May	31
Days in June	30
Days in July	31
Days in August	31
Days in September	15
Total days	184

It frequently happens that even though the period is less than one year it covers parts of two calendar years To find the time between two dates in different calendar years, find the number of days remaining in the first year, and add to this the number of days in the second year For example, to find the exact number of days between November 24 and February 20, look up the two dates in the table and make the following steps

TABLE 1. THE NUMBER OF EACH DAY OF THE YEAR COUNTING FROM JANUARY 1

Day of Month	JAN	FEB	MAR	APR	MAY	JUN	JUL	AUG	SEP	OCT	NOV	DEC
1	1	32	60	91	121	152	182	213	244	274	305	335
2	2	33	61	92	122	153	183	214	245	275	306	336
3	3	34	62	93	123	154	184	215	246	276	307	337
4	4	35	63	94	124	155	185	216	247	277	308	338
5	5	36	64	95	125	156	186	217	248	278	309	339
6	6	37	65	96	126	157	187	218	249	279	310	340
7	7	38	66	97	127	158	188	219	250	280	311	341
8	8	39	67	98	128	159	189	220	251	281	312	342
9	9	40	68	99	129	160	190	221	252	282	313	343
10	10	41	69	100	130	161	191	222	253	283	314	344
11	11	42	70	101	131	162	192	223	254	284	315	345
12	12	43	71	102	132	163	193	224	255	285	316	346
13	13	44	72	103	133	164	194	225	256	286	317	347
14	14	45	73	104	134	165	195	226	257	287	318	348
15	15	46	74	105	135	166	196	227	258	288	319	349
16	16	47	75	106	136	167	197	228	259	289	320	350
17	17	48	76	107	137	168	198	229	260	290	321	351
18	18	49	77	108	138	169	199	230	261	291	322	352
19	19	50	78	109	139	170	200	231	262	292	323	353
20	20	51	79	110	140	171	201	232	263	293	324	354
21	21	52	80	111	141	172	202	233	264	294	325	355
22	22	53	81	112	142	173	203	234	265	295	326	356
23	23	54	82	113	143	174	204	235	266	296	327	357
24	24	55	83	114	144	175	205	236	267	297	328	358
25	25	56	84	115	145	176	206	237	268	298	329	359
26	26	57	85	116	146	177	207	238	269	299	330	360
27	27	58	86	117	147	178	208	239	270	300	331	361
28	28	59	87	118	148	179	209	240	271	301	332	362
29	29		88	119	149	180	210	241	272	302	333	363
30	30		89	120	150	181	211	242	273	303	334	364
31	31		90		151		212	243		304		365

Note. For leap years the number of any day after February 28 is one greater than the tabular number.

Days remaining in the first year:

December	31 is	365th day
November	24 is	328th day
Difference	is	37 days

Add days in second year:

February	20 is	51st day
		88

The exact time between the two dates is 88 days.

To find the exact number of days between dates during a leap year, add 1 to each number shown in the table after February 28 Thus to find the exact number of days between January 8 and April 15 of any leap year, look up the two dates in the table, add 1 to the later date and proceed as before

April	15 is	106th day	(105th day plus 1 day)
January	8 is	8th day	
Difference	is	98 days	

Without a table the time would have been computed as follows

Days remaining in January	(31 − 8)	23
Days in February		29
Days in March		31
Days in April		15
	Total days	98

The number of days may be counted not as the exact number of days, but rather as the approximate number of days Considering each month to have 30 days, the time from March 15 to September 15 is 180 days When this method is used, the time is called the *approximate time* In computing bond interest on corporate bonds, it is customary to consider a year as 360 days and a month as 30 days Thus a bond bought on August 8 and sold on December 15 has been held 4 months and 7 days, or 127 days, computed as follows

From August 8 to December 8 there are 4 months or	120 days
From December 8 to December 15 there are	7 days
Total days	127 days

Since the method of approximate time is commonly used for computing accrued interest on bonds, it is sometimes referred to as the "bond method"

Since only one month has less than 30 days, and seven months have 31 days each, the exact number of days is usually greater than the approximate number of days between any two given dates In computing simple interest for fractional parts of a year the time between dates is the numerator of the time factor Since a higher or larger numerator results in a higher interest payment, the computation of the exact number of days tends to favor the one receiving the interest payment, since it is usually larger than the approximate number of days Three methods of counting time are in common use, and consequently three kinds of simple interest *ordinary interest*, *exact interest*, and *bankers' interest*

In computing ordinary interest, the year is considered to have 12 months of 30 days each, with a total of 360 days in a year; in computing exact interest, 365 days are counted in a year and the actual number of days in each month is counted; in computing bankers' interest, the year is considered to have 360 days, but the actual number of days in each month is counted.

The methods of computing time in calculating these three types of simple interest can be summarized as follows:

Type of	*Number of Days Counted*	
Simple Interest	*In Each Month*	*In Each Year*
Ordinary	30	360
Exact	exact	365
Bankers'	exact	360

EXERCISE 10.1

Find the time between the dates in the following:

	From	*To*	*Exact Time*	*Approximate Time*
1.	February 28, 1958	October 3, 1958		
2.	October 3, 1957	February 3, 1958		
3.	April 15, 1958	July 6, 1958		
4.	January 1, 1958	August 15, 1958		
5.	October 10, 1958	April 15, 1959		
6.	May 1, 1958	July 3, 1958		
7.	March 2, 1959	November 10, 1959		
8.	June 13, 1959	October 26, 1959		
9.	August 7, 1959	January 19, 1960		
10.	September 7, 1960	February 26, 1961		
11.	October 18, 1959	March 4, 1960		
12.	December 5, 1959	February 29, 1960		
13.	February 24, 1960	April 24, 1960		
14.	January 10, 1959	May 4, 1960		
15.	February 29, 1960	June 8, 1960		

Bankers' vs. exact interest

In computing simple interest, whether by the exact, the ordinary, or the bankers' method, the basic formula is $I = Prt$. There is no difference in the amount of interest among the three methods when interest for a whole year is computed. The difference arises when the time is a fractional

part of a year The factor t then becomes a fraction with a denominator of either 360 or 365, according to the method used

Bankers' interest for 1 day is equal to $\frac{1}{360}$ of a year's interest, and exact interest for 1 day is equal to $\frac{1}{365}$ of a year's interest The relation between exact interest and bankers interest, then, resolves itself into a simple proportion

$$\frac{\text{One day exact interest}}{\text{One day bankers' interest}} = \frac{\frac{1}{365}}{\frac{1}{360}} = \frac{360}{365} = \frac{72}{73}$$

We see therefore that exact interest is equal to $\frac{72}{73}$ of bankers' interest Thus we can state *exact interest is equal to bankers interest decreased by $\frac{1}{73}$ of itself*

Conversely, bankers' interest is equal to $\frac{73}{72}$ of exact interest, or *bankers interest is equal to exact interest increased by $\frac{1}{72}$ of itself*

It can be seen readily that bankers' interest is more than exact interest As a general rule, it can be said that when a contract is drawn up by the lender, such as a bank, the bankers' method of calculating interest is customarily used The few cents more paid by a borrower makes little difference to him individually, but taken collectively by the lender it may amount to a sizable sum

In the case of a government, such as a state or our federal government, which pays interest on billions of dollars for short periods of time, the amount of interest under the exact method is less Consequently, the exact method is almost always used when the borrower draws up the contract Thus in computing interest on municipal securities, United States government securities, and loans made by the Federal Reserve banks, exact or accurate interest is customarily used

In calculating simple interest, the work is often simplified if the problem is set up with the interest rate shown as a fraction whose denominator is some multiple of 100, and the time shown as a fraction with an appropriate denominator of 360 or 365

Illustrations

a Find the bankers' interest at 3% on $1,000 for 180 days

The formula is $I = Prt$ Here $P = \$1{,}000$, $r = 3\%$ or $\frac{3}{100}$ (i e , stated as a fraction with a denominator of 100), $t = \frac{180}{360} = \frac{1}{2}$ Therefore

$$I = \$1{,}000 \times \tfrac{3}{100} \times \tfrac{1}{2} = \$15\ 00$$

b Find the exact interest on $1,000 for 180 days at 3%

The formula is $I = Prt$ Here $P = \$1{,}000$, $r = 3\%$ or $\frac{3}{100}$, $t = \frac{180}{365}$ Therefore

$$I = \$1{,}000 \times \tfrac{3}{100} \times \tfrac{180}{365} = \$14\ 79$$

By using the process of cancellation such computations can be rapidly made. Some find it convenient in finding exact interest to first compute bankers' interest, and decrease it by $\frac{1}{73}$.

Illustration: Find the exact interest on \$1,000 for 180 days at 3%.

$$\text{Bankers' interest} = \$1{,}000 \times \tfrac{3}{100} \times \tfrac{180}{360} = \$15.00$$
$$\tfrac{1}{73} \times \$15.00 = \$\ 0.21$$
$$\text{so exact interest is} \quad \$15.00 - 0.21 = \$14.79.$$

EXERCISE 10.2

Solve the following:

1. Find the ordinary interest on \$1,750 at $4\frac{1}{2}$% for 132 days.
2. Find the bankers' interest on \$830 at 5% from March 8 to May 12.
3. Find the exact interest on \$1,780 at $3\frac{1}{2}$% for 112 days.
4. Find the ordinary interest on \$784.56 at 5% from February 8 to June 5.
5. Find the bankers' interest on \$768 at 7% for 197 days.
6. Find the exact interest on \$384.27 at $6\frac{1}{2}$% from March 4 to April 24.
7. What is the exact interest for the month of March at 4% on \$100,000?
8. What is the bankers' interest for the month of July at $4\frac{1}{2}$% on \$150,000?
9. What is the ordinary interest for $4\frac{1}{2}$ months at $3\frac{1}{2}$% on \$1,250?
10. Mr. Jones borrows \$850 from his bank on May 8 at 7%. He pays off the loan on July 18. How much does he pay?
11. The federal treasury borrowed \$2,500,000 at $1\frac{1}{2}$% for the month of May. How much did it pay back?
12. To take advantage of a special offer, the Acme Hardware Company on April 4 borrowed \$1,800 from the Citizens National Bank at 5%. The debt was paid off on May 18. How much was needed to clear the debt at that time?
13. Charles Williams borrows \$750 from his bank on April 10 at 7%. On May 10 he paid half the amount of the loan and the total interest charge to date. How much did he pay?
14. \$827.34 is borrowed for the time from May 12 to September 4 at $5\frac{1}{2}$%. What is the ordinary interest, the bankers' interest, and the exact interest?
15. What is the difference between the exact interest and the bankers' interest on \$248,000 at $3\frac{1}{2}$% for 238 days?

16. The penalty levied by Fulton County on all delinquent tax bills is 6% per year A tax of $5,190, due April 2, was not paid until June 14 If the penalty is computed on the basis of exact interest, what is the amount of the penalty

17. On July 21 the sum of $10,000,000 in gold was shipped from the New York mint to Buenos Aires It was delivered to the bank there on July 31 How much was lost in exact interest if money was worth 3%?

18 On March 15 Mr Warren borrowed $850 at 6% How much should he repay 6 months later?

19. Charles Wilson borrowed $900 at his bank If interest is 7%, how much should he pay the bank at the end of 120 days?

20. Douglas Perry bought 10 shares of stock for $240 which he held for 9 months During the period he received $20 in dividends He received $230 when he sold the stock What rate of interest did he receive on his investment?

Short cuts in calculating bankers' interest and ordinary interest

The fact that 360 is a number with many multiples, and that 6% is a widely used rate of interest, has led to the development of so-called short-cut methods

The interest on $1,000 at 6% for 60 days is equal to

$$\$1{,}000 \times \tfrac{6}{100} \times \tfrac{60}{360} = \$10$$

Thus it is readily apparent that when all possible cancellations have been carried out, the interest at 6% for 60 days is equal to $\frac{1}{100}$ of the principal Since this is true the following rule may be applied

To find the interest on any amount at 6% for 60 days, move the decimal point in the principal two places to the left

Since 6 days is $\frac{1}{10}$ of 60 days, to find the interest for 6 days at 6% it is necessary only to move the decimal point 3 places to the left To find the interest for one day at 6% it is necessary only to move the decimal point 3 places to the left and then divide by 6 The quotient will be the interest at 6% for 1 day on the principal

Illustration How much interest will be received on a loan of $6,484 30 at 6% for 6 days?

By moving the decimal point 3 places to the left, we find that $6 48 is the interest for 6 days

By the use of fractional parts, one can readily find the interest at 6% for any number of days

Illustrations:

a. Find the interest on \$645.26 at 6% for 40 days.

Moving the decimal point 2 places to the left, we find that \$6.45 is the interest at 6% for 60 days. Forty days equals $\frac{2}{3}$ of 60 days. Dividing \$6.45 by 3, the interest for 20 days is \$2.15. Therefore the interest for 40 days is \$4.30 (either \$6.45 − \$2.15, or \$2.15 × 2).

b. Find the interest on \$1,286.75 at 6% for 75 days.

Interest at 6% for 60 days	\$12.8675
The interest for 15 days is $\frac{1}{4}$ of the interest for 60 days	3.2169
Therefore interest for 75 days is	\$16.0844, or \$16.08

The same method can be used also for rates other than 6%. Usually in using it for more complex problems, much time can be saved by presenting the solution in orderly form so that it can be readily checked.

Illustrations:

a. Find the interest on \$4,520 at 3% for 60 days.

Interest at 6% for 60 days	\$45.20
3% is $\frac{1}{2}$ of 6%. Therefore the interest at 3% is	\$22.60.

b. Find the interest on \$892.50 at 4% for 72 days.

Interest at 6% for 60 days	\$8.925
Interest at 6% for 12 days (divide by 5)	1.785
Interest at 6% for 72 days	\$10.710
$\frac{1}{3}$ of 6% is 2%. (Divide by 3 to get the interest at 2% for 72 days)	\$3.57
Subtracting the interest at 2% from that at 6% leaves interest at 4% for 72 days	\$7.14

It must be emphasized that these short-cut methods are based on the application of the principles of fractions and cancellation. In many instances the so-called short-cut methods do not actually save time. The preceding illustration, written in fractional form, becomes:

$$\$892.50 \times \frac{4}{100} \times \frac{72}{360}$$

When 10 is canceled into \$892.50 and into 360, and when the quotient, 36, is canceled into 72, and 100 into \$89.25, the problem resolves itself into

$$\overset{\overset{\$0.8925}{\cancel{\$89.25}}}{\cancel{\$892.50}} \times \frac{4}{\cancel{100}} \times \frac{\overset{2}{\cancel{72}}}{\underset{\cancel{36}}{\cancel{360}}} = \$7.14$$

Since the short-cut methods are intended only as timesaving devices, they should be utilized only if and when the student is convinced that they do save his time

Dollars-times-days method

The dollars-times days method an adaptation of the 6% method, is commonly used by accountants, and has much value when calculating machines are used It has previously been shown that the interest at 6% for 1 day can be found in the following way

1 Move the decimal point 3 places to the left to find the interest for 6 days

2 Divide by 6 to find the interest for 1 day

If it is desired to find the interest for a given number of days, such as 37, the same basic procedure of finding the interest for 1 day and then multiplying by the number of days, here 37, could be used In the interest of accuracy, however, it is better to adopt the following procedure

1 Multiply the principal by the number of days

2 Point off 3 places to the left

3 Divide by 6 (This quotient is the interest for the stated number of days at 6%)

4 Convert to the desired rate

Illustration What is the interest on \$420 for 50 days at 5%?

1 Multiply the principal by the number of days \$420 × 50 = \$21,000

2 Point off 3 places to the left \$21 00

3 Divide by 6, giving \$3 50, interest at 6% for 50 days

4 Since 5% is $\frac{1}{6}$ less than 6%, divide \$3 50 by 6, giving \$0 58 Then \$3 50 − 0 58 = \$2 92 the interest on \$420 for 50 days at 5%

Interchange of principal and days

The fundamental relationships used in the 6% method are sometimes more easily applicable to the amount than to the number of days Thus it is more difficult to compute the interest at 6% on \$7,200 for 37 days than to compute the interest at 6% on \$37 for 7,200 days, but the results are the same The interest on \$37 for 60 days is \$0 37, for 600 days it amounts to \$3 70, and for 6,000 days it amounts to \$37 Since 6,000 days + 600 days + 600 days = 7,200 days, the sum of \$37 + \$3 70 + \$3 70, or \$44 40, is equivalent to the interest on \$7,200 for 37 days

Illustration Find the interest on \$4,200 for 47 days at 6%?

Interchanging principal and days, we have the interest on \$47 for 4,200 days

Interest on \$47 for 60 days is...........	\$0.47
Since $60 \times 70 = 4{,}200$, multiply by......	70
	\$32.90

This gives the interest on \$47 for 4,200 days, which is equivalent to the interest on \$4,200 for 47 days. Observe, however, that the problem can be written in fractional form and cancellation can be used, with the following results.

$$\overset{\$0.70}{\cancel{\$4{,}200}} \times \frac{\cancel{6}}{\cancel{100}} \times \frac{47}{\underset{\cancel{60}}{\cancel{360}}} = \$32.90$$

This solution can be arrived at just as readily as the other. Again it must be emphasized that the 6% method and its variations are intended as short cuts. If, after some practice, it is found that they do not save time over the method of stating the problem in arithmetical form and applying the principles of cancellation, they should not be used.

EXERCISE 10.3

Using any short cut, find the ordinary or the bankers' interest on the following:

	Principal	*Time in Days*	*Annual Rate %*	*Interest*
1.	\$ 223.45	60	6	
2.	1,242.00	40	6	
3.	1,800.00	12	4	
4.	397.20	72	5	
5.	2,040.00	15	$4\frac{1}{2}$	
6.	1,872.40	50	3	
7.	636.00	20	2	
8.	4,800.00	29	6	
9.	6,600.00	39	4	
10.	7,200.00	72	5	
11.	25,000.00	90	4	
12.	3,250.00	105	7	
13.	1,680.00	127	$4\frac{1}{2}$	
14.	384.27	68	5	
15.	1,600.00	79	5	
16.	2,250.00	84	6	
17.	258.49	32	3	
18.	48,600.00	120	$4\frac{1}{2}$	
19.	328,562.00	78	5	
20.	48,000.00	119	2	

True or simple discount

In discussing simple interest it was pointed out that people ordinarily *must* be paid for lending funds. The amount to be returned is greater than the amount lent. One could say that the future amount is greater than the present value. Often debts or commitments are stated in terms of their future value.

In the settlement of A's estate, B is to receive \$1,000 when he reaches the age of 21, one year from now. If money is worth 5%, what is the present value of B's legacy? Another way of stating this is: What sum invested at 5% today will amount to \$1,000 a year hence?

Using the simple interest formula, $S = P(1 + rt)$, then $P = \dfrac{S}{1 + rt}$. Since $S = \$1{,}000$, $r = 5\%$, $t = 1$,

$$P = \frac{\$1{,}000}{1 + 5\%} = \frac{\$1{,}000}{1.05} = \$952.38$$

Anyone seeking a 5% return on his money would not pay B more than \$952.38 *for his legacy of \$1,000 due one year hence, since by investing* \$952.38 now at 5% he would have \$1,000 in one year. It is said that the discount on \$1,000 for one year at 5% is \$47.62 (\$1,000 − \$952.38).

Discount may be defined as the difference between the present value of a debt and its maturity value.

Attention should be called to the fact that the simple interest on P is equal to the simple discount on S. Thus \$47.62 is the simple interest on \$952.38 at 5%, or \$47.62 is the simple discount on \$1,000 for one year. That is, \$952.38 + \$47.62 = \$1,000, and \$1,000 − \$17.62 = \$952.38.

Present value of an interest-bearing note

In the preceding illustration, in which the present value of a future sum was found, it was assumed that the debt did not bear interest. Often *it is necessary to find the present value of an interest-bearing debt.* Suppose, for example, that a debt of \$1,000 bears interest at 4%, and is due in one year. What is the amount of the debt?

Given that $S = P + I$ and $I = Prt$. Here $P = \$1{,}000$, $r = 4\%$, and $t = 1$. Thus $S = \$1{,}000 + \$1{,}000 \times 4\% \times 1 = \$1{,}000 + \$40 = \$1{,}040$

If money is worth 4%, what is the present value of the debt? That is, what is the present worth of the maturity value of the debt? The maturity *value or amount S of the debt is \$1,040. This amount discounted for 1 year* at 4% gives

$$P = \frac{\$1{,}040}{1 + 4\%} = \frac{\$1{,}040}{1.04} = \$1{,}000$$

When the discount rate on a note equals the interest rate, the present value is always equal to the face amount of the debt. More often than not, however, the two rates are not the same. It is necessary to find the present value of a debt at one rate, when it bears interest at another rate. Thus two problems are involved. First, it is necessary to find the maturity value of the debt using the rate of interest; then it is necessary to use the rate of discount to find the present value of the maturity value of the note. The rate of discount is usually expressed either as the discount rate, or is designated by the expression *money is worth.*

Illustration: Find the value on April 24, 1958, of the following note if money is worth 4%.

> Pittsburgh, Pennsylvania
> April 24, 1958
>
> $1,000.00
>
> One year after date I promise to pay to the order of Dean Morgan the principal sum of one-thousand and no/100 Dollars, together with interest from the date hereof at the rate of 5 % per annum.
> Payable at Pittsburgh, Pa.
>
> Shearson Kay

The *face* of the note is $1,000; the date of the note is April 24, 1958; the term or period of the note is one year; and the interest rate is 5%. The maturity date is April 24, 1959.

The first step in the solution is to find the maturity value of the note: $S = \$1{,}000 + \$1{,}000 \times 5\% = \$1{,}050$. This could be illustrated diagrammatically as follows:

(First step)

$$\begin{array}{ccc} P & \dashrightarrow & S \\ \$1{,}000 & & \$1{,}050 \\ \left(\begin{array}{c}\text{Original value}\\ \text{of the debt}\end{array}\right) & & \left(\begin{array}{c}\text{Maturity value}\\ \text{of the debt at } 5\%\end{array}\right) \end{array}$$

The second step is to find the present worth of the maturity value at the stated rate: $P = \dfrac{\$1{,}050}{1+4\%} = \dfrac{\$1{,}050}{1.04} = \$1{,}009.62$. If an illustration of the second step is combined with the illustration of the first step it appears as follows:

(First step)

P ————————————————→ S

$1,000	5%	$1,050
(Original value of the debt)	(Rate of note)	(Maturity value of the debt at 5%)

0 1 2 3 4 5 6 7 8 9 10 11 12 months

(Present value of the debt at 4%)	(Rate money is worth)	(Maturity value of the debt)
$1,009 62	4%	$1,050

P ←———————————————— S

(Last step)

Not only the rates but also the periods of time may differ

Illustration Compute the discount and the proceeds (that is, the sum received) of discounting a $1,500, 7% note for 90 days, dated April 21, on June 8 at 6%

First step Determine the maturity date—July 23

Second step Compute the maturity value of the note

$$S = \$1{,}500 + \$1{,}500 \times \tfrac{7}{100} \times \tfrac{1}{4} = \$1{,}500 + \$26\ 25 = \$1{,}526\ 25$$

This step of the solution can be illustrated diagrammatically as follows

Face of Note	Term	Rate	Maturity Value
$1,500	90 days	7%	$1,526 25

————————————————→

Third step Determine the period of discount There are 45 days between June 8 and July 23

Fourth step Compute the proceeds

$$P = \frac{\$1{,}526\ 25}{1 + 6\% \times \frac{45}{360}} = \frac{\$1{,}526\ 25}{1 + 6\% \times \frac{1}{8}} = \frac{\$1{,}526\ 25}{1\ 0075} = \$1{,}514\ 89$$

This final step may be diagrammed as follows

Maturity Value
$1,526 25

←————————————————

Proceeds	Discount Period	Rate
$1,514 89	45 days	6%

These two diagrams can now be combined into one diagram.

Face of Note	Term	Rate	Maturity Value
$1,500	90 days	7%	$1,526.25

Date of Note	Discount Date	Discount Period	Rate	Maturity Date
4/24	6/8	45 days	6%	7/23
	Proceeds $1,514.89			

The procedure to adopt in solving such problems can be summarized as follows:

1. Determine the maturity date.
2. Compute the maturity value.
3. Determine the discount period.
4. Compute the proceeds.

In computing the proceeds for a fractional part of a year, the divisor may appear as a repeating decimal. Since the value of a fraction is not changed if both numerator and denominator are multiplied by the same number, the difficulty of dividing by a repeating decimal may often be avoided by multiplying the decimal by one of its factors.

Illustration: Find the proceeds of a $1,000 note drawn for 6 months at 6%, which is discounted at 4%, 4 months before maturity.

Maturity value of the note: $\$1{,}000 + 1{,}000 \times \frac{6}{100} \times \frac{6}{12} = \$1{,}030$.

The present value 4 months before maturity at 4% is

$$P = \frac{\$1{,}030}{1 + 4\% \times \frac{4}{12}} = \frac{\$1{,}030}{1.0133\ldots}$$

Rather than divide $1,030 by 1.0133..., both numerator and denominator may be multiplied by 3, since $\frac{4}{12}$ equals $\frac{1}{3}$.

$$P = \frac{\$1{,}030 \times 3}{(1 + 4\% \times \frac{1}{3}) \times 3} = \frac{\$3{,}090}{3.04} = \$1{,}016.45$$

The division of $3,090 by 3.04 is not difficult. This method may be used to save time in solving several of the problems in the next exercise.

EXERCISE 10.4

Find the proceeds of the following notes

	Face Value	Date of Note	Interest Rate	Term	Date of Discount	Discounting Rate
1.	$1,000	3/7	5%	60 days	4/12	6%
2.	$2,500	1/20	1%	60 days	5/18	5%
3.	$ 850	5/1	7%	90 days	7/3	8%
4.	$ 125	9/12	6%	90 days	11/8	5%
5.	$1,200	1/21	none	120 days	7/12	7%
6.	$1,800	5/22	none	120 days	7/18	8%
7.	$2,000	5/12	none	72 days	6/18	5%
8.	$2,100	8/13	none	72 days	9/21	$1\frac{1}{2}$%
9.	$ 100	6/27	$1\frac{1}{2}$%	15 days	7/12	6%
10.	$ 325	10/8	5%	45 days	11/12	$6\frac{1}{4}$%

11. Find the present value of a noninterest-bearing note for $1,000, due in 8 months, if money is worth 1%

12. Find the value of a 120-day, 6% note for $3,500, one month before the due date, if money is worth 3%

13. Find the present value of a noninterest-bearing note drawn for 6 months in the amount of $106.08, if money is worth $1\frac{1}{2}$%

14. A 180-day, 5% note, for $1,850 is discounted on the day it is written What are the proceeds if money is worth 8%?

15. A 90-day, 7% note, dated May 18, for $800 is discounted on July 6 What are the proceeds if money is worth $7\frac{1}{2}$%?

Bank discount

The concept of true discount is important in analytical reasoning on financial problems In dealing with banks, however—and banks are the principal lending agencies—true discount, as it is defined, is not used Instead, bankers figure interest on the maturity value of a note, deduct *the interest from the maturity value of the note, and call the amount* deducted the discount A bank given a $200 note for 90 days, to be discounted at 6%, figures as follows interest on $200 at 6% for 90 days is $3, $200 − $3 = $197 The amount received by the borrower, $197, is called the *proceeds*, but it is understood to mean bank proceeds The $3 deduction is called the *discount*, meaning bank discount In this transaction, it may be said that the borrower *discounted* his note at the bank, or that the bank *discounted* the borrower's note This is confusing to those who do not realize that the word may be used in either way.

Notes discounted by a bank generally fall into one of the two classifica-

tions already considered; that is, either interest-bearing or noninterest-bearing. In the preceding illustration, the borrower gave his own non-interest-bearing note to the bank to be discounted. The maturity value of the note was its face value, the period of discount was the same as the time of the note, and the proceeds were equal to the difference between the maturity value of the note and the discount.

Before these relationships can be shown in the form of an equation, it is convenient to adopt the following symbols:

S = the maturity value of the note;
t = the number of years, or the fractional part of a year, between the date of discount and the maturity date of the note;
d = the annual bank discount rate;
D = the bank discount.

Inasmuch as bank discount is found in the same way as simple interest, $D = Sdt$. If P_b (read P sub b) represents the bank proceeds of the note, then $P_b = S - D$. Since $D = Sdt$, this value may be substituted for D in the second equation: $P_b = S - Sdt$.

A comparison of the formula $P_b = S - Sdt$ with the formula for present value, $P = \dfrac{S}{1 + rt}$, shows the difference between the computation of present worth and that of bank discount. In figuring present worth it is necessary to divide, but in computing bank discount it is necessary to multiply.

If S is the unknown in the formula $P_b = S - Sdt$, then it can readily be seen by rewriting the formula as $P_b = S(1 - dt)$ that

$$S = \frac{P_b}{1 - dt}$$

This formula should be used in finding the face amount of a note when the proceeds are known.

Illustration: The Union Bank agrees to discount a customer's note for 90 days at 6%. If the customer needs \$2,800, what should be the face amount of the note?

Here the proceeds (P_b) is \$2,800, d is 6%, and t is 90/360, and S is unknown. Hence

$$S = \frac{\$2{,}800}{1 - 6\% \times \frac{1}{4}} = \frac{\$2{,}800}{0.985} = \$2{,}842.64$$

The note should be drawn for \$2,842.64.

Bank discount on interest-bearing notes

When a payment is deferred, a debtor may agree to pay interest on the amount due and to give the creditor a promissory note as evidence of the agreement The face value of the note is equal to the amount of the debt, but its maturity value is greater The creditor need not wait for his funds until the note matures He may discount the note at a bank in order to get immediate use of the funds

Thus if A owes B some money, he may give B a note for a specified term at an agreed on interest rate However, B may not choose to wait until the maturity date of the note Instead B may discount A's note at a bank at a stipulated rate and get the proceeds immediately

When bank discount is figured on an interest-bearing note, two problems are involved The first, a problem in simple interest, is to find the maturity value of the note It is only in this respect that the calculations differ from that on a noninterest-bearing note The second is a problem in bank discount, in which both the time and the rate of bank discount will probably differ from the time and rate used in finding the maturity value of the note The same symbols can be used in calculating the bank discount even though the maturity value of the note is not the same as its face value

Illustration The St Louis Heavy Hardware Company accepts a 120 day, 6% note for $1,000 from one of its customers Being in need of funds 90 days before the note is due, the company discounts it at the bank at 5% What are the bank proceeds?

The maturity value of the note is $\$1,000 + 1,000 \times \frac{6}{100} \times \frac{120}{360} = \$1,020$ The bank discount on the maturity value of the note for 90 days at 5% is $\$1,020 \times \frac{5}{100} \times \frac{90}{360} = \$12\ 75$ The bank proceeds are equal to $\$1,020\ 00 - \$12\ 75 = \$1,007\ 25$

Banks usually compute the terms of discount in the exact number of days In most states, the first day of the period is not included in the time of discount, but the final day of maturity is included In several states, however, both days are counted

Since a bank looks to the person discounting the note for payment rather than the original maker of the note, the practice of discounting third party paper by banks occurs less frequently than formerly

EXERCISE 10.5

Find the bank proceeds of the following notes:

	Face Value	*Date of Note*	*Interest Rate*	*Term*	*Date of Discount*	*Bank Discount Rate*
1.	$2,500	4/16	4%	60 days	5/14	5%
2.	$1,800	5/14	5%	90 days	5/24	7%
3.	$4,000	3/11	5%	60 days	4/16	6%
4.	$850	5/20	7%	90 days	7/19	8%
5.	$1,200	4/24	none	120 days	7/12	7%
6.	$3,200	5/24	none	135 days	7/20	6%
7.	$2,400	8/12	none	72 days	9/23	$4\frac{1}{2}$%
8.	$520	5/22	none	120 days	7/18	8%
9.	$360	6/27	$4\frac{1}{2}$%	45 days	7/12	6%
10.	$475	10/8	5%	45 days	11/12	$6\frac{1}{2}$%

11. The First National Bank charges 7% discount on loans of less than $5,000. How much does a borrower receive who signs a $600 note for 90 days?

12. A receives a noninterest-bearing note from B for $400. Find the proceeds if A discounts the note at a bank at 6%, 60 days before it falls due.

13. The Calplastic Company receives a $1,200, 5%, 90-day note from a customer. The note falls due on July 22. Find the proceeds if on April 26 the Calplastic Company discounted the note at a bank at 6%.

14. Six months before a $500, 5% note, drawn for one year, is due, the holder discounts it at a bank for 4%. Find the proceeds.

15. A $2,500 note, dated March 12, is due 80 days from date with interest at 4%. What are the proceeds if it is discounted on May 4 at 7% simple discount?

16. What are the proceeds of the note in Problem 15 if discounted on May 4 at a bank discount rate of 7%? Which is the greater amount and why?

17. A $250, 5% note, for 90 days, is dated May 5. If discounted on June 12 at a bank discount rate of 7%, what are the proceeds?

18. Fred Marer desires $750 as the proceeds of a bank loan for 90 days. What is the amount of the note if the bank discount rate is 8%?

19. Fred Essig desires $600 as the proceeds of a bank loan for 120 days. What is the amount of the note if the bank discount rate is 7%?

20. A $6,500, 5% interest-bearing note for 3 months, dated October 5, is discounted on December 2. What are the proceeds if the bank discount rate is 7%?

Finding the rate of interest

It is impossible to anticipate and to treat specifically all the various problems that a student may later encounter One goal in teaching mathematics of finance is to train the student to analyze and to solve diverse problems, thus giving him confidence to attempt solutions to all problems which he will later face

Though a businessman rarely is concerned with computing what a simple interest rate in the past has been, he may well choose to make certain comparisons before making a decision Thus many problems arise in analytical reasoning which are seldom met in routine operations

Anyone who deals with a bank has perfect freedom of contract If he prefers, a customer can usually pay interest rather than to have the bank discount his note If one banker will not agree, perhaps the next one will The small difference in the cost of money under a contract based on bank discount and one based on simple interest is usually not large enough to warrant spending much time in shopping among banks, or enough to have the banker gain the displeasure of a customer Nonetheless it is often desirable to make accurate comparisons It has been shown that

$$S = \frac{P_b}{1 - dt}$$

If the proceeds P_b amount to the maturity value S during the period t, the interest earned on P_b during that time is $S - P_b$ We know that $S = \frac{P_b}{1 - dt}$ Therefore the interest earned on P_b would be $\frac{P_b}{1 - dt} - P_b$,

$$\text{or } \frac{P_b - P_b(1 - dt)}{1 - dt}, \text{ or } \frac{P_b dt}{1 - dt}$$

If the interest for 1 year is desired, divide by t

$$\frac{P_b dt}{1 - dt} \div t = \frac{P_b d}{1 - dt}$$

If the interest on \$1 of proceeds is desired, divide by P_b

$$\frac{P_b dt}{1 - dt} \div P_b = \frac{d}{1 - dt}$$

Thus the rate of interest r corresponding to a discount rate d is $r = \frac{d}{1 - dt}$

In like manner

$$d = \frac{r}{1 + rt}$$

Illustrations:

a. What interest rate is equivalent to a bank discount rate of 7% on a 90-day note?

Since $d = 7\%$ and $t = \frac{1}{4}$,

$$r = \frac{7\%}{1 - \frac{1}{4} \times 7\%} = \frac{0.07}{0.9825} = 0.071246\ldots = 7.125\%$$

b. What bank discount rate is equivalent to an interest rate of 6% on an 180-day note?

Since $r = 6\%$ and $t = \frac{1}{2}$,

$$d = \frac{6\%}{1 + \frac{1}{2} \times 6\%} = \frac{0.06}{1.03} = 0.0582524\ldots = 5.825\%$$

Much misunderstanding has arisen because banks compute their charges as they do. In the preceding illustration the difference between the interest rate of 6% and the equivalent bank discount rate was 0.175% (6.000% − 5.825% = 0.175%). At this rate the customer borrowing \$5,000 for 6 months would have been charged \$4.36 more at a discount rate of 6% than he would have been at simple interest of 6% (\$5,000 $\times$ 0.175% $\times \frac{180}{360}$ = \$4.36).

The banker makes a charge for lending money. Regardless of what the banker calls the charge, or how he computes it, the person wanting to borrow money should compare the charges made by alternative lending institutions. If all institutions compute their charges in the same way, a simple comparison of rates will suffice. When, however, charges are not computed in a uniform manner, all must be reduced to a similar basis before accurate comparisons may be made. Even then, the factors of personal relationship, service, and convenience must be weighed as well as the interest charges. As a borrower your alternative may not be between a 6% interest charge, and a 6% discount. Your alternative may be between a note discounted by the bank at 6%, or no loan.

EXERCISE 10.6

Solve the following problems:

1. Immediately after school begins, a university controller has surplus funds to invest. If he seeks a return of 2%, how much should he pay for a 90-day noninterest-bearing note of \$100,000?

2. A private lender has two prospective borrowers. One is willing to sign a 6-month note for \$10,000 in exchange for \$9,700 in cash, the other is willing to pay $6\frac{1}{4}$% simple interest for \$10,000 for 6 months. On which loan would the lender receive the greater return?

3. The discount rate of the Merchant's National Bank is 7% on all loans less than $5000 What should be the face of a note if the proceeds for a 6 month loan are to be $2,300?

4. The discount rate of the First National Bank is 8% on all loans less than $500, and 7% on the excess of $500 What are the proceeds of a 90-day $850 loan?

5. Determine the interest rate which would correspond to a bank discount rate of 7% on a 90-day loan

6. *Determine the bank discount rate which would correspond to an* interest rate of 8% for a 6-month loan

7. A bank discounts a 3-month note at 8% What rate does the bank earn on its money?

8. The stockholders of a bank have $1,520,000 invested in the stock In the past few years the bank has had average annual earnings of $136,800 If $\frac{2}{3}$ of the earnings have been paid to the stockholders in the form of dividends, what rate of return have the stockholders received on their investment?

9. A manufacturer buys some material at terms 2/10 n/30 His accountant tells him that such a discount is equivalent to simple interest of 36% Show whether his accountant is right

10 A paper dealer buys $1,000 worth of paper, terms 3/15 n/30 What is the interest rate corresponding to this discount rate?

Partial payments on interest-bearing debts

In most lines of business, when goods are bought on open account, cash discounts are allowed if payment is made during the discount period If payment is not made during the discount period the debt bears no interest until the due date and any payment made is deducted from the sum due A payment on account which is less than the total amount owed is called a *partial payment*

In such short-term transactions as buying goods on open account or repaying a seasonal bank loan, debtors are normally expected to pay both principal and interest on the stated date Situations arise, however, in which partial payments are made on interest-bearing debts Partial payments may be made on debts with definite maturity dates, either before or after maturity, or they may occur in relation to transactions which have no fixed or predetermined date of maturity—for example, merchandise bought on open account when interest is charged on the account

The relationship between the debtor and the creditor is contractual If the debt has a definite maturity date, the creditor may not choose to

accept payment before the maturity date. On the other hand, he may want to encourage the debtor to pay early. If the debtor wants to make a partial payment on an interest-bearing debt before maturity and the creditor agrees to accept such payment, the question arises of just how much of the payment should be considered a payment of principal and how much a payment of interest.

Many years ago a case regarding the question of partial payment reached the Supreme Court. The Court's decision included what is generally known as the *United States Rule.* It states that the payment received in partial settlement must first be applied to the payment of interest, and that only the balance of the payment shall go to reduce the principal. For the next payment, interest shall be calculated on the balance of the principal from the date of the preceding payment.

Illustration: A 4% note for \$500 fell due on January 1, 1958. Being unable to pay the full amount, the borrower reached an agreement with the holder of the note to pay the principal in 3 semiannual payments of \$170 each, the balance to be paid with the 3rd installment. Find the amount of the 3rd payment.

Face of note	\$500.00
Interest for 6 months at 4%	10.00
	510.00
1st payment of \$170	170.00
	340.00
Interest for 6 months at 4%	6.80
	346.80
2nd payment of \$170	170.00
	176.80
Interest for 6 months at 4%	3.54
	180.34
Last payment	180.34
	00.00

This can be shown more concisely as follows.

Semiannual Payments	*Payment*	*Interest Accrued*	*Reduction of Principal*	*Balance*
				\$500.00
1	\$170.00	\$10.00	\$160.00	340.00
2	170.00	6.80	163.20	176.80
3	180.34	3.54	176.80	00.00

Under the United States Rule, if a partial payment received does not exceed the accrued interest, the payment is held until additional payments are received When the payments received taken together total more than the interest computed from the date of the last reduction in the principal, the amount of the debt is reduced by the excess of the payments over the accrued interest

Illustration A 6% note for $1,000 is to be repaid in partial payments over the next year Payments are received as follows End of 2nd month, $30, end of 3rd month $800, end of 7th month, $50, end of 8th month, $100, final payment at the end of year What is the amount of the final payment?

Amount		$1,000 00
Interest at 6% for 2 months	$10 00	
Payment of	30 00	
Interest at end of 3rd month		60 00
Total debt and interest		4,060 00
Deduct payments of $30 and $800		830 00
Balance		3,230 00
Interest for 4 months	64 60	
Payment of	50 00	
Interest on balance for 5 months		80 75
Total debt and interest		3,310 75
Less payments of $50 and $100		150 00
Balance		3,160 75
Interest on balance for 4 months		63 22
		3,223 97
Final payment		3,223 97
		$ 00 00

The problem of partial payments may occur because the debtor, wanting to save interest charges, makes partial payments before the maturity of the debt A person temporarily in need of funds to complete a transaction borrows $5,000 from his bank for 6 months at 6% If at the end of 2 months a partial payment of $2,000 is made to the bank, the bank might proceed as follows

Total debt	$5,000 00
Interest at 6% for 2 months	50 00
Total due	5,050 00
Less payment of	2,000 00
Balance	$3,050 00

The usual practice is for the bank to issue a new note for \$3,050. It can be seen that this is an application of the United States Rule.

The steps to follow in computing the amount due under the United States Rule at any time may be summarized as follows.

1. Compute the amount of interest on the face of the note from the date of the note to the time of the first payment.
2. Find the sum of the face of the note and the interest.
3. If the payment exceeds the amount of the interest added, deduct the total payment from the sum of the debt and the interest. The balance found is the amount of the debt still to be paid. The entire debt could be discharged at this time by the payment of this balance.
4. If the partial payment received is less than the amount of interest accrued on the debt, the payment is simply held until another payment is received.
5. When such payments taken together exceed the interest accrued on the debt from the date of the note, or from the date of the last payment which was deducted from the face of the note, they are deducted from the sum of the principal and the accrued interest.
6. For any subsequent payments, interest is computed on the outstanding principal from the date of the last reduction in principal, and the preceding procedure is repeated.

Although the discussion of the United States Rule here has been restricted to past events, the same basic principles apply in determining what credit should be allowed for future partial payments.

Another method in general use for determining the amount of payment necessary to discharge equitably the balance of an interest-bearing debt on which partial payments have been made, is the use of the *Merchants Rule.* Under it, the interest for the entire period is added to the debt; as partial payments are received, the payment, plus simple interest on the amount of the payment to the maturity of the debt, is deducted from the sum of the debt plus the interest to maturity, Consequently, its use always results in a smaller total payment than the United States Rule.

Illustration: A 6% note for \$4,000 is to be repaid in partial payments over the next year as follows: end of 2nd month, \$30; end of 3rd month, \$800; end of 7th month \$50; end of 8th month, \$100; final payment at the end of year. What is the amount of the final payment under the Merchants Rule?

Original debt		$4,000 00
Add the interest for the period of the debt		210 00
		4,210 00
Payment	$30 00	
Interest on payment for 10 months	1 50	31 50
Balance		4,208 50
Payment	800 00	
Interest on payment for 9 months	36 00	836 00
Balance		3 372 50
Payment	50 00	
Interest on payment for 5 months	1 25	51 25
Balance		3 321 25
Payment	100 00	
Interest on payment for 4 months	2 00	102 00
Final payment at end of year		$3 219 25

The Merchants Rule is definitely advantageous to the debtor When partial payments are permitted the creditor should protect himself by specifying that the United States Rule is to be followed

EXERCISE 10.7

Solve the following

1. Assume that on February 28, 1957, you sign a 2 year, 5% note for $1,400, with the provision that the interest is to be paid semiannually and that the principal may be reduced on any interest date If payments are made as follows, find the amount which must be paid on February 28, 1959

Date	*Payment*
August 31, 1957	$ 35
February 28, 1958	275
August 31, 1958	120

2. The following payments are to be made on a note for $3,200 dated March 1, 1958, due in 3 years, with interest at 5% September 1, 1958, $1,000, March 1, 1959, $50, September 1, 1959, $480, September 1, 1960, $600 Find the amount due at maturity under the United States Rule

3. Using the Merchants Rule, find the amount due at maturity on a 9-month, 5% note for $600 dated January 14 on which the following payments were made February 16, $50, May 30, $100, July 26, $140

4. Find by the Merchants Rule the amount due at maturity on a note for $1,690 dated April 1, due in 1 year, with interest at 6%, if the following payments were made: July 31, $500; December 31, $500; March 1, $500.

5. Paul Neher buys a house for $15,000. He pays $5,000 down and agrees to pay the balance at the rate of $120 a month with interest at 6%. By the time he has made the fourth monthly payment, how much of the principal has been discharged under the United States Rule?

6. Richard Whitlo borrows $6,000 from the bank for 6 months at 7%. At the end of the second month he pays $3,000 on the note. At the end of the fourth month he wants to pay the balance. How much should he pay under the United States Rule?

7. What is the balance due on a 6% note for $4,800 due six months hence if it is reduced by equal payments of $2,000 made two months and four months prior to the due date (1) using the Merchants Rule; (2) using the United States Rule?

8. A 180-day, 5% note for $1,500 was due July 7. Don Glenn paid $600 on September 7, and $400 on December 1. How much must he pay to settle the debt on December 31, by the United States Rule? By the Merchants Rule?

9. On March 14 a principal customer of the Clementine Corporation owed $21,000 which was past due. The customer agreed to pay 7% interest on the balance from that date. On May 15, the customer paid $8,000; on June 10 he paid $6,000. Using the exact number of days and ordinary interest, how much would be required to pay the balance on August 1 by the United States Rule? By the Merchants Rule?

10. A debt of $3,000 was due April 2, 1957. On August 1, 1957, a payment of $500 was made; on October 3, 1957, $500; and on February 15, 1958, $2,000. Find the balance due on April 2, 1958, using the exact number of days and ordinary interest at 5% by the United States Rule.

Installment buying

The supreme confidence of the average American in his own future income and security is well illustrated by his growing willingness to assume obligations to be paid in future installments. Indeed it is argued by some that without the use of installment credit, productive capacity would not have reached its present proportions, and the attendant savings of mass production and mass marketing would not yet have been achieved. As it is, however, the purchase of automobiles, refrigerators, washing machines, and home repairs on the installment plan method is widely accepted as one of the common characteristics of our way of life.

Even though installment credit is commonplace it is often not clearly understood In the first place it should be recognized that there are many different types of installment credit and that charges are not always figured in the same way Under some plans the charges appear high, while under other systems the charges are fair and equitable In any case, the wise buyer should be able to measure the charges of one plan against those of another At the same time he will want to be able to make comparisons between what he is paying for the use of credit on the installment plan, and the amount he is receiving on the savings he has in a savings bank or in a savings and loan association It stands to reason that the cost of installment credit will be much higher than the amount received on one s savings But it does not always follow that a person is unwise to use such credit A comparison between the costs of the credit and the benefits derived should be made by each buyer The fact that the cost to the buyer is high does not mean that the one who receives the payment is making an above average return

One procedure used in installment selling is to add to the cash price of the item an amount known as the *carrying charge* The size of the carrying charge may be determined as a fixed percentage of the cost of the item, as a fixed percentage of the debt, as a monthly percentage of the credit granted, as a flat charge, or by some other method

The charge made on an installment contract is usually compared with an interest rate Actually the cost represents many factors, such as the expense of investigation and bookkeeping, as well as certain inevitable losses The reason that such losses are termed inevitable is that in the retailing of credit, goods are often sold to very poor credit risks If such credit were restricted only to those who are sure to pay, there would in effect be no installment credit On the other hand, the person who is buying anything on the easy payment plan is likely to be more concerned about the total charge than he is about the final distribution of the extra dollars he pays To him all costs over and above the cash price of the item might just as well be interest

There are several methods of computing the rate charged for installment buying If the period of installment runs for more than a year or two it is expected generally that the rate charged should be measured on the basis of each unpaid balance, a procedure not unlike that followed under the terms of the United States Rule Interest computed in such a manner is equivalent to compound interest, and will always be less than the corresponding rate computed as simple interest Since compound interest and annuities are considered in detail in Chapters 12 and 13, the computation of interest rates under such methods is omitted at this point

Computation of rates on installment purchases

One method, referred to as the *residuary method* or *merchants method*, of computing the rate charged on an installment purchase, or the amount of money borrowed and repaid by equal installments, is to assume that each payment goes to repay the principal until the entire amount owed has been paid. Subsequent payments are made to pay the interest.

This method may be illustrated by the schedule of loans from the Union Bank, which shows that a loan of $500 may be repaid by 12 equal monthly installments of $43.86 each. Since 12 times $43.86 is $526.32, the total charge for the loan is $26.32. The borrower who repays in monthly installments has the use of the full $500 for 1 month. At the end of the month he pays $43.86 and has the use of $456.14 for 1 month; at the end of the second month he pays $43.86 and has the use of $412.28 for 1 month. The outstanding amount which he can thus use may be scheduled as follows:

	Amount	*Period*
	$ 500.00	1 month
	456.14	1 month
	412.28	1 month
	368.42	1 month
	324.56	1 month
	280.70	1 month
	236.84	1 month
	192.98	1 month
	149.12	1 month
	105.26	1 month
	61.40	1 month
	17.54	1 month
Total	$3,105.24	12 months

If the amount outstanding is added it is found that the borrower thus has had the use of the equivalent of $3,105.24 for 1 month. The interest he paid of $26.32 was thus the interest for $\frac{1}{12}$ of a year for the use of $3,105.24. This is at the equivalent simple interest rate of 10.17% since

$$\$26.32 = \$3{,}105.24 \times \tfrac{1}{12} \times r$$

$$r = \frac{\$26.32 \times 12}{\$3{,}105.24} = 10.17\%$$

This method of computing the interest charge on installment credit may be justified on the grounds that it involves exactly the same principles as the Merchants Rule.

This relationship may be illustrated by considering the preceding example as a problem to be solved by using the Merchants Rule

Illustration A borrower owes $500 on which interest is charged at the rate of 10 17% At the end of each month for 11 months, he makes equal partial payments of $43 86 How much must he pay at the end of the year to discharge the balance of debt?

Original debt		$500 00
Interest on $500 for 12 months at 10 17%		50 85
Balance		550 85
1st payment	$43 86	
Interest on $43 86 for 11 months at 10 17%	4 09	47 95
Balance		502 90
2nd payment	43 86	
Interest on $43 86 for 10 months at 10 17%	3 72	47 58
Balance		455 32
3rd payment	43 86	
Interest on $43 86 for 9 months at 10 17%	3 34	47 20
Balance		408 12
4th payment	43 86	
Interest on $43 86 for 8 months at 10 17%	2 97	46 83
Balance		361 29
5th payment	43 86	
Interest on $43 86 for 7 months at 10 17%	2 60	46 46
Balance		314 83
6th payment	43 86	
Interest on $43 86 for 6 months at 10 17%	2 23	46 09
Balance		268 74
7th payment	43 86	
Interest on $43 86 for 5 months at 10 17%	1 86	45 71
Balance		223 03
8th payment	43 86	
Interest on $43 86 for 4 months at 10 17%	1 49	45 35
Balance		177 68
9th payment	43 86	
Interest on $43 86 for 3 months at 10 17%	1 12	44 98
Balance		132 70

10th payment	43.86	
Interest on \$43.86 for 2 months at 10.17%	0.74	44.60
Balance		88.10
11th payment	43.86	
Interest on \$43.86 for 1 month at 10.17%	0.37	44.23
Balance		43.87
12th payment		\$43.87

Another method of computing the interest charge, known as the *FHA method* or the *constant-ratio method,* involves the assumption that each payment is made up of two parts: first, a payment of principal; and second, a payment of interest. The assumption is that the ratio between these two parts is constant.

Using the preceding illustration in which a \$500 home-improvement loan is to be paid in 12 equal monthly installments of \$43.86, it is seen that total repayments are \$526.32. If only the \$500 were to be repaid in 12 equal monthly installments the monthly payment would be \$41.67, and if the \$26.32 (\$526.32 − \$500) were to be paid in 12 monthly installments, each would be \$2.19. Thus it is assumed that \$41.67 of each payment of \$43.86 is to be applied on the principal, and that the \$2.19 is to be considered interest.

Using these assumptions and preceding as before we have:

Amount outstanding	*Term*
\$ 500.00	1 month
458.33	1 month
416.66	1 month
374.99	1 month
333.32	1 month
291.65	1 month
249.98	1 month
208.31	1 month
166.64	1 month
124.97	1 month
83.30	1 month
41.63	1 month
\$3,249.78	12 months

Under these assumptions the conclusion can be reached that the debtor had the equivalent of \$3,249.78 for 1 month. Referring to the formula for simple interest, the rate could now be computed as amounting to 9.72%, since

$$\$26\ 32 = \$3{,}249\ 78 \times r \times \tfrac{1}{12}$$

or

$$r = \frac{\$26\ 32 \times 12}{\$3\ 249\ 28} = 9\ 72\%$$

A shorter method of computing the rate under the constant-ratio method is to compute the cost on the amount lent for the average period it was out at 1% per month, and then to compare this cost with the actual cost to find the actual cost as a per cent per month The rate per month is converted to an annual rate by multiplying it by 12

The average time can be found by adding 1 to the number of payments and dividing by 2 Thus if 12 payments are to be made at regular monthly intervals, $\frac{12+1}{2}$ gives $6\frac{1}{2}$ months as the average period of time

Illustration If a loan of $100 is to be repaid in 6 equal payments of $18 18, what would be the rate under the constant-ratio method?

The total payment will be $18 18 × 6, or $109 08 Thus the cost of borrowing the $100 is $9 08 If the principal loan had been repaid in 6 equal payments of $16 67 the outstanding balances would have been the following, and had the interest on the balances been computed at 1% per month the interest costs would have been the following

Outstanding Balances	*Interest Costs*
$100 00	$1 00
83 33	0 83
66 66	0 67
49 99	0 50
33 32	0 33
16 65	0 17
00 00	$3 50

Using the formula $t = \frac{n+1}{2}$, this interest charge would have been found easily by the simple interest formula, $I = Prt$ Since $P = \$100$, $r = 1\%$, $t = \frac{6+1}{2} = \frac{7}{2}$,

$$I = \$100 \times \tfrac{7}{2} \times \tfrac{1}{100} = \$3\ 50$$

It is known that the actual cost was $9 08 The monthly rate can be found by dividing the actual amount by what the charge would have been at 1% Thus $9 08 ÷ $3 50 = 2 594, the ratio between the rate per month and the rate at 1% The actual monthly rate was thus 2 594%

If the rate was 2.594% per month, the annual rate must have been 31.13% (2.594% × 12).

In the preceding illustration, computing the rate by the constant-ratio method, the equivalent amount outstanding for 1 month is found to be \$349.85. Since the charge was \$9.08, the equivalent charge for 12 months would be \$108.96. The annual rate would be 31.14% $\left(\frac{\$108.96}{\$349.85}\right)$, a figure virtually the same as that found by the shorter method. Once it is decided to use the constant-ratio method the rate may be computed as follows:

1. Find the total carrying charge, or interest charge.

2. Compute what the total charge would have been had interest been charged at the rate of 1% per month by the use of the formula:

$$\frac{\text{Number of payments} + 1}{2} \times \frac{1}{100} \times \text{Principal of the loan}$$

or the amount of the original unpaid balance.

3. Divide the actual interest by the amount of interest at 1%.

4. Multiply this quotient by 12 and by 1%.

The last step is necessary since in Step 3 the quotient found is either a whole number, or a mixed decimal fraction which shows the relationship between the *amount* of actual interest paid and the amount at 1%. The answer wanted is an annual rate, and hence the monthly rate must be stated as a per cent and multiplied by 12 to get an annual rate.

EXERCISE 10.8

Solve the following:

1. A radio listed at \$59 cash is sold on the installment plan for \$5 down and 12 monthly payments of \$5 each. What is the equivalent interest rate to the nearest 0.1% under the constant-ratio method?

2. The Housewife's Friendly Finance Company offers a loan of \$300 to be repaid in 8 monthly installments of \$41.25 each. What is the annual interest rate as computed under the residuary method? Under the constant-ratio method?

3. A boat is offered for sale for \$900 cash or a down payment of \$300 and the balance in 9 monthly payments of \$75 each. Find the equivalent interest rate to the nearest 0.1% under the constant-ratio method.

4. The Security-First National Bank offers to lend \$500 to be repaid in 6 equal monthly installments of \$86.50 per month. What is the annual interest rate as computed under the constant-ratio method?

5 On orders totaling $111 Sears required a down payment of $11, a carrying charge of $10 was added, and the balance of $110 was paid in equal monthly payments of $10 Find the equivalent interest rate to the nearest 0 1% under the constant ratio method

6 A deepfreeze is offered for sale for $450, with a down payment of $150 and the balance in 9 monthly installments of $37 50 The buyer has $150 cash and can borrow $300 from a finance company to be repaid in 8 monthly installments of $41 25 each Which method is at the lower rate of interest and by how much?

7 A television set is listed for $299 50 cash It may be bought with a down payment of $49 50 The balance of $250 plus the carrying charge is to be paid in 12 equal monthly payments of $25 Find the rate under the constant ratio plan to the nearest 0 1%

8 After buying a used car, Casey Winstead found that he owed Honest John $250 Under the terms of his contract he is to pay $30 at the end of each month for the next 9 months What simple interest rate is he paying under the Merchants Rule? Under the constant ratio method?

9 Ralph McConachie published the following schedule of payments charged on auto repairs Compute the simple interest rate using the constant ratio plan

Amount Owed	*Monthly Payment*	*Number of Payments*
$ 50 00	$ 9 00	6
100 00	14 00	8
175 00	16 25	12

10 An industrial bank offers to lend $1,000 to be repaid in 12 monthly installments of $90 32 Find the simple interest rate charged under the constant-ratio plan

11. A paper boy, saving his money to go to college, deposited it in a savings bank at 2% interest After he had saved $150 he bought a $55 bicycle on the installment plan under the following terms down payment of $10, balance to be paid in 5 monthly installments of $10 each What simple interest rate did he pay under the Merchants Rule? Under the constant ratio plan?

Equation of value

Since money can be used to produce income, it is to be expected in business transactions that debts not paid when due should bear interest, and that a reduction should be made in the amount due if payment is made early It is a fundamental principle of the mathematics of finance

that the present value of a sum of money due in the future depends both upon the time which must elapse before the sum is due, and the rate of interest which is used in determining the present value. Thus if money is worth 6%, \$1,000 today is worth exactly the same as \$1,030 to be received 6 months hence. Expressed in another way, a person to whom money is worth 6% would just as soon have \$970.87 $\left(\frac{\$1,000}{1 + \frac{1}{2} \times 6\%}\right)$ today as \$1,000 six months later.

Three months from now—that is, three months before the \$1,000 is to be received—if money is still worth 6%, the person would just as soon have \$985.22 $\left(\frac{\$1,000}{1 + \frac{1}{4} \times 6\%}\right)$ as the \$1,000 three months later.

When the \$1,000 is received at the end of the 6 months, it would be worth exactly \$1,000. If it were not received when due, but were paid 3 months later, the one to whom it was owing would feel—and justly so—that he should receive not \$1,000 but \$1,000 plus interest for 3 months at 6%, or \$1,015.

In business it is often necessary to make comparisons between two or more sums of money due at different times and perhaps bearing interest at different rates. The value of such payments cannot be compared unless the value of one is discounted to or accumulated to the date the other is due, or unless they are both evaluated on a common date. The date chosen for such a comparison is called the *focal date.* The equation used to relate two or more sums of money on a focal date is called an *equation of value.*

Illustration: Richard Glenn is planning to accept a position abroad for 2 years. His salary will be high enough that within five months he can pay off all his debts. He now wants to sign a note due in 5 months to discharge three debts: one of \$300 due today; one of \$750 due 4 months hence; and one of \$750 due 8 months hence. If money is worth 5%, what should be the amount of a noninterest bearing note which he gives today payable 5 months hence to discharge the three debts?

These facts can be shown on a time diagram as follows:

Old Obligations						Focal Date			
	\$300				\$750	↓			\$750
	0	1	2	3	4	5	6	7	8 months hence
New Obligation						X at 5%			

If the focal date is selected as the date payment is to be made, and X is the amount of the note, then X must equal the sum of the values of the three obligations on that date These values are

(a) The value of \$300 in 5 months at 5%,

$$\$300\ 00 + \$300\ 00 \times \tfrac{5}{12} \times 5\% \qquad = \$\ 306\ 25$$

(b) The value of \$750 one month after the due date,

$$\$750\ 00 + \$750\ 00 \times \tfrac{1}{12} \times 5\% \qquad = \quad 753\ 12$$

(c) The value of \$750 three months before the due date,

$$\frac{\$750\ 00}{1 + \frac{1}{4} \times 5\%} \qquad = \quad \underline{740\ 74}$$

$$\$1{,}800\ 11$$

Therefore a note for \$1,800 11 payable 5 months hence would be equal in value to the three debts

Average due date

If a series of debts is owed it may be necessary to know on what date the total amount may be paid without interest or discount A similar problem is involved in settling an account made up of several debts on which partial payments have been made The problem may be one of finding a single payment, or determining what series of payments will equitably discharge the debt

To find the date on which a series of debts may be equitably discharged by a single payment equal to the face value of the sum of the debts is to find the *average due date*, or the *equated date* If there are any past due debts in the account, the average due date may actually be a date that has already passed *Once the date is known, however, adjustments may be made from that date by adding interest if the payment is to be made later, or discounting the amount if the value is to be found at an earlier date*

To find the average due date choose any convenient date as the focal date As a matter of practice, the earliest due date is generally selected Since the objective of the computation is to find the *average due date*, the date of purchase or the date the obligation is incurred does not enter into the problem In solving such problems a concept of weighted dollars, known as *dollar-day*, is used The theory is that at any given interest rate a dollar for 1 day is just as valuable as another dollar for 1 day The debts are converted into dollar days and the payments are converted into dollar-days by multiplying the amount of each debt or payment by the number of days from the focal date The difference between the

dollar-days of the debt and the dollar-days of the payments is divided by the net balance of the debt to find the number of days from the focal date. The addition or subtraction of this number of days from the focal date will give the average due date.

Illustrations:

a. Find the average due date for the following purchases by John Dunn from The Desert Catalyst Company: July 30, $750; August 15, $400; and August 25, $300.

Select July 30 as the focal date.

Due Date	*Amount*	*Days from Focal Date*	*Dollar-Days*
July 30	$ 750	0	0
August 15	400	16	6,400
August 25	300	26	7,800
	$1,450		14,200

Then $14,200 ÷ $1,450 = 10, to the nearest whole number. Thus the average due date is 10 days beyond the focal date of July 30—in other words, the average due date is August 9. If payment is made before August 9, anticipation may be deducted; if payment is made later, say August 30, interest should be added to the amount paid.

b. The Michael Thomas Printing Company's account appears on the books of the Kelly Paper Company as follows:

Debits		*Credits*	
Due Date	*Amount*	*Date*	*Payment Received*
January 15	$ 1,200	February 2	$1,000
February 25	1,800	March 5	1,500
March 15	3,000	March 15	2,000
April 15	5,000		
	$11,000		$4,500
Less	4,500		
Balance	$ 6,500		

The accountant at the printing company wants to make full payment to the Kelly Paper Company on the last date which will make an equitable settlement without interest or discount. On what date should he pay the balance of $6,500?

Select January 15 as the focal date.

	Debits			Credits	
Days	Amount	Dollar-Days	Days	Amount	Dollar-Days
0	1,200	0,000	18	1,000	18 000
11	1 800	73,800	19	1,500	73,500
59	3 000	177,000	59	2,000	118,000
90	5,000	150,000			
	Total	700,800		Total	209,500
	Less	209,500			
	Balance	491,300 debit dollar-days			

Now 491,300 dollar days ÷ \$6,500 = 75 days (to the nearest whole day) The focal date was January 15 Therefore payment of \$6,500 on the 75th day following January 15, that is, on March 31, would settle the debt equitably

EXERCISE 10 9

Solve the following

1. A debt of \$120 is due 6 months from now If money is worth 6%, what is the value of the debt 3 months from now, 12 months from now?

2. A debt of \$840 is due 10 months hence If money is worth 4%, what is the value of the debt 6 months hence, 10 months hence, 12 months hence?

3. Ray Morrison owes \$300 due in 4 months and \$500 in 8 months He wants to pay the debts in 2 equal installments, one at the end of 3 months and one at the end of 6 months If money is worth 5%, what should be the size of the 2 payments?

4. What should be the size of 3 equal payments to be made at 3-month intervals to discharge a debt of \$3,000 due today, if money is worth 6%? The first payment is to be made 3 months from now

5. Find the average due date for the following account which represents purchases by St. Mary's Academy from the American Hospital Supply Company October 1, \$1,600, October 15, \$580, November 1, \$3 000

6. Western Fertilizer Company sold to the White and White Nursery material which was to be paid for as follows: June 30, \$1,500, July 15, \$500, July 30, \$2,200 On what date may the account be paid with the net amount of the purchases?

7. Find the average due date for an account which contains the following amounts: June 15, $440; June 25, $385; September 10, $4,200.

8. From what date should interest be paid on the total of the following accounts: January 15, $625; January 28, $420; February 5, $200?

9. A ledger account for the Champaign Detinning Company showed the following:

Debits		*Credits*	
Due Date	*Amount*	*Payment Date*	*Amount*
March 15	$ 620	March 20	$ 500
March 30	1,200	April 3	600
April 5	2,150	April 12	1,000
April 20	200		

On what date may the balance be equitably paid without interest?

10. What is the last date upon which the balance of the following account should be paid without interest?

Debits		*Credits*	
April 5	$ 225	April 2	$ 200
April 20	1,050	April 15	1,100
July 15	3,200		

11. Johnson and Johnson owe $450 which falls due on September 25. On August 20 they pay $200 on the debt. If money is worth 6%, what sum will equitably discharge the debt on the due date?

12. If interest at 6% is to be added to an account from the average due date, what amount should be paid to settle the following account on June 30?

Debits		*Credits*	
April 5	$1,200	April 25	$400
April 30	1,800	June 10	600
June 15	3,200	June 18	225

13. If interest is to be added at 6%, what will settle the following account on December 1?

Debits		*Credits*	
July 7	$ 420	August 25	$300
August 28	1,200	September 3	600
September 10	1,800		
November 15	2,500		

14. A schoolteacher decided to build and sell a house, with the expectation of making a profit He borrowed three sums of money $3,000 due 2 months from now, $1,000 due 5 months from now, and $7,000 due 9 months from now The house is now finished and is to be sold How much should he pay the lender now to discharge equitably the three debts if money is worth 7%?

15. A contractor is engaged to carry certain grading operations in a subdivision The progress payments of $2,000 each are to be made at intervals of every 2 months for a total of 5 payments If no payments have been made, if money is worth 6%, what payment made 6 months hence would equitably discharge this obligation?

11

Merchandising Mathematics

Introduction

Mathematical principles and techniques have applications in almost every type of business. Because of the complexity and diversity of modern industry it is impossible to show such applications in even the major types of business, and difficult to justify concentrating attention on the use of mathematics in only one. To overcome this problem, an attempt is made in this chapter to show the applications of mathematics in one type of operation which permeates all business activity. While there is great variation in the size of enterprises and in types and quantities of commodities used, the purchase and sale of goods is an important operation in almost every line of business, and one in which the mathematical problems encountered are similar.

Merchandising mathematics

By and large, merchandising mathematics entails the use of the fundamental operations of arithmetic, some simple algebraic principles, and an understanding of the use of percentage. In Chapter 5, the general subject of trade discounts and cash discounts was discussed as an application of the principles of percentage. The following points were developed:

1. Trade discount is deducted from the list price, which does not include the cost of transportation.

2. If two or more trade discounts are to be taken they must be taken successively. It is customary to compute the complement of a series of discounts which is referred to as the "on" percentage. The product of the list price and the "on" percentage gives the net price.

3. The invoice shows the net price of the goods bought after the deduction of trade discount.

4 If the invoice is paid within the discount period the cash discount is deducted Cash discount is the product of the discount rate allowed and the amount of the invoice (excluding transportation charges)

Anticipation

It is not necessary to describe all the various terms of cash discount found in business The fundamental fact is that cash discount is figured on the amount of the invoice, provided that payment is made *before* a specified date The element of time plays no part in the calculation of cash discount Thus on an invoice dated April 20, *Terms* 3/10 E O M cash discount may be taken if the payment is made at any time between April 20 and May 10 Under these terms, a buyer has no inducement to pay before the last date on which discount can be taken Payment before this last day is called *anticipation* The practice is growing, particularly among retailers of deducting from the bill the exact simple interest *for the exact number of days between the date of the invoice and the last day of the discount period* The amount deducted is the *anticipation*, or *anticipation interest* as it is sometimes called

To find the amount payable on a given date, find first the amount payable on the last day of the discount period by deducting the cash discount from the amount of the invoice With this figure as the base, find the anticipation precisely as exact interest would be found by multiplying the base by the rate for the exact number of days between the day that payment is made and the last day of the discount period The amount of the anticipation is then deducted from the base

Illustration An invoice for \$327 40 is dated April 26, terms are 3/10 E O M The anticipation rate is 5% What amount is necessary for full payment of the bill on May 11?

Discount may be taken any time up to June 10 Payment on May 11 is 30 days before the end of the discount period

Face amount of invoice	\$327 40	
Less cash discount of 3%	9 82	\$317 58
Less anticipation (\$317 58 × 5% × $\frac{30}{365}$)		1 30
		\$316 28

To make it easier to calculate deductions which can be made for anticipation at various rates and periods, a table can be constructed showing the amount which should be deducted for each \$100 at varying rates and days Such a table, in effect a table of exact interest on \$100 for the days and rates indicated, is shown as Table 2

TABLE 2. DEDUCTIONS ON $100 FOR ANTICIPATION AT VARYING RATES AND PERIODS

Days	1%	2%	3%	4%	5%
1	.00274	.00548	.00822	.01096	.01370
2	.00548	.01096	.01644	.02192	.02740
3	.00822	.01644	.02466	.03288	.04110
4	.01096	.02192	.03288	.04384	.05479
5	.01370	.02740	.04110	.05479	.06849
6	.01644	.03288	.04932	.06575	.08219
7	.01918	.03836	.05753	.07671	.09589
8	.02192	.04384	.06575	.08767	.10959
9	.02466	.04932	.07397	.09863	.12329
10	.02740	.05479	.08219	.10959	.13699
11	.03014	.06027	.09041	.12055	.15068
12	.03288	.06575	.09863	.13151	.16438
13	.03562	.07123	.10685	.14247	.17808
14	.03836	.07671	.11507	.15342	.19178
15	.04110	.08219	.12329	.16438	.20548
16	.04384	.08767	.13151	.17534	.21918
17	.04658	.09315	.13973	.18630	.23288
18	.04932	.09863	.14795	.19726	.24658
19	.05205	.10411	.15616	.20822	.26027
20	.05479	.10959	.16438	.21918	.27397
21	.05753	.11507	.17260	.23014	.28767
22	.06027	.12055	.18082	.24110	.30137
23	.06301	.12603	.18904	.25205	.31507
24	.06575	.13151	.19726	.26301	.32877
25	.06849	.13699	.20548	.27397	.34247
26	.07123	.14247	.21370	.28493	.35616
27	.07397	.14795	.22192	.29589	.36986
28	.07671	.15342	.23014	.30685	.38356
29	.07945	.15890	.23836	.31781	.39726
30	.08219	.16438	.24658	.32877	.41096

In the example just considered, the deduction made for early payment is $1.34. To calculate this from the table, it is necessary to look under the 5% column to find the amount of anticipation on $100 for 30 days (0.41096). This number (0.41096), multiplied by the amount of the invoice less the cash discount, and divided by 100, gives the amount of deduction for anticipation. That is, $\$0.41096 \times \frac{\$317.58}{100} = \$1.30$.

To operate a business efficiently and to keep costs at a minimum, mathematical operations should be combined whenever possible. Any time that many multiplications are entailed it is essential from the standpoint of costs to have calculating machines of some kind available.

The machines can handle large numbers quickly and accurately It is necessary, however, that someone establish the method to be followed

The computation of cash discount and anticipation can be shortened in the following manner

1 Subtract the cash discount rate from 100% and change into its decimal equivalent

2 Move the decimal point 2 places to the left in the figure taken from the table and subtract from 1

3 Multiply the amount of the invoice which is subject to discount by the figures found in Steps 1 and 2

By this method the preceding example would be solved as follows

Step 1 $100\% - 3\% = 97\% = 0\ 97$

Step 2 Tabular figure is 0 41096, $1 - 0\ 0041096 = 0\ 9958904$

Step 3 $\$327\ 40 \times 0\ 97 \times 0\ 9958904 = \$316\ 24$

The anticipation table is a simple interest table based on a 365 day year Thus it is possible to find the anticipation for any number of days by adding the amounts shown in the table For example, to find the anticipation at 5% for 42 days, combine the tabular figures for 30 days and 12 days

Anticipation on \$100 for 30 days = 0 41096
Anticipation on \$100 for 12 days = 0 16438
Anticipation on \$100 for 42 days = 0 57534

EXERCISE 11.1

Solve the following

1. An invoice for \$932 60 is dated September 14, terms 2/10 30 Extra If payment is made September 19, how much should be remitted if anticipation at 4% is allowed?

2. An invoice for \$1,237 56 is dated July 17, and carries terms 2/10 n/30 If payment is made July 21, what is the amount payable to the vendor if anticipation at 5% is permitted?

3. An invoice for \$327 84, dated January 4, carries terms 2/15 E O M Anticipation at 3% is permitted What amount should be remitted if the bill is paid January 19?

4. An invoice for \$3,827 60 is dated May 12, terms 3/20 45 Extra If payment is made May 20, how much should be remitted if anticipation at 3% is allowed?

5. An invoice for $437.64, dated December 1, carries terms 3/10 E O M 30 Extra. Anticipation at 4% is permitted. What amount should be remitted if the bill is paid December 5?

6. An invoice, dated December 28, for $278.25 carries terms 4/15 E O M. If payment is made January 20, how much should be remitted if the anticipation interest allowed is 2%?

7. An invoice for $860.50, dated January 20, terms: 2/10 60 Extra. Anticipation at 7% is permitted. What amount should be remitted if the bill is paid February 1?

8. An invoice for $2,160, dated February 16, has the following terms: 4/10 2/30 n/60. Anticipation at 3% is permitted. How much should be remitted if the bill is paid February 18? If the bill is paid March 1?

Markup

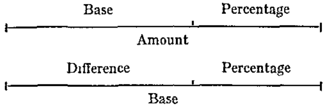
The purchaser of merchandise for resale hopes to gain by selling it at a price high enough to cover the cost of the goods, the expenses incurred in selling it, and the amount of desired profit. It is essential, therefore, that his selling price be higher than his cost. The amount that he adds to the cost is known as the *markup*. The markup may be stated as a certain per cent either of the cost or of the selling price. The cost, plus the markup, equals the selling price, or what is commonly known to the retail merchant as the *retail*.

The manufacturer has the problem of determining the price he will charge for the goods manufactured. It is a generally accepted practice for the manufacturer, as well as for many small retailers, to refer to markup as a percentage based on *costs*, whereas larger retailers refer to markup as a per cent of *selling prices*.

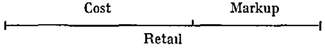
Some people who are unaware of this difference or who fail to realize its significance tend to think the retailer's markup is exorbitant. An examination of the facts shows the difficulty of making accurate comparisons. For example, a manufacturing concern makes 100,000 units at a total cost of $1.00 per unit. If these units are sold at $1.10 there has been a 10% markup based on the cost of the article. A retailer receiving this article at $1.10 may set his selling price at $1.65—that is, a markup based on the cost of the article plus 50%. Obviously this appears much higher than the 10% added by the manufacturer. If the retailer states his markup of 55 cents on the article as a per cent of the selling price of $1.65 it appears to be much lower, since it is now only $33\frac{1}{3}\%$ ($\frac{55}{165}$) of the selling price. Hence the retailer may feel less open to criticism. Actually the comparison should be made between the profits involved in the two operations. Whereas all the manufacturer's costs were included in the

figure on which he based his markup, the retailer must pay all his costs of operation out of the markup he receives Thus his net profit out of every dollar of sales may be much less than the profit of the manufacturer

To show the differences between computing the markup based on cost and the markup based on retail the similarity between these relationships and those included in an earlier chapter should be observed In discussing percentages, the base, the percentage, the amount, and the difference were represented by a diagram as follows

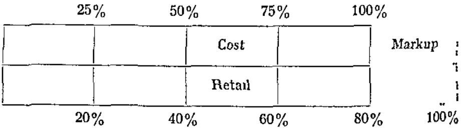


The relationship between cost, markup, and retail can be shown in much the same way



Here, when markup is stated as a percentage of cost, the cost is the base, the markup is the percentage, and the retail is the amount But when the markup is stated as a percentage of retail, the cost is the difference, the markup is the percentage, and the retail is the base

If, for example, an article costs \$1 00 and sells for \$1 25, the markup on cost is $\frac{\$0\ 25}{\$1\ 00}$, or 25%, and the markup on retail is $\frac{\$0\ 25}{\$1\ 25}$, or 20%



This diagram shows that the markup in this example is equal to $\frac{1}{4}$ of the cost or $\frac{1}{5}$ of the retail It can be seen that since the retail is more than the cost, the markup based on the retail price will always be a smaller per cent than the markup based on the cost price In merchandising mathe-

matics, markup is virtually always considered as based on the retail price. If no statement is made to the contrary, it may be assumed that the per cent of markup refers to the retail price.

By and large, the basic types of markup problems fall into fairly well-defined categories. To facilitate an explanation of the problems and to demonstrate their solutions, the following symbols are used:

C = Cost of merchandise
M = Markup or gross profit
R = Retail or selling price
$C\%$ = Per cent markup on cost
$R\%$ = Per cent markup on retail or selling price

To state the relationships in words and in symbols, it should be quickly recognized that:

Cost plus markup equals retail, or $C + M = R$.
Retail minus markup equals cost, or $R - M = C$.
Retail minus cost equals markup, or $R - C = M$.

The per cent markup on cost equals markup divided by cost, or $C\% = \frac{M}{C}$. Therefore $M = C \times C\%$, and $C = \frac{M}{C\%}$.

The per cent markup on retail equals markup divided by retail, or $R\% = \frac{M}{R}$. Therefore $M = R \times R\%$, and $R = \frac{M}{R\%}$.

Since $M = C \times C\%$, then $C + M = R$ becomes $C + C \times C\% = R$, or $C(1 + C\%) = R$. Therefore $C = \frac{R}{1 + C\%}$.

Since $M = R \times R\%$, then $R - M = C$ becomes $R - R \times R\% = C$, or $R(1 - R\%) = C$. Therefore $R = \frac{C}{1 - R\%}$.

Finding the per cent of markup when cost and retail price are known

A common problem in merchandising is that of finding the per cent of markup when cost and retail are given.

Illustration: The billed cost of goods is \$118. Transportation charges are \$2. If the goods are sold for \$180, what is the per cent of markup received on the goods? (Transportation charges are ordinarily considered as part of the cost, and the per cent of markup is figured on the net price of goods, excluding any deduction for cash discount.)

$$R - C = M, \quad C\% = \frac{M}{C}, \quad R\% = \frac{M}{R}$$

Substituting $R = \$180$, $C = \$120$

$$\$180 - \$120 = \$60 \text{ markup}$$

$$C\% = \frac{\$60}{\$120} = 50\% \text{ markup on cost}$$

$$R\% = \frac{\$60}{\$180} = 33\tfrac{1}{3}\% \text{ markup on retail}$$

EXERCISE 11.2

Solve the following

	C	R	M	$C\%$	$R\%$
1.	$ 150	$ 300	?	?	?
2.	$ 25	$ 40	?	?	?
3.	$ 8 50	$ 12	?	?	?
4.	60 cents	85 cents	?	?	?
5.	$ 3 50	$ 4 90	?	?	?
6.	$ 18 50	$ 25	?	?	?
7.	$1,200	$1,650	?	?	?
8.	$ 2 20	$ 3 50	?	?	?
9.	$ 1 25	$ 1 80	?	?	?
10.	$ 12 50	$ 17 25	?	?	?

11. Nylon hose were bought at $20 per dozen and sold at $2 50 per pair What was the markup per cent on cost, and on retail?

12. Lawnmowers were bought at $8 50 each Transportation charge for each is 25 cents If sold at $12 50 each, find the per cent markup on cost, and on retail

13. A merchant bought 20 dozen tumblers at $4 50 per dozen They were sold at $6 30 per dozen Find the per cent markup on retail

14. Toasters that cost $10 50 were sold for $15 Find the per cent markup on cost

15. Fruit that cost 12 cents per pound was sold for 15 cents per pound Find the per cent markup on cost, and on retail

Finding the cost when retail and per cent markup on retail are known

Department stores usually sell at established price lines For example, a store may sell men's suits only at the established prices of $47 50, $62 50, $79 50, and $94 50 Each of these figures could be referred to as

a price line. In buying goods, a buyer must keep two things in mind: first, the established price at which the merchandise must be sold; and second, the per cent of markup which must be maintained in the department. Consequently the buyer is often faced with the problem of calculating how much he may pay for an article when he knows the price at which it must be sold and the markup which must be obtained. An example of finding the cost when the retail and per cent of markup on retail are known is shown in the following illustration.

Illustration: A buyer from a department store sees collegiate dresses which he estimates will retail at $29.95. In his department he must have a markup of 45% of retail. How much may he pay for the dresses?

$$C = R - M; \quad M = R \times R\%$$

Substituting: $R\% = 45\%$; $R = \$29.95$

$$\$29.95 \times 45\% = \$13.48 \text{ markup}$$
$$\$29.95 - \$13.48 = \$16.47, \text{ cost}$$

EXERCISE 11.3

Solve the following:

	R	$R\%$	M	C
1.	$400	35%	?	?
2.	$250	$37\frac{1}{2}\%$	?	?
3.	$ 75	$33\frac{1}{3}\%$	?	?
4.	$ 45.50	35%	?	?
5.	$ 32.50	50%	?	?
6.	$ 27.95	30%	?	?
7.	$ 18.75	$36\frac{1}{4}\%$	?	?
8.	$ 8.79	25%	?	?
9.	$ 4.29	45%	?	?
10.	$815.50	32%	?	?

11. A department store buyer sees sweaters which he estimates will retail at $12.50. In his department he must have a markup of 35% on retail. How much may he pay for the sweaters?

12. A buyer finds that one of the most popular prices for women's gloves is $4.50. He must have a markup of 37% on retail. How much may he pay for the gloves to sell at $4.50 if he is to obtain his 37% markup?

13 From past experience the buyer of men's accessories knows that \$3 50 is a popular price for men s ties at Christmas In his department he must have a markup of 52% on retail How much may he pay for ties to be sold at \$3 50 if he is to make the necessary markup?

14. The buyer for a yarn shop must sell fingering yarn used to make socks for 75 cents per ounce In her department she must have a markup of 45% of retail How much may she pay for each pound box?

15. The buyer for a stationery store finds an unusual item, special stationery for men which will sell for \$2 50 If stationery markup in his store is $37\frac{1}{2}$% of retail, how much may he pay for this unusual item?

Finding retail when cost and per cent of markup on retail are known

Often a merchant is interested in finding the retail when the cost and the per cent of markup on retail are known This is a daily problem in most retail stores

Illustration A price on ceramic pins of 80 cents each is quoted by a manufacturer The merchant estimates that he needs 60% on retail to cover expenses and shortages and still leave a fair profit At what price should he sell the pins?

$$\begin{aligned} 100\%R &= \text{Retail} \\ -\ 60\%R &= \text{Markup} \\ \hline 40\%R &= \text{Cost} = \$0\ 80 \\ R &= \$2\ 00 \end{aligned}$$

EXERCISE 11.4

Solve the following

	C	$R\%$	R		C	$R\%$	R
1.	\$40	40%	?	6.	\$ 2 10	25%	?
2.	\$32 50	32%	?	7.	\$ 62 50	$33\frac{1}{3}$%	?
3.	\$ 8 50	28%	?	8.	\$125	65%	?
4.	60 cents	45%	?	9.	\$ 12 50	$27\frac{1}{2}$%	?
5.	\$18 75	$37\frac{1}{2}$%	?	10.	\$ 38 75	42%	?

11. In a garden supply department, the buyer anticipates a $37\frac{1}{2}$% markup on retail If garden spray guns cost \$5 25 each, find the retail price needed

12. If a buyer needs a markup of $42\frac{1}{2}$% on retail, find the selling price of an article which costs 63 cents

13. The buyer pays \$8 for a 1 pound box of yarn. In her shop she must have a 40% on retail. At what price must she sell each ounce of the yarn to have the desired markup?

14. A hardware merchant bought 1,000 feet of rope for \$8.30, to be sold by the pound. On the average there are 25 feet per pound for rope of this size. If he expects a markup of 35% on retail, at what price per pound should he sell the rope?

15. If a buyer needs a markup of $62\frac{1}{2}\%$ on retail, find the selling price of an article which costs \$45.60.

Finding retail when cost and markup on cost are known

One of the simplest problems in markup is to find the retail price when cost and markup per cent on cost are known.

Illustration: A merchant buys a number of items at 75 cents each. He wants a markup of $33\frac{1}{3}\%$ on cost. What is retail?

$$R = C + M; \quad M = C \times C\%$$

Substituting: $C\% = 33\frac{1}{3}\%; \quad C = \$0.75$

$$\$0.75 \times 33\tfrac{1}{3}\% = \$0.25 \text{ markup}$$
$$\$0.75 + \$0.25 = \$1.00\text{, retail}$$

EXERCISE 11.5

Solve the following:

	C	$C\%$	R		C	$C\%$	R
1.	\$45	50%	?	**6.**	\$ 1.25	140%	?
2.	\$62.50	40%	?	**7.**	38 cents	250%	?
3.	\$25.50	35%	?	**8.**	\$ 5.75	$37\frac{1}{2}\%$	?
4.	\$ 1.80	85%	?	**9.**	\$ 18.80	75%	?
5.	\$42.50	100%	?	**10.**	\$142.50	$87\frac{1}{2}\%$	?

11. If an item costs \$5 and the markup per cent on cost is 60%, find the retail price.

12. Find the retail price of a ceramic pin that costs 80 cents if the markup on cost is 125%.

13. If the markup is 25% on cost, find the selling price for each item bought at \$1.80 per dozen.

14. If the markup is 80% on cost, find the selling price for each item bought at \$36 per gross.

15. If a buyer needs a markup of 120% on cost, find the selling price of an article which costs 24 cents.

Finding the cost when retail and markup per cent on cost are known

In department stores, markups for particular departments are sometimes established on cost In purchasing goods, it is essential that a buyer consider the price line—that is, the retail price—and the per cent markup on cost so that he will be able to determine readily just how much he can afford to pay for any product The problem resolves itself into one of finding the cost when retail and markup per cent on cost are known

Illustration Hats are offered to a buyer which he will have to sell at his established price of \$7 95 He needs a markup on cost of $66\frac{2}{3}\%$ How much can he afford to pay for the hats?

$$C = R - M, \quad M = C \times C\%$$

Substituting $C\% = 66\frac{2}{3}\%$, $R = \$7\ 95$

$$C = \$7\ 95 - 66\tfrac{2}{3}\% \times C, \quad 166\tfrac{2}{3}\% \times C = \$7\ 95, \quad C = \$4\ 77$$

EXERCISE 11.6

Solve the following

	R	$C\%$	C		R	$C\%$	C
1.	\$ 40	50%	?	6.	\$ 44 50	120%	?
2.	\$120	80%	?	7.	\$1,750	75%	?
3.	\$ 25	$42\frac{1}{2}\%$	?	8.	\$ 35	60%	?
4.	\$ 16 50	35%	?	9.	\$ 150	85%	?
5.	\$ 39 95	$62\frac{1}{2}\%$	?	10.	\$2,460	200%	?

11. If the price for doeskin gloves has been established at \$4 20 and the markup on cost is 40%, what is the maximum amount a buyer can pay for the gloves?

12. A buyer is offered children's shoes which he wants to sell at an established price of \$5 95 He needs a markup on cost of 30% How much can he afford to pay for the shoes?

13. A buyer has found small bed lamps which he believes will sell readily at \$6 75 He expects a markup on cost of 70% How much can he afford to pay for the lamps?

14. If the price for a private brand television set has been established at \$180 and the markup on cost is 35%, what is the maximum amount a store can pay for the set?

15. A buyer has found small radios which will sell at \$27 95 He expects a markup on cost of $62\frac{1}{2}\%$ How much can he afford to pay for 20 radios?

Finding equivalent markups

If it is frequently necessary to find the equivalent of the markup on cost when the markup on retail is known, a table of markup equivalents can easily be constructed. The following illustration shows a method which can be followed in making such a table or in calculating a single equivalent.

Illustration: Find the markup on cost that is equivalent to 40% markup on retail.

$$R - M = C \qquad \frac{\text{Markup}}{\text{Cost}} = C\%$$

$$\begin{array}{r} 100\% \times R = \text{retail} \\ 40\% \times R = \text{markup} \\ \hline 60\% \times R = \text{cost} \end{array} \qquad \frac{40\% \times R}{60\% \times R} = C\% = \frac{40}{60} = 66\tfrac{2}{3}\%$$

EXERCISE 11.7

Solve the following:

	$R\%$	$C\%$		$R\%$	$C\%$		$R\%$	$C\%$
1.	20%	?	**6.**	45%	?	**11.**	$42\frac{1}{2}\%$	?
2.	25%	?	**7.**	50%	?	**12.**	75%	?
3.	30%	?	**8.**	44.4%	?	**13.**	$56\frac{1}{4}\%$	?
4.	35%	?	**9.**	$66\frac{2}{3}\%$	?	**14.**	48%	?
5.	40%	?	**10.**	80%	?	**15.**	$37\frac{1}{2}\%$	?

One other type of problem is to find the markup on retail when the markup on cost is known.

Illustration: Find the markup on retail that is equivalent to 65% markup on cost.

$$C + M = R \qquad \frac{\text{Markup}}{\text{Retail}} = R\%$$

$$\begin{array}{r} 100\% \times C = \text{cost} \\ 65\% \times C = \text{markup} \\ \hline 165\% \times C = \text{retail} \end{array} \qquad \frac{65\% \times C}{165\% \times C} = R\% = \frac{65}{165} = 39.4\%$$

EXERCISE 11.8

Solve the following:

	$C\%$	$R\%$		$C\%$	$R\%$		$C\%$	$R\%$
1.	50%	?	**6.**	90%	?	**11.**	180%	?
2.	60%	?	**7.**	100%	?	**12.**	120%	?
3.	70%	?	**8.**	$37\frac{1}{2}\%$	?	**13.**	$112\frac{1}{2}\%$	?
4.	75%	?	**9.**	400%	?	**14.**	650%	?
5.	$83\frac{1}{3}\%$	?	**10.**	320%	?	**15.**	800%	?

Averaging markup

Usually the buyer in each department of a store is allowed considerable latitude in determining the amount of markup for each particular item but the store management generally determines the policy for the entire department For example, the buyer in the book and stationery department may be told that his department must have an average markup of 40% The buyer then knows that any markup of less than average must be balanced by a markup greater than average if he is to achieve the desired average markup Several different types of problems may arise in averaging markup The more common ones are illustrated

One problem is to find the markup needed on new purchases in order to have an average markup on all purchases planned

Illustrations

a A buyer plans to buy books and stationery to be sold for $10,000 during the month of December He pays $2,200 for stationery which customarily bears a 45% markup on retail What is the minimum markup he needs to get on the remaining purchase, in order to have an average markup of 40%?

	Cost	*Retail*	*Markup %*
Total purchases estimated for month	$6 000	$10,000	40%
Amount of purchases already made	2,200	4,000	45%
Balance to buy	$3,800	$ 6,000	

The buyer still has $3,800 to spend on goods to sell for $6 000 He needs a markup of $2,200 ($6,000 − $3,800 = $2,200), which is equal to $36\frac{2}{3}\%$ $\left(\frac{\$2\,200}{\$6,000} = 36\frac{2}{3}\%\right)$, the markup he must have on the balance of the purchases

b *During the month of June, the buyer in the sundries department* plans to purchase goods costing $10,000 with an average of $37\frac{1}{2}\%$ markup on retail By the middle of the month he has purchased $6,000 worth at cost and $9,000 at retail What markup at retail must be realized on the balance of the June purchases?

	Cost	*Retail*	*Markup %*
Estimated purchases for month	$10 000	$16,000	$37\frac{1}{2}\%$
Purchased to date	6 000	9,000	$33\frac{1}{3}\%$
Balance to purchase	$ 4,000	$ 7,000	

The balance of \$4,000 to be purchased must be sold for \$7,000 to achieve the desired markup. In other words, it must be sold at a markup of about 43% $\left(\frac{\$3,000}{\$7,000} = 42.86\%\right)$.

EXERCISE 11.9

Solve the following:

1. A buyer plans to purchase goods to be sold for \$16,000 during a given month. On these purchases he plans to maintain an average markup of 40% on retail. He makes an initial purchase of \$3,600 at cost, which he believes will sell at an average markup of 50% on retail. What is the minimum markup he needs to get on the remainder of his purchase in order to have an average markup of 40%?

2. In the first quarter of the year a buyer plans to purchase goods having a total cost of \$11,700. He plans to have an average markup of 35% on retail. By the end of January he has purchased \$4,000 worth of goods which he estimates will retail at \$6,000. What markup must he obtain on the balance of his purchases to average 35% on retail?

3. A buyer plans to buy television sets to be sold for \$32,000 during the month of April. He pays \$12,000 for one make which customarily bears a 40% markup on retail. What is the minimum markup he needs to get on the remaining purchase, in order to have an average markup of 45%?

4. During May a buyer plans to purchase goods having a total cost of \$18,000. He plans to have an average markup of $33\frac{1}{3}\%$ on retail. By May 20 he has bought goods to sell for \$22,500 which have a markup on retail of 35%. What markup must he obtain on the balance of purchases to average $33\frac{1}{3}\%$ on retail?

5. A buyer plans to purchase items that will be sold for \$21,250 during the second quarter of the year. On these purchases he plans to maintain an average markup of 60% on retail. He makes an initial purchase , \$5,000 at cost, which he is sure will sell at an average markup of $66\frac{2}{3}\%$ on retail. What is the minimum markup he needs to get on the remainder of his purchase in order to have his desired average markup?

6. A department buyer expects to make purchases totaling \$12,000 at cost. Purchases to date are \$3,000 at cost and \$3,900 at retail. What markup must he gain on the remainder of his purchase to gain $37\frac{1}{2}\%$ average markup on retail?

Averaging markup

Usually the buyer in each department of a store is allowed considerable latitude in determining the amount of markup for each particular item, but the store management generally determines the policy for the entire department For example, the buyer in the book and stationery department may be told that his department must have an average markup of 40% The buyer then knows that any markup of less than average must be balanced by a markup greater than average if he is to achieve the desired average markup Several different types of problems may arise in averaging markup The more common ones are illustrated

One problem is to find the markup needed on new purchases in order to have an average markup on all purchases planned

Illustrations

a A buyer plans to buy books and stationery to be sold for $10,000 during the month of December He pays $2,200 for stationery which customarily bears a 45% markup on retail What is the minimum markup he needs to get on the remaining purchase, in order to have an average markup of 40%?

	Cost	*Retail*	*Markup %*
Total purchases estimated for month	$6,000	$10,000	40%
Amount of purchases already made	2,200	4,000	45%
Balance to buy	$3,800	$ 6,000	

The buyer still has $3,800 to spend on goods to sell for $6,000 He needs a markup of $2 200 ($6 000 − $3,800 = $2,200), which is equal to $36\frac{2}{3}\%$ $\left(\frac{\$2,200}{\$6,000} = 36\frac{2}{3}\%\right)$, the markup he must have on the balance of the purchases

b During the month of June, the buyer in the sundries department plans to purchase goods costing $10 000 with an average of $37\frac{1}{2}\%$ markup on retail By the middle of the month he has purchased $6,000 worth at cost and $9,000 at retail What markup at retail must be realized on the balance of the June purchases?

	Cost	*Retail*	*Markup %*
Estimated purchases for month	$10,000	$16,000	$37\frac{1}{2}\%$
Purchased to date	6,000	9,000	$33\frac{1}{3}\%$
Balance to purchase	$ 4,000	$ 7,000	

The balance of \$4,000 to be purchased must be sold for \$7,000 to achieve the desired markup. In other words, it must be sold at a markup of about 43% $\left(\frac{\$3{,}000}{\$7{,}000} = 42.86\%\right)$.

EXERCISE 11.9

Solve the following:

1. A buyer plans to purchase goods to be sold for \$16,000 during a given month. On these purchases he plans to maintain an average markup of 40% on retail. He makes an initial purchase of \$3,600 at cost, which he believes will sell at an average markup of 50% on retail. What is the minimum markup he needs to get on the remainder of his purchase in order to have an average markup of 40%?

2. In the first quarter of the year a buyer plans to purchase goods having a total cost of \$11,700. He plans to have an average markup of 35% on retail. By the end of January he has purchased \$4,000 worth of goods which he estimates will retail at \$6,000. What markup must he obtain on the balance of his purchases to average 35% on retail?

3. A buyer plans to buy television sets to be sold for \$32,000 during the month of April. He pays \$12,000 for one make which customarily bears a 40% markup on retail. What is the minimum markup he needs to get on the remaining purchase, in order to have an average markup of 45%?

4. During May a buyer plans to purchase goods having a total cost of \$18,000. He plans to have an average markup of $33\frac{1}{3}\%$ on retail. By May 20 he has bought goods to sell for \$22,500 which have a markup on retail of 35%. What markup must he obtain on the balance of purchases to average $33\frac{1}{3}\%$ on retail?

5. A buyer plans to purchase items that will be sold for \$21,250 during the second quarter of the year. On these purchases he plans to maintain an average markup of 60% on retail. He makes an initial purchase of \$5,000 at cost, which he is sure will sell at an average markup of $66\frac{2}{3}\%$ on retail. What is the minimum markup he needs to get on the remainder of his purchase in order to have his desired average markup?

6. A department buyer expects to make purchases totaling \$12,000 at cost. Purchases to date are \$3,000 at cost and \$3,900 at retail. What markup must he gain on the remainder of his purchase to gain $37\frac{1}{2}\%$ average markup on retail?

7. A buyer for a drug store expects to make purchases totaling $18,500 at retail To date he has purchased $7,000 at cost and $11,500 at retail Determine the markup he must gain on the remainder of his purchases to gain 40% average markup on retail

8. A buyer for a shoe store expects to make purchases totaling $8,250 at cost Purchases to date are $5,000 at cost and $9,500 at retail What markup is needed on the remainder of his purchases to gain 45% average markup on retail?

9. A buyer desires to make 35% on retail for purchases totaling $8,500 at retail To date he has purchased $2,000 at cost, which were sold for $3,000 What markup is needed on the remainder of his purchases?

10. A buyer desires to make 150% on cost for purchases totaling $5,525 at retail To date he has spent $800 at cost for goods which were sold for $2,000 What markup on cost is needed on the remainder of his purchases?

Buying at one cost to sell at two retails

In retailing, the fact that there are established price lines sometimes makes it necessary for the person running a department to have to purchase at one price items which must be sold at different retails To achieve the desired average markup he must be able to ascertain the proportion in which the different price lines should be bought

Illustration The cost of men's shirts is $2 50, and the necessary markup is 40% The nearest retail prices are $3 59 and $4 75 The buyer decides to offer shirts with one style of collar at $3 59 and shirts with another style of collar at $4 75 In what proportion should he buy two styles?

Established prices	$3 59	$4 75
Retail with 40% markup should be	4 17	4 17
Loss (−) or gain (+)	− $0 58	+ $0 58

Since he gains more than his minimum markup on one type and lacks by an equal amount gaining as much as he wants on the other type, the *number of each type purchased should be equal Thus on each shirt he sells* at $3 59 he will lack 58 cents making as much as he should, while on every shirt sold at $4 75 he will be 58 cents above what his average needs to be Thus if he buys the shirts in equal amounts he will achieve his desired average markup

Suppose, however, that his price line has been established at $3 95 and $4 95, in what proportion should he have bought the shirts?

Established prices	\$3.95	\$4.95
Retail with 40% markup should be	4.17	4.17
Loss (—) or gain (+)	— \$0.22	+ \$0.78

Since he gains 78 cents more on one than he must average and lacks 22 cents gaining as much on the other as he wants, he can afford to sell $3\frac{1}{2}$ ($78 \div 22$) shirts at the lower price for each one he sells at the higher price. Thus he should buy them in such a proportion, namely about 7 to 2. That is, he will sell 7 shirts at \$3.95 for every 2 shirts sold at \$4.95.

EXERCISE 11.10

Solve the following:

1. The established retail prices for women's dresses in a store are \$37.50 and \$42.50. If a manufacturer has a uniform cost price of \$28, and the store desires an average markup on retail of 30%, in what proportion should the store stock them?

2. The established retail prices for men's shirts in a store are \$3 and \$3.50. A manufacturer has two styles available, both at \$1.95. The buyer estimates that one style will sell at \$3.50 and the other at \$3. In what proportion should he buy the two styles if he seeks an average markup of 40%?

3. A certain style of portable radio sells at established retail prices of \$32.50 and \$37.50 depending on the trade name. Both are available at \$22.50 from the manufacturer. If an average markup of $37\frac{1}{2}$% is desired, in what proportion should a store stock them? If 30 radios are ordered, how many of them should be sold at each retail price?

4. Two styles of tricycles can be sold at \$37.50 and \$44.50. They are available from a manufacturer for \$26.00. In order to gain an average markup of 35% on 56 tricycles, how many should a store plan to sell at each price?

5. A buyer expects to have a markup of $33\frac{1}{3}$% on handbags. She can buy bags for \$8 each, some of which she expects to sell at \$11.50 and the others at \$12.75 each. In what proportion should she buy the two types in order to gain the desired markup?

6. A store plans to buy 80 tables that will sell for either \$67.50 or \$79.50, depending on the style. If both styles are available from a manufacturer for \$25 each, how many of each style should be bought if the store is to achieve an average markup of $66\frac{2}{3}$%?

7. *The established retail prices for women's dresses in a store are* $18 50 and $22 50 If a manufacturer has a uniform cost price of $12, and the store desires an average markup of 40%, on a purchase of 40 dresses how many will be purchased to sell at each price?

8 Two styles of pajamas sell for $4 50 and $5 45 If they can be purchased at a single cost of $3 60, and if a 30% markup is desired, in what proportion should they be purchased?

9 Two tools retail at $6 95 and $8 25, despite the fact that each costs $5 80 If an average markup of 25% is desired, in what proportion should a store stock them to assure this markup?

Buying at two costs to sell at one retail

In some departments there may be only one established price line and to get adequate stock it may be necessary to buy at various prices The problem then arises of how much to buy at each price

Illustration In the toy department, the established price of tricycles is $7 95 and the required markup is 35% The buyer at Christmas time wants to buy two styles of tricycles to sell in this price range, and he wants to limit his purchase to 500 The quoted wholesale prices are $5 00 and $5 50 How many should he buy at each wholesale price?

If the retail price is $7 95 *a markup of* 35% *would give an average* cost of $5 17 [$7 95 × (100% − 35%) = $5 17]

Actual cost	$5 00	$5 50
Average cost should be	5 17	5 17
Amount lost (−) or gained (+)	− $0 17	+ $0 33

Since the sale of one tricycle with the higher than average markup would be balanced by the sale of two with the lower than average markup, the buyer will need to buy in the ratio of two tricycles at $5 for each $5 50 tricycle in order to get the desired average markup

EXERCISE 11.11

Solve the following

1. A store has an established retail price of $4 25 for ladies' gloves The department needs an average markup of 40% on retail Wholesale prices are $2 85 and $2 45 In buying 100 pairs of gloves, how many should be bought at each price to gain an average markup of 40%?

2. The men's department of the ABC organization has an established price of $60 for men's suits. One manufacturer will sell them one style for $35 each, and another manufacturer will sell them another style for $45 each. They desire to have a markup of 35%. In what proportion should they be stocked?

3. Ties are to be sold for $2.20 each. In order to get a good supply, two wholesale houses were contacted. One offer was $1 each and the other offer was $1.30 each. Since neither house can supply the entire order desired, and if a markup of 45% is needed, in what proportion should the purchases be made?

4. Shirts cost $3.21 and $3.75 each. They are to be retailed at $5.95. For each 100 shirts bought, what should be the number purchased at each price in order to maintain the desired average markup of 40% on retail?

5. A certain grade of candy is to sell for an established price of $4.45 per box. Supplies are available at $2.50 per box and $3 per box. If 88 boxes per day can be sold and if $37\frac{1}{2}$% on retail is desired, how many at each cost price should be ordered?

Maintained markup

Although goods have an original markup, say 40%, of retail, this *initial markup*, as it is called, may not actually be received. If merchandise does not sell readily, the price may be reduced. The difference between the cost and the actual selling price is called the *maintained markup*. (In accounting the terms, *net markup* and *gross margin*, are used as synonymous with *maintained markup*.) The reduction in selling price is called the *markdown*.

Initial markup

A buyer must be able to ascertain the maintained markup percentage which will result from a given markdown. In planning the initial markup of stock, it should be realized that the markup must be high enough to cover not only the cost of the goods, expenses, and profits, but also any shortages. The initial markup is expressed as a per cent of retail price at which the goods are originally offered. The original retail price is not the same as the retail price received; the original retail price includes any reduction in price which may be necessary later. Consequently, in planning sales, the markup expressed as a per cent of original retail price must be established sufficiently high to gain the desired markup on sales. Sales

are equivalent to original retail less reductions From past experience a buyer is able to estimate the per cent reduction which will probably be necessary *in the course of a year* Therefore in calculating the initial markup he will use the following formula

$$\text{Initial markup per cent} = \frac{\text{Maintained markup \%} + \text{Reductions \%}}{100\% + \text{Reductions \%}}$$

That is, if a merchant wants a 40% markup on sales of \$10,000 (100%) and knows from past experience that he can anticipate reductions of \$500 (5%) on such sales, he would mark the merchandise originally at \$10,500 (105%) The problem in such a case is to determine the required initial markup (42 9%) when reductions can be estimated Thus

$$\text{Initial markup \%} = \frac{40\% + 5\%}{100\% + 5\%} = 42\ 9\%$$

EXERCISE 11.12

Solve the following

	Maintained Markup %	*Estimated Markdown %*	*Initial Markup %*
1.	45%	5%	?
2.	35%	6%	?
3.	$37\frac{1}{2}$%	$4\frac{1}{2}$%	?
4.	50%	$7\frac{1}{2}$%	?
5.	$66\frac{2}{3}$%	10%	?

6. Department X of the Beta Gamma Department Store desires a maintained markup of $42\frac{2}{3}$% of retail If the manager estimates markdowns of 10%, find his required initial markup

7. Find the initial markup needed to maintain a markup of $32\frac{1}{2}$% on retail if an average markdown of $2\frac{1}{2}$% is expected

8. A merchant wants a 35% markup on retail on sales of \$20,000 and knows from past experience that he can anticipate reductions of \$1,500 on such sales Determine the initial markup per cent

Original retail and markdown

Sometimes, after a buyer knows his initial markup on retail, he is interested in determining the maximum amount of markdown which may be taken without reducing his maintained markup below the established minimum

$$\text{Original retail} = \frac{\text{Complement of maintained markup \%}}{\text{Complement of initial markup \%}} \times \text{Sales price}$$

Original retail − Sales price = Reduction

Reduction ÷ Original retail = Percentage markdown

Illustration: The initial markup on certain merchandise was 40% on retail. After looking the goods over, the buyer is willing to accept a 25% markup. What per cent reduction should be made in the retail price of the goods? Assume $100 sales price.

$$\text{Original retail} = \frac{100\% - 25\%}{100\% - 40\%} \times \$100 = \frac{75}{60} \times \$100 = \$125$$

So, $\$125 - \$100 = \$25$; and $\$25 \div \$125 = 20\%$ markdown.

EXERCISE 11.13

Solve the following:

	Initial Markup %	*Maintained Markup %*	*Estimated Markdown %*
1.	35%	30%	?
2.	45%	$33\frac{1}{3}$%	?
3.	$42\frac{1}{2}$%	35%	?
4.	60%	45%	?
5.	$37\frac{1}{2}$%	$33\frac{1}{3}$%	?

6. So far this season the Town and Country department has had an average initial markup of 35.24%. If the buyer is expected to have a maintained markup of 32%, what is the maximum reduction which may be made?

7. Because of delayed delivery, a buyer knows that he will not be able to obtain the $37\frac{1}{2}$% initial markup on merchandise. He believes, however, that he can sell the goods if he reduces the price to a 30% markup. What per cent reduction should he make in the retail price of the goods?

8. A buyer is willing to accept 35% on retail on certain goods that had an initial markup of 45%, because they are a little out of style. What is the per cent markdown?

Markdown per cent for balance of sales

The per cent of markdown a department can take at any time is the difference between the total markdown permissible and the amount already taken.

$$\text{Original retail } \% = \frac{\text{Complement of maintained markup } \%}{\text{Complement of initial markup } \%}$$

$$\text{Reduction } \% = \textit{Original retail } \% - 100\%$$

$$\text{Markdown } \% = \frac{\text{Reduction } \%}{\text{Original retail } \%}$$

Total markdown = Total anticipated sales × Markdown %

Balance of markdown which may still be taken

= Total markdown − Markdown already taken

Markdown per cent possible for balance of sales

$$= \frac{\textit{Balance of markdown which may still be taken}}{\text{Sales for the rest of the period}}$$

Illustration A department has an initial markup of 43% and a maintained markup of 40% Sales to date are $100,000 Markdowns are $3,000 Planned sales for the rest of the period are $50,000 How much markdown can still be taken without affecting the maintained markup?

$$\text{Original retail } \% = \frac{60\%}{57\%} = 105\ 2613\%, \quad \text{Reduction } \% = 5\ 2613\%$$

$$\text{Markdown } \% = 5\ 2613\% - 105\ 2613\% = 5\%$$

Therefore

Total markdown (5% on $150,000)	$7,500
Less markdown taken	3,000
Balance of markdown which may still be taken	*$4,500*

Finally, $4,500 − $50,000 = 9%, markdown per cent possible for balance *of sales*

EXERCISE 11.14

Solve the following

1. The Acme Jewelers has an initial markup of 60% and wants a maintained markup of 45% Sales to date are $5,400 and markdowns totaling $1,200 have been made If the planned sales for the rest of the period are $5,600, how much markdown can be taken?

2. One department has an initial markup of 40%, and a maintained markup of 35% Sales to date are $74,000, and markdowns total $6,000 If the planned sales for the remainder of the years are $30,000, how much markdown can be taken if the department is to achieve a maintained markup of 35%?

3. The Knit Yourself Shop has an initial markup of 50% and wants an average markup of 45% on total sales of \$33,000. To date sales have been \$18,000 and markdowns of \$1,200 have been allowed. How much markdown per cent can be allowed on remaining sales?

4. Department C of the Williams Department Store tries to maintain a markup of $42\frac{1}{2}$% by having an initial markup of 45%. The total sales for the first quarter of the year will be \$80,000. To date, sales have been \$45,000 and markdowns total \$2,500. What markdown per cent can be permitted on remaining sales?

5. The meat department of the Crown City Food Mart tries to maintain a markup of 20% by having an initial markup of $22\frac{1}{2}$%. The total sales for the fiscal period will be \$45,000. Sales of \$30,000 and markdowns of \$950 have been made so far. What markdown and markdown per cent can be permitted on remaining sales for the period?

REVIEW PROBLEMS

Chapters 10 and 11

1. *Find the ordinary interest on a* 5% note for $3,200 from October 19 to March 1

2. Find the proceeds on April 12 of a $1,200 noninterest-bearing *90-day note dated March 27, at simple discount rate of* 4%

3. Find the proceeds of a $475 noninterest-bearing note dated July 17, for 120 days, discounted at a bank on August 4, at 5%

4. *Accountants must often calculate how much interest has been* earned in a given month, even though the notes do not mature until later What is the interest earned on the following notes during the month of *April?*

Amount of Note	*Interest Rate*	*Period of Note*	*Due Date*
$420	5%	90 days	May 15
600	6%	6 months	June 30
540	4%	60 days	April 15

5. If books are kept on a calendar year basis, how much should be reported as interest income this year, and how much for next year, on the following notes?

Amount of Note	*Interest Rate*	*Period of Note*	*Date Due Next Year*
$6,400	6%	90 days	March 15
420	$5\frac{1}{2}$%	6 months	May 10
1,640	$6\frac{1}{2}$%	30 days	January 18

6. *On March 21, A accepts B's note for $1,500 for 6 months with* interest at 6% On June 21, A discounts the note at his bank at 5% Find the proceeds

7. A city incurs an indebtedness of *$140,000,000* At $2\frac{1}{2}$%, what is the ordinary simple interest per day? What is the exact simple interest per day?

8. A customer who had received $500 from the First National Bank on June 17 agreed to repay $519 48 on December 17 What rate of discount had the bank charged?

9. The Merchants National Bank accepted from a customer his 90-day noninterest-bearing note for $10,000 at 6% discount Find the proceeds

10. On December 8, a debt was incurred at interest of $5\frac{1}{2}$% Six months later the debt was discharged by the payment of $1,157 40 What was the original amount of the debt?

11. A man with inadequate funds to complete a commercial building borrows \$12,000 and agrees to repay \$12,600 at the end of 4 months. What rate of interest did he pay?

12. In order to assure the completion of public improvements the developer of some property can put up the necessary cash to complete the job, \$9,960, or he can furnish a completion bond at a cost of \$199.26. How much will he save by borrowing the money from the bank for 3 months at 7% interest, rather than put up the bond?

13. A man has an account with his broker. At the beginning of the month he owed the broker \$1,280. On March 8 the debt was reduced to \$420; on the 17th it was increased to \$1,860. No further changes were made during the month of March. The broker charges exact interest at $4\frac{1}{2}$%. What was the interest charge for March?

14. On January 18 my \$1,000 savings bond will be redeemed. If money can be borrowed from the bank at 5% simple interest, when may \$975 be borrowed to be repaid with the \$1,000?

15. On December 1, the bank will pay out the \$500 in a customer's Christmas savings fund. The customer needs \$490 before December 1. If he can borrow from the bank at 4%, how long before December 1 may he borrow the \$490 with the expectation of repaying it with the \$500?

16. Tax anticipation warrants are sold on a discount basis. If a person expects to earn $3\frac{1}{2}$% on his investment, how much should he pay for a \$1,000 noninterest-bearing warrant 3 months before it is to be paid?

17. The Mayway Washing Machine Company has issued notes of \$5,000 due in 180 days. If a bank buys one of these notes at \$4,900, what is the bank discount rate? What is the true discount rate?

18. A note for \$1,500 dated March 6 with interest at 6% is given by a borrower who makes the following payments: May 10, \$250; June 20, \$350; August 16, \$400. Find the amount due under the Merchants' Rule if the note was paid in full December 1.

19. A loan of \$4,800 bears interest at 4%. Payments of \$130 each are due on the last day of each month. The first payment is due January 31. If interest is figured by the banker's method, and if the United States Rule is used, find the amount of the debt just after the fifth payment is made on May 31.

20. A vacant lot was sold on February 2 with the understanding that the balance of \$2,000 was to bear interest at 6% and was to be paid over the next 2 years. On March 15, \$500 was paid; on May 18, \$1,000. The buyer wants to discharge the balance at the end of the first year. How much should he pay under the United States Rule? Under the Merchants Rule?

21. A builder very short of funds obtained a loan of $6,000 for 6 months at 2% per month from a finance company He paid $425 at the end of each of 5 months How much must he pay at the end of the sixth month to discharge the debt? Use both methods

22. L C Smith borrowed $3,000 from the bank for 6 months with interest at 5% He has the privilege of making partial payments At the end of 2 months he paid $750, and at the end of 4 months he paid $1,000 Under the United States Rule how much should he pay when the note is due?

23. A small-loan company may charge $2\frac{1}{2}$% per month on the unpaid balance for loans of less than $300 This is equivalent to what simple interest rate?

24. Find the equated time for the payment of $450, $600, and $500 due in 1 month, 2 months, and 6 months, respectively

25. Payments on 3 invoices fall due as follows May 18, $125, May 31, $325, July 1, $450 At what date may these items be paid with $900?

26. An automobile is bought for $2,700 The customer is allowed $1,700 on his old car and agrees to pay the balance in 12 monthly payments of $90 What was the simple interest charged (a) under the Merchants Rule, (b) under the constant-ratio plan?

27. Henry Florence agreed to pay $3,000 one year from today. If he pays $600 at the end of each 3-month period, what will the final payment be under the Merchants Rule if money is worth 5%?

28. A clothier who has heretofore sold only for cash inaugurated a credit plan For purchases of $50, a credit charge of $2 50 is added At the time a purchase is made, $5 must be paid and the balance paid in monthly installments of $9 50 each, the first being due one month after the date of purchase Find the rate by the constant-ratio plan

29. Rich Brothers make a credit charge of $5 for purchases of $150 to $160 On a purchase totaling $160 the buyer paid $15 at the time of purchase and the balance in 6 equal monthly installments of $25 each Find the rate he paid by the constant-ratio plan

30. The May Company offers shopping coupons worth $75 for a down payment of $25 and 4 monthly payments of $12 75 In effect there is thus a service fee of $1 for $50 credit What is the rate computed by the residuary method?

31. A loan of $1,250 from the Rite Way Finance Company may be repaid in 6 equal monthly payments of $220 26 Find the rate under the constant-ratio plan

32. The Citizens Bank makes a loan of $240 to be repaid in 9 equal monthly installments of $28 31. What is the rate charged under the residuary method? The constant-ratio method?

33. In settling an estate 4 pieces of property were sold partly for cash and partly for notes. The trustee now holds 3 notes: one $5,000 note at 6% due in 9 months; one $3,000 note at $5\frac{1}{2}$% due in 112 days; and one for $6,400 at 5% due in 18 months. An investor offers to buy the three notes to yield him 6%. How much should he pay?

34. Ten thousand dollars was borrowed on a 5-year note at 5%. Payments of $1,000 were made at the end of each of the first 5 months. How much should be paid at the end of the sixth month to discharge the debt under the United States Rule?

35. Dwight Roybals signed a 1-year note for $2,700 with interest at 5% to be paid on September 18. On January 1 he paid $25; on March 1 he paid $1,050; on July 1, $700; and on August 1, $175. How much does he owe at the maturity of the note under the Merchants Rule?

36. The trustee of an estate was given possession of the assets of the estate amounting to $16,000 on February 14. The funds are invested at 5%. The assets are distributed among the 4 heirs in payments of $2,500 to each on May 1, $500 to each on July 1, and the balance on September 1. How much should be distributed to each heir on September 1?

37. When a new issue of government bonds was offered for sale at $3\frac{1}{4}$% interest, Richard Ryniker borrowed $92,500 at $3\frac{1}{2}$% and with $7,500 of his own money bought $100,000 worth of the bonds, which he hypothecated to secure the loan. At the end of the year he received interest on the bonds, paid the interest on the loan, and sold the bonds for $106,000. What per cent had he earned on the amount of his own money invested?

38. Fenn Erickson purchased a piece of earth-moving equipment for $22,000. To buy it on the installment plan he is required to pay $5,000 down and $1,550 a month for 12 months. He can borrow $17,000 from his bank at 6% by signing 2 notes for $9,500 each, one maturing in 6 months and the other at the end of 1 year. How much will he save by using bank credit if the bank charges ordinary interest on the face of the notes?

39. The owner of a business invested $5,000 for 30 days in order to take advantage of a special order. On an equated time basis what amount would he invest for 120 days?

40. On an equated time basis, how much invested in a business for 90 days is equivalent to $8,000 invested for 160 days?

41. A man invests $1,800 at 3% for a certain period of time. At what rate would he get the same return if he invested $1,500?

42. A man invested $3,200 at 4% for a certain period of time. How much would he have to invest at $3\frac{1}{2}$% to get the same return?

43. Find the equated date for paying the balance of the following account

Debits		*Credits*	
Due date		July 10	$250
June 30	$168 50		
July 15	211 50		

44. Find the average due date for the following account in which all purchases are on terms of n/30

Debits		*Credits*	
Balance due March 1	$600	March 12	$700
March 10	132		
March 15	168		
March 30	300		

45. On what date may the following account be settled equitably by the payment of the balance?

Debits		*Credits*	
January 12 Term n/30	$2,200	February 15	$1,600
March 10 Term n/30	1,300		

46. If money is worth 5%, what amount will equitably discharge the balance of the following account on May 30?

Debits		*Credits*	
February 15 (2/10 n/30)	$520	March 20	$400
March 10 (2/10 n/30)	680		
May 25 (2/10 n/30)	500		

47. From what date should interest be figured on the balance of the following account?

Debits		*Credits*	
August 1 (n/30)	$1,200	August 20	$1,500
August 15 (n/30)	600	September 5	500
August 30 (n/30)	2,400		
September 20 (n/30)	800		

Find the amount of trade discount, cash discount, and the net amount of the following bills if paid within the discount period:

	List Price of Goods	*Trade Discount*	*Terms*
48.	$2,425	20% and 10%	2/10n/30
49.	$ 218	25%, 16%, and 10%	2/15n/60
50.	$1,535	40%	1/5n/30
51.	$ 720	25%, 20%, and 10%	5/10n/60
52.	$3,275	20%, 20%, and 5%	4/10–60 Extra
53.	$8,425	30%, 10%, and 10%	3 E O M
54.	$ 637	20% and 5%	3/15n/60
55.	$ 480	20%, 20%, and 10%	2/10n/30

56. An electric drill costing $21.50 sold for $42.50 less discounts of 20% and 10%. The gain is what per cent of the selling price?

57. An invoice for $720 dated March 15 carries terms of 2/10n/30. If payment is made on March 18, and anticipation at the rate of 5% is allowed, what amount should be remitted?

58. An invoice for $1,642, dated April 17, carries terms of 2/10n/60. If payment is made May 17, how much should be remitted to the vendor?

59. If terms are 2/10n/30, would it be economical for the buyer to borrow at the rate of 10% per annum to take advantage of the cash discount?

60. An invoice for $2,400 dated November 1 carries terms of 2/10–30 Extra. If payment is made November 7, how much should be remitted if the anticipation rate is 5%?

61. The markup on cost in the furniture department of a department store is 50%. If dining-room chairs cost $8.50, what is the retail price?

62. A hardware jobber expects a markup of 12½% on cost. If he sells 1,000 units of a given item at $2.40 each, what is the cost per unit?

63. The retail price of bathing suits is $2 more than the cost. The markup on cost was 66⅔%. Find the cost. Find the retail.

64. If the markup on skis in the sporting goods department is 40% of retail, what is the markup on cost?

65. If the markup on picture frames in the art department is 100% on cost, what is the markup on retail?

66. The buyer for the Happy Day Dress Department finds dresses offered by a manufacturer at $16.50. The markup expected in the department is 40% on retail. The nearest established retail prices are $29.95 and $24.95. The buyer anticipates that among the styles offered, a limited number of dresses will sell at the $29.95 price. If 50 dresses are to be bought, how many should be bought to sell at each price?

67. In the linen department the initial markup is $37\frac{1}{2}\%$ on retail If reductions of 5% are necessary, what is the maintained markup?

68. The buyer in the linen department sees handkerchiefs which he believes will retail at $1 50 each The markup in this department is $37\frac{1}{2}\%$ on retail What is the maximum price he may offer per dozen for the handkerchiefs if he is to achieve his desired markup?

69. A department expects total sales to be $70,000 during the year The average markup in the department is 40% on retail Purchases so far this year have totaled $30,000 at cost, and the markup on purchases to date has been 45%, with reductions of 6% What minimum markup must the buyer obtain on the remainder of his purchases in order to have an average markup of 40%?

70. The initial markup in a department is 38% Shortages equal 2% of sales If the buyer is expected to have a maintained markup of 35%, what is the maximum reduction which may be made?

71. Compute the respective selling price of the manufacturer, wholesaler, and retailer, given manufacturing costs and markups (all based on selling price rather than cost) as follows manufacturing cost per unit, 70 cents, manufacturer's markup 40%, wholesaler s markup 15%, retailer s markup 30%

72. A manufacturer has fixed the list price of an item at $80 A dealer is permitted a trade discount of 35% and cash discount of 2% of the net What is the cost to a dealer who takes his cash discounts?

73. A retail merchant buys basketballs at $4 80 each He wants to sell them on a basis which will yield a gross profit of 40% of his selling price Determine the selling price

74. A store has an established price of $64 on topcoats The department is expected to make an average markup of 35% on retail The buyer has an opportunity to buy two different styles of coats, one at $37, and one at $46 In buying 100 topcoats, how many should he buy at each price to gain an average markup of 35%?

75. A buyer plans to purchase $14,000 worth of goods with a markup on retail of 30% The purchases made to date total $10,000 at cost and $13,500 at retail Find the minimum markup necessary on the balance of purchases to obtain the desired average markup

76. A department has an established retail price of $10 for purses An average markup of 40% on retail is desired in the department A buyer is offered purses at $5 25 and $6 25 If 50 purses are to be bought, how many should be bought at each price to obtain an average markup of 40%?

77. The Collegienne Shop offers handbags at established retail prices of \$7.95 and \$8.95. If 100 bags are bought at a cost of \$5.25 each, how many should be retailed at each price in order to obtain an average markup of 40% on retail?

78. The retail price of a glass pitcher is \$8.20. The markup on cost is 60%. Find the cost.

79. The markup on a globe is 41% of retail. Find the per cent markup on cost.

80. Before merchandise which cost \$60 and priced at \$100 could be sold, the price was reduced to \$95. What was the maintained markup? What was the markdown?

81. What initial markup should a buyer have for merchandise which cost \$60 if the buyer anticipates a markdown of 10% and desires an average maintained markup of 40% on retail?

82. The initial markup in a department is 45%. If the buyer expects to have a maintained markup of 40%, what is the maximum markdown he can make?

83. The buyer in a department seeks an average markup of 38%. He finds that $\frac{1}{4}$ of his stock has a markup of 30%, that $\frac{1}{2}$ of his stock has an average markup of 40%, and that the remainder of his stock has a markup of 35%. Is he achieving his desired average markup?

84. A merchant sold 80 pairs of skis at \$20 per pair. This was a reduction of 15% from the original price. His initial markup had been $66\frac{2}{3}$% of cost. Compute the percentage of gross margin received on the skis.

85. Compute the percentages of markup on cost which correspond to the following per cent markups on selling price: 20%; $37\frac{1}{2}$%; 50%; $66\frac{2}{3}$%.

12

Compound Interest and Discount

Introduction

An understanding of the theory of compound interest is important to the serious student as a basis for investment and business decisions Some students will be interested primarily in methods of computing compound interest, and in the use of tables for the solution of routine problems It is important both to understand the theoretical basis of compound interest and to know how it is computed When the theory is understood, its application is greatly simplified

The theory of compound interest

When loans are made for short periods of time, the lender anticipates that at the maturity of the loan he will receive a sum equal to the amount of the loan plus the interest for the period of the loan In other words, the sum lent will have grown during the period of the loan At the time of repayment, the lender has a choice of relending, investing, or spending the original principal, as well as the income, received If he chooses to relend the entire sum—that is, both principal and interest—for the next period, he will receive interest on both the principal and the interest which he has lent Thus if a loan of \$1,000 is made at 6% for 6 months, the lender receives \$1,030 when the loan is repaid If he immediately lends this sum to another borrower at 6%, he will receive \$1,060 90 at the end of the following 6 months Thus in two 6-month periods he will have received \$60 90 in interest If, however, he lent the \$1,000 for one year at 6% with interest payable at maturity, he would receive only \$1,060 at the end of the year

Since the lender receives less by making a loan covering the longer period, he has an incentive to do one of two things Either he refrains

from making long-term loans except at higher rates of interest, or he insists that interest be paid to him periodically. If the lender of the \$1,000 for 1 year had received interest at the rate of 6.09%, he would have received \$60.90 in income at the end of the year; or if he had been paid \$30 interest at the end of the first 6 months and had invested it at 6% for the 6-month period following, he would have had a total income of \$60.90. When interest is received on a principal which is increased periodically by interest for the period, the interest is called *compound interest.*

The difference between simple and compound interest is illustrated by a comparison of the procedure followed by United States Postal Savings and by mutual savings banks, respectively. If \$2,000 is deposited in Postal Savings for 5 years at $2\frac{1}{2}$% and left undisturbed, the interest at the time of withdrawal is calculated as follows: $\$2{,}000 \times 0.025 \times 5 = \$250$. The amount paid at the time of withdrawal is \$2,250 (\$2,000 + \$250). Here interest is figured as simple interest for 1 year, and is multiplied by 5, to find the amount for 5 years.

Had an equal amount of money been deposited in the Savings Fund Society of Germantown, a mutual savings bank, or in any other similar institution, which pays $2\frac{1}{2}$% compounded annually, the depositor could have withdrawn \$2,262.81. The total amount is referred to as the *compound amount,* and the difference between the original principal (\$2,000) and the compound amount (\$2,262.81) is called the *compound interest.*

What the bank actually does is to add $2\frac{1}{2}$% to the principal at the end of the first year, showing a balance of \$2,050. At the end of the second year, the interest is computed at $2\frac{1}{2}$% of this balance, and the total becomes \$2,101.25. At the end of the third, fourth, and fifth years, the same procedure is followed. The amount grows as follows:

At the Beginning of	*Principal*	*Interest at $2\frac{1}{2}$% Added at End of Year*	*Compound Amount at End of Year*
First year	\$2,000.00	\$50.00	\$2,050.00
Second year	2,050.00	51.25	2,101.25
Third year	2,101.25	52.53	2,153.78
Fourth year	2,153.78	53.84	2,207.62
Fifth year	2,207.62	55.19	2,262.81

Comparison of symbols used in compound and simple interest

In dealing with compound interest, certain terms and symbols are generally employed.

The principal, or the present value As in simple interest, the term *principal* is used to represent the sum of money lent, borrowed, or invested, it is represented by the symbol P.

The compound amount In the simple interest formula the symbol S refers to the amount which is the sum of the simple interest I, plus P. In compound interest, *the symbol S refers to the total interest plus the* principal Though this sum is sometimes referred to simply as the *amount*, it is generally referred to as the *compound amount*.

The rate of interest In the formula for simple interest, the rate of interest is represented by r, which is always an annual rate. Although the rate for compound interest is generally stated as an *annual rate*, the symbol used to represent the rate of interest is i rather than r. The rate i represents the *rate per period*, which may be an annual rate or a rate for any other period *The time* In simple interest, t is used to represent the time period stated in years or fractional parts of years. In compound interest the time factor is represented by the symbol n, which indicates the *number of interest periods* rather than the number of years or fractional parts of years

Frequency of conversion

If the interest is paid, or is added to the principal once a year, it is said to be *compounded*, or *converted*, *annually* Interest may be converted annually, semiannually, quarterly, monthly, or at any other regular period The *frequency of conversion* indicates the number of times that interest is converted each year. If interest is converted semiannually, the frequency of conversion is 2 and the conversion period is 6 months. If interest is converted quarterly, the frequency of conversion is 4, and the conversion period is 3 months.

The frequency of conversion is sometimes indicated by the symbol m. When interest is converted semiannually, it can be stated as $m = 2$, if *converted quarterly*, $m = 4$.

Rate per period

The interest rate is almost always stated as an annual rate, known as the *nominal annual rate*, or simply as the *nominal rate* In calculating compound interest, the *rate per conversion period*, or the *periodic rate*, is used

The periodic rate, represented by the symbol i, is found by dividing the nominal rate by the frequency of conversion. If interest is compounded annually, the periodic rate is equal to the nominal rate.

In the preceding illustration showing how money accumulates at compound interest, the rate paid by the bank was $2\frac{1}{2}\%$ *compounded annually.* Had the bank compounded its interest semiannually, the rate would have been stated as $2\frac{1}{2}\%$ *compounded semiannually,* but the rate per period would have been $1\frac{1}{4}\%$, and the number of periods would have been 10 instead of 5.

Often the stated nominal (annual) rate is represented by the symbol j. If m is used to represent the frequency of conversion, then the rate per period—that is, the rate i—is equal to $\frac{j}{m}$. In many tables of compound interest these symbols are used. In this text, however, reference is usually made to the annual rate, and the frequency of conversion is usually given.

If the nominal rate is 6%, but the interest is converted every 6 months, the rate i is 3%. A principal of \$1 lent for 1 year at 6% converted semiannually would not be increased by 6% at one time. At the end of 6 months, it would be increased from \$1 to \$1.03, the 3 cents interest being added for one period. During the next 6 months, the new principal of \$1.03 would be increased by 3%. At the end of the year, the original \$1 of principal would be augmented by further interest, making a total of \$1.0609. Thus in the period of one year, the principal would be increased by 6.09%, an amount greater than 6%.

The actual rate of increase during the year is called the *effective rate;* and if the frequency of conversion is greater than 1, the effective rate of interest will always exceed the nominal rate. If interest is converted annually, the effective rate and the nominal rate are equal. When the frequency of conversion is not stated, it may be assumed to be annual.

The number of periods

Generally the time is stated in years or fractional parts of years, but because interest is not always converted annually, n, as used in the compound interest formulas, refers not to the number of years but rather to the number of conversion periods. To compute compound interest it is necessary to know the rate per period and the number of periods.

The number of periods—n—is found by multiplying the time in years by the frequency of conversion. Thus if compound interest is to be computed on a sum of money for 5 years at 4% converted quarterly, n becomes not 4, but 5×4, or 20. The frequency of conversion is 4, the conversion period is 3 months, and the interest rate per period is 1%.

EXERCISE 12.1

State the value of i and n

	Time	*Stated Rate*	*Frequency of Conversion*	*Periodic Rate, i*	*Number of Conversion Periods, n*
1.	2 years	4%	annually		
2.	5 years	$3\frac{3}{4}$%	semiannually		
3	6 years	$3\frac{1}{2}$%	quarterly		
4.	*4 years 2 months*	3%	*monthly*		
5.	3 years 4 months	2%	monthly		
6.	1 year 6 months	$1\frac{1}{2}$%	semiannually		
7.	*4 years 3 months*	5%	*quarterly*		
8	3 years	$2\frac{1}{2}$%	semiannually		
9	17 years 1 months	3%	monthly		
10.	12 years	2%	quarterly		
11.	18 months	6%	semiannually		
12.	10 years	5%	quarterly		
13.	25 years	$1\frac{1}{2}$%	monthly		
14.	15 years	6%	annually		
15.	20 years	$5\frac{1}{2}$%	monthly		

Computing the compound amount

If, as was assumed earlier in the chapter, \$2,000 was invested at $2\frac{1}{2}$%, the amount S at the end of the first year would be the sum of \$2,000 and the product obtained by multiplying \$2,000 by $2\frac{1}{2}$% Expressed as a formula, it is $P + Pi$, or simply $S = P(1 + i)$ At the end of the second period, the amount would be the amount at the end of the first period, $P(1 + i)$, *times* $(1 + i)$ *Since* $P(1 + i)(1 + i)$ *is equivalent to* $P(1+i)^2$, we can write the amount S for 2 periods as $P(1 + i)^2$

At the end of the third period, the amount would be the principal at the beginning of the period, $P(1 + i)^2$, times $(1 + i)$, or $P(1 + i)^3$ In the same manner, it can be determined that at the end of the fifth period the amount would be $P(1 + i)^5$ If n represents the number of periods, the amount S at the end of n periods is always shown by the general formula $S = P(1 + i)^n$

The calculation of the compound amount can be carried out in one of three ways In the case of the \$2,000 it was carried out by actually multiplying \$2,000 by 1 025 (i e , $1 + 2\frac{1}{2}$%) by 1 025 by 1 025, etc , for 5 times, the product being \$2,262 81

The amount can be computed more readily by using logarithms to

solve for S. It is known that $S = P(1 + i)^n$ and that $P = \$2{,}000$, $n = 5$ periods, and $i = 2\frac{1}{2}\%$. In substituting these values in the formula, it facilitates computation to show $1 + 2\frac{1}{2}\%$ as $1 + 0.025$ or simply as 1.025. If this procedure is followed, it is seen that $S = \$2{,}000\,(1.025)^5$.

Solving by logarithms,

$$\begin{aligned}
\log (1.025)^5 &= 5 \times \log 1.025\\
&= 5 \times 0.010724\\
&= 0.053620\\
\log 2{,}000 &= 3.301030 \quad (+)\\
\hline
\log S &= 3.354650\\
S &= 2{,}262.81
\end{aligned}$$

The formula $S = P(1 + i)^n$ can be used to solve any problem in compound interest if a table of logarithms is available. It is important to know this method of solving such problems, particularly for the solution of problems involving an extremely large number of periods, or problems involving unusual rates of interest.

A third method of solving problems in compound interest, a method often stressed in accounting books, is to compute the amount to which \$1 will accumulate at the given interest rate for the stated number of periods. This amount may then be multiplied by the principal. This is an advantageous method to use when it is necessary to find a compound amount for different sums of money for the same period of time and at the same rate.

Illustration: What is the compound amount of \$2,000 left at interest of $2\frac{1}{2}\%$ compounded annually for 5 years?

The following table can be constructed by simple multiplication:

Period	*Balance at Beginning of Period*	*Ratio of Increase per Period*	*Accumulated Total at End of Period*
1	1.000000000	1.025	1.025000000
2	1.025000000	1.025	1.050625000
3	1.050625000	1.025	1.076890625
4	1.076890625	1.025	1.103812890
5	1.103812890	1.025	1.131408213

Rounded to 7 places beyond the decimal point, the compound amount of 1 at interest of $2\frac{1}{2}\%$ per period is 1.1314082 at the end of the fifth year. Since the sum invested was \$2,000, the compound amount is \$2,000.00 × 1.1314082, or \$2,262.81.

The compound amount table

The third method illustrated exemplifies the most frequently used method of computing the compound amount Tables can be constructed by multiplication When based on the amount of 1, such tables can be used for any amount or for any kind of units The units may represent persons, different currencies, such as dollars, pounds, pesos, or yen, or any other unit

To aid in the calculation of the compound amount, various tables have been developed The Financial Publishing Company, Boston, Mass, has computed and published a comprehensive set of tables under the title *Financial Compound Interest and Annuity Tables* The tables reproduced in the appendix of this book, by permission of the copyright owner, are representative of the comprehensive coverage of this complete book of tables These tables have been constructed and are made available to save time They should be used whenever possible to avoid unnecessary computations

Certain fundamental facts must be understood in their use The rate per period is shown in the upper left-hand corner and the upper right-hand corner of the tables in the appendix It is stated both as a per cent and as a decimal equivalent per period Thus the rate of $\frac{1}{4}\%$ per period is also shown as 0025, the rate of 6% per period as 06 Since in compound interest the rate per period is the significant rate, the rate shown at the upper corner of the table is the one to be observed

Along each side of the table are shown the equivalent nominal rates Thus the table for $\frac{1}{2}\%$ per period is used if the annual nominal rate of $\frac{1}{2}\%$ is compounded annually, the same table is used if the annual nominal rate of 1% is converted semiannually, or if the annual nominal rate of 2% is converted quarterly, or if the annual nominal rate of 6% is converted monthly

In the lower corner of the table for $\frac{1}{2}\%$ appears the following notation

$$i = 005$$
$$j(2) = 01$$
$$j(4) = 02$$
$$j(12) = 06$$

That is, the decimal equivalents are given for the different nominal rates at various frequencies of conversion Thus this table is to be used if the nominal annual rate of $\frac{1}{2}\%$ is compounded annually, if the nominal annual rate is 1% converted semiannually, if the nominal annual rate is 2% converted quarterly, or if the nominal annual rate is 6% converted monthly

The number of periods is shown in the first column. In the complete book of tables the number of periods shown is 360 for the lower rates, a length sufficient to compute monthly periods for 25 years. For illustrative purposes in the text, the tables have been reproduced for only 120 periods.

The second column in the table is headed AMOUNT OF 1.

AMOUNT OF 1
How $1 left at
compound interest
will grow

At the bottom of the column is shown the formula $s = (1 + i)^n$. Note that P is not shown in the formula. It is assumed that P is 1, and since multiplication by 1 does not change the value of $(1 + i)^n$, the compound amount of 1 is the same as $(1 + i)$ for 1 period; $(1 + i)^2$ for 2 periods; and $(1 + i)^n$ for n periods. It is customary in such tables to use a small letter s to designate the compound amount to which any unit accumulates for any number of periods.

To find the compound amount by the use of the table, first find the number of periods by multiplying the time in years by the number of conversions per year. Next find the table for the desired rate per period. That is, if the rate is 6%, converted monthly, look through the tables under monthly rate for 6%. This will be found in the $\frac{1}{2}\%$ table—that is, $i = \frac{1}{2}\%$.

A second method, which would obtain the same result, is to compute the rate per period and to look for the table showing that rate. By either method the same table is used.

Find the tabular value from the table for the number of periods involved, and multiply this value by the principal (P). The answer, correct to the nearest cent, is obtained *if the tabular values are rounded to include the same number of decimal places as there are significant places (dollars and cents) in the multiplier.* Since the tables are shown to 10 places, they may be used to give answers correct in cents for amounts up to $10,000,000.00.

Illustration: Using the table, find the amount of $326.40 for 10 years at 6% compounded semiannually.

Since the period of time is 10 years and the interest is converted semiannually, the number of periods is 20. Look in the table for the marginal notation SEMIANNUALLY for the rate of 6%. That is the table for 3% per period.

The tabular value for 20 periods is 1 806 111 2317 Since there are only 5 digits in the principal ($326 40), this number is rounded to 1 80611 The amount is $589 51

Finding the compound interest

The answer given in the preceding illustration includes both the original principal and the compound interest for the times involved If one wanted to find only the compound interest, the tabular value could have been reduced by 1, since that is the principal on which the tabular value was computed, and the difference multiplied by the principal of $326 40 An alternative method would be to compute the compound amount and deduct the principal from this amount

Illustration Find the compound interest on $326 40 for 10 years at 6% compounded semiannually

Here again $n = 20$, $i = 3\%$ Therefore

Tabular value of $(1 + 3\%)^{20}$	1 80611
Less the original principal	1 00000
	0 80611,

the compound interest on 1 at 3% for 20 periods Therefore the compound interest on $326 40 for 20 periods at 3% is $263 11 ($326 40 × 0 80611 = $263 11)

Or by the alternative method The $589 51 found in the previous illustration less $326 40 also gives $263 11

EXERCISE 12.2

Solve the following Give (a) the per cent and decimal values of $(1 + i)$, (b) the value of n, (c) the compound amount of 1 for n periods in each of the following

	Time	*Nominal Rate*	*Compounded*
1.	$20\frac{1}{2}$ years	4%	Semiannually
2.	10 years	$2\frac{1}{2}\%$	Annually
3.	4 years 3 months	3%	Monthly
4.	$5\frac{3}{4}$ years	4%	Quarterly
5.	$2\frac{1}{8}$ years	6%	Semimonthly

6 By the use of logarithms, find the amount of $1 invested for 15 years at 6% compounded annually Compare your answer with the amount of 1 for 15 periods at 6% as shown in the compound amount table

7. Find the amount due a depositor who has invested $500 at 3% compounded semiannually for $8\frac{1}{2}$ years.

8. The sum of $1,400 was borrowed at 4% compounded monthly. How much will it take to discharge this debt 2 years later?

9. The sum of $10,000 was borrowed at 9% with the understanding that the interest was to be paid monthly. If the borrower did not make monthly payments of the interest, the principal with interest at 9% compounded monthly was to be paid at the end of the year. If the borrower did not make monthly payments, how much was due at the end of the year?

10. Compare the compound amount of $6,500 at 5% compounded semiannually for 5 years, with the amount at 5% simple interest for the same length of time.

11. Find the compound amount of $2,250 for 8 years at $5\frac{1}{2}$% converted annually.

12. When George was born his father deposited $1,000 in his account at the Dollar Savings Bank. When George is 21 years old, how much interest will have accumulated on the deposit if interest has been compounded annually at 3%?

13. On his 25th birthday a man inherited $5,000. If he invested this amount at 4%, compounded annually, how much would he receive when he retires at age 65?

14. The Clementine Corporation offers to sell a piece of land for $10,000 cash, or to accept a noninterest-bearing note due in 5 years for $12,500. A buyer who has the funds available knows he can invest his money at 5% compounded semiannually. Which is more advantageous to him, and by how much?

15. Find the compound interest on $1,500 left at interest of 4% compounded semiannually for 6 years.

Finding the unknown time

The tables can be used to find not only the amount but also the rate or the number of periods. It is possible to interpolate from the tables in finding any of the three factors: time, rate, or amount. It may be recalled that straight-line interpolation assumes that corresponding differences are proportional. Since in compound interest, corresponding differences are not exactly proportional, ordinary straight-line interpolation is not exactly accurate. For most purposes, if the table available progresses only by small intervals, interpolation is sufficiently accurate. The actual practice when large sums are involved is to use tables which show exactly what is wanted or to resort to longer methods of computation.

Illustration Find the time required for money to double itself at 5% interest compounded annually

Looking at the table it is seen that in 14 periods \$1 will amount to \$1 97 and in 15 periods will amount to \$2 07 Thus for all practical purposes it can be said that it takes more than 14 periods and fewer than 15 for money to double itself at 5% compounded annually When the time involved is in shorter intervals such as months no greater precision than the number of periods would be expected Greater precision can be gained from interpolation as follows

Difference		*Difference*
x	In 14 years \$1 amounts to \$1 97993160	0 02006840
	In ? years \$1 amounts to \$2 00000000	
1	In 15 years \$1 amounts to \$2 07892818	0 09899658

If in 1 year the amount of \$1 at compound interest of 5% increases from \$1 97993160 to \$2 07892818 a total increase of \$0 09899658 what proportionate part of a year is required for the amount of \$1 97993160 to increase only \$0 0200684 that is to \$2? If ordinary straight line interpolation is assumed to be sufficiently accurate for this purpose it would require only $\frac{2006840}{9899658}$ of a year Set up as a proportion

$$\frac{x}{1} = \frac{2\,006\,840}{9\,899\,658} \qquad x = 0\ 203 \text{ year}$$

This is equivalent to about $2\frac{1}{2}$ months Thus it takes money 14 years $2\frac{1}{2}$ months to double itself at 5% compounded annually

While the unknown time may be found by interpolation from the compound amount table it can also be found by using logarithms Thus to solve the same problem using logarithms

The original form is $(1\ 05)^n = 2$ Therefore

$$n \times \log (1\ 05) = \log 2 \text{ or } n \quad \frac{\log 2\ 00}{\log 1\ 05} = \frac{0\ 301030}{0\ 021189}$$
$$= 14\ 21 \text{ years or } 14 \text{ years } 2\tfrac{1}{2} \text{ months}$$

When as frequently happens the period of time involved is a period outside the range of the tables it is necessary to depend on the use of logarithms

Finding the rate of interest

To compare different kinds of investments it is often desirable to determine the rate of interest actually received on the amount invested either by using logarithms or by interpolating from the tables

Illustration: An investor may purchase a United States Savings bond for \$75. No interest is paid on the bond periodically, but at the end of 10 years the government will pay the holder of the bond \$100. If an investor has funds to invest for a period of 10 years, would he earn a higher rate by investing his money in a savings bank which pays $2\frac{1}{2}\%$ compounded semiannually or by buying the bond? What rate compounded semiannually would he earn on the bond?

If he bought the bond for \$75, in 10 years the compound amount would be \$100. If he invested in the savings bank,

$$S = \$75\,(1 + 1\tfrac{1}{4}\%)^{20} = \$75 \times 1.28204 = \$96.15$$

The compound amount of \$75 in the savings bank at the end of 10 years would be \$96.15, or \$3.85 less than the amount of the bond.

The actual rate earned on the bond is necessary only for purposes of comparison with other rates. If the other rates are compounded semiannually, then we should compute the rate on the bond as if the number of conversion periods in the 10 years is 20. If on the other hand, the comparison is with a type of investment which pays interest monthly, then the rate should be computed for 120 periods for the 10 years.

Since interest on investments is generally paid semiannually, the problem will be considered one in which the number of periods is 20. To find, by the use of the table, the rate actually paid on the bond, proceed as follows:

$$\$100 = \$75\,(1 + i)^{20}$$
$$(1 + i)^{20} = \tfrac{100}{75} = 1.33\cdots$$

Look in the tables for the amount of 1 for 20 periods and see if any rate in the tables shows the value as 1.33333333. Since the tables do not show this rate, look at the interest rates which show the values both immediately above and below this rate. Thus from the table we find that the amount of \$1 for 20 periods is:

Difference			*Difference*
x	$1\frac{1}{4}\%$	1.28203723	0.05129610
	i	1.33333333	
0.0025	$1\frac{1}{2}\%$	1.34685501	0.06481778

$$\frac{x}{0.0025} = \frac{5129610}{6481778}$$
$$6481778x = 12824.02$$
$$x = 0.001978$$
$$= 0.198\%,$$
$$1\tfrac{1}{4}\% + x = 1.25\% + 0.198\% = 1.448\%$$

This rate is 1 448% per period Since we want the annual (nominal) rate, the 1 448% is doubled to give 2 896% or about 2 9% compounded semiannually

To solve the same problem by logarithms

$$100 = 75(1+i)^{20}$$

$$\log 100 = \log 75 + 20 \times \log (1+i),$$

or $$20 \log (1+i) = \log 100 - \log 75$$

$$\log (1+i) = \frac{\log 100 - \log 75}{20}$$

$$\log 100 \quad = 2\ 000000$$

$$\log\ 75 \quad = \underline{1\ 875061\ (-)}$$

$$\log (1+i) = 0\ 124939 - 20 = 0\ 006247$$

$$1 + i = 1\ 01449$$

$$i = 1\ 449\%$$ per period, or 2 9% per year, compounded semiannually

EXERCISE 12 3

Solve the following

1. (a) By the use of logarithms, find the time required for money to double itself at 6% converted annually

(b) By interpolation from the tables find the time required for money to double itself at 6% converted annually

2 One class of United States government bonds is sold at $74 for each $100 of maturity value No interest is paid on the bond, but 12 years later the bond is redeemed at maturity value Find the rate of interest earned on the investment, assuming that interest is compounded annually

3. A city is expanding A man owning nonincome producing property not far from the city estimates that his land now worth $600 an acre, can be subdivided in 20 years at $1,500 an acre Assuming that his estimation is correct, would he gain more by holding his land for the next 20 years, or by selling the land now and investing the proceeds at 5% compounded semiannually?

4. If a nondividend-paying stock purchased 2 years ago at 5 can be sold 1 year hence at 7, what is the annual rate of appreciation?

5. The population of Ourville has grown in the past 10 years from 106 000 to 120 000 If the growth has been at a constant annual rate, what has been the annual rate of increase?

6. How long will it take for a sum of money to triple itself at 6% compounded semiannually?

7. An investor can receive $5\frac{1}{2}\%$ annually, which he reinvests at the same rate. If he has \$34,272.90 at age 45, how much will he have at age 65?

8. A man buys a nondividend-paying share of stock today with the expectation that it will double in price in 5 years. What rate of annual return is he anticipating?

9. It is anticipated that the value of certain western lands now offered for sale by the government at \$5 an acre will be \$30 an acre 20 years hence. An investor has an opportunity to buy income property which will give him a return of 10% per year, payable semiannually. Assuming that he can reinvest his income at a similar rate each year, which alternative would furnish the greater amount 20 years hence?

10. In a speculative enterprise it is anticipated that certain property bought now at \$2,000 can be sold for \$10,000 at the end of 3 years. If such a speculation is successful, what is the rate compounded monthly that has been received?

Finding values higher than those shown in the tables

Although tables are constructed to cover long periods of time, it is sometimes necessary to find values for periods greater than those shown in the tables. In such circumstances, it may be necessary to use either logarithms or an alternative based on the law of exponents, which was considered in an earlier chapter.

In the study of exponents it was seen that a^{m+n} was equal to $a^m \times a^n$. Thus if a tabular value for 100 periods is desired and the table shows only 60 periods, the value for 100 periods may be found by multiplying any tabular values together for periods totaling 100. Thus it could be the product of $(1+i)^{60}(1+i)^{40}$ or $(1+i)^{50}(1+i)^{50}$ or $(1+i)^{20}(1+i)^{40}(1+i)^{40}$, etc.

A knowledge of exponents is useful in developing tables when the problem requires a rate of interest not included in available tables. If, for example, it is necessary to find the compound amount of any sum for 15 periods at $1\frac{1}{4}\%$ per period, one might proceed as follows:

$$
\begin{aligned}
(1+i) & & &= 1.0125 \\
(1+i)^2 &= (1+i)\ (1+i) &&= 1.0125 \times 1.0125 = 1.02515625 \\
(1+i)^3 &= (1+i)^2\ (1+i) &&= 1.02515625 \times 1.0125 = 1.0379707031 \\
(1+i)^6 &= (1+i)^3\ (1+i)^3 &&= 1.0379707031 \times 1.0379707031 \\
& &&= 1.0773831805 \\
(1+i)^{12} &= (1+i)^6\ (1+i)^6 &&= 1.0773831805 \times 1.0773831805 \\
& &&= 1.1607545177 \\
(1+i)^{15} &= (1+i)^{12}(1+i)^3 &&= 1.1607545177 \times 1.0379707031 \\
& &&= 1.2048291829
\end{aligned}
$$

The same figure, to five significant digits, could be obtained by logarithms as follows

$$x = (1.0125)^{15}$$
$$\log x = 15 \times \log 1.0125 = 15 \times 0.005395 = 0.080925$$
$$x = 1.2018$$

The two methods are demonstrated in the following illustrations

Illustration Find the amount of $1,200 invested at 1% converted quarterly for 30 years, (a) by the use of the tables, (b) by logarithms

a By tables

$$S = \$1,200\,(1 + 1\%)^{120}$$

Assume that the table does not show a value for 1% for 120 periods It is known that $(1 + i)^{60}(1 + i)^{60} = (1 + i)^{120}$

$$(1 + 1\%)^{60} = 1.81669670$$
$$(1 + 1\%)^{120} = 1.81669670 \times 1.81669670 = 3.30038689$$
$$S = \$3,960.16$$

b By logarithms

$$\log 1.01 = 0.004321$$
$$120 \times \log 1.01 = 0.518520$$
$$\log 1,200 = 3.079181$$
$$\log S = 3.597701$$
$$S = \$3,960.05$$

EXERCISE 12.4

Do not carry out the multiplication, but show the necessary layout, using (a) the tables, (b) logarithms, to find the compound amount of the following

	Principal	*Rate*	*Time*	*Compound Amount*
1.	$ 2,100	2% compounded monthly	11 years	
2.	$ 2,100	2% compounded quarterly	40 years	
3.	$ 2,100	2% compounded monthly	15 years	
4.	$ 4,000	1% compounded quarterly	31 years	
5.	$ 1,000	5% compounded annually	10 years	
6.	$15,000	6% compounded quarterly	50 years	
7.	$ 100	6% compounded monthly	15 years	
8.	$ 600	1% compounded monthly	12 years	
9.	$ 8,750	3% converted semiannually	75 years	
10.	$10,000	1% converted quarterly	38 years	

Finding values when the time is not an integral number of conversion periods

It should be realized that all calculations of compound interest are not made for an integral number of conversion periods. Often when interest is compounded annually, it is necessary to find a compound amount for a period, such as 3 years 3 months, which is not an integral number. In practice, the customary procedure is to find the amount of the debt at compound interest for the integral number of periods and then to add the simple interest on this amount for the fractional part of the period.

Illustration: A debt of \$12,000 with interest at 3% compounded annually is to be paid at the end of 3 years 4 months. What is the amount of the debt?

From the table, the compound interest of 1 at 3% for 3 years is 1.092727.

Therefore the amount of the debt at the end of 3 years is

$$\$12{,}000 \times 1.092727 = \$13{,}112.72$$

Add simple interest on this amount for 4 months at 3%.

$$\$13{,}112.72 \times \tfrac{3}{100} \times \tfrac{4}{12} = \quad 131.13$$

Therefore the total amount of the debt is $\$13{,}243.85$

EXERCISE 12.5

Solve the following:

1. A debt of \$2,000 with interest at 4% compounded annually is paid at the end of one year 6 months. What is the amount of the debt?

2. A debt of \$1,000 was due 2 years ago today, but payment was not made. Interest has accumulated on the debt at the rate of 6% compounded semiannually since that time. If full payment is to be made at the end of 3 months from today, how much should the payment be?

3. If \$7,000 is lent at 4% compounded annually, how much should be returned at the end of 5 years 2 months?

4. A debt of \$10,000 at 5% converted annually is paid at the end of 25 months. What is the amount?

5. A \$4,000 note bears interest at 4% converted semiannually. If the note is paid after 2 years 9 months, how much should be paid?

6. Find the compound amount of \$1,000 left at 5% compounded semiannually for 7 years 4 months.

7. Find the compound interest on \$1,200 left at interest of 6% converted quarterly for 4 years 1 month.

8 If Gary Clark deposits $380 in a savings bank which pays 3% interest converted semiannually how much will he have at the end of $5\frac{1}{4}$ years?

9 Compute the compound amount of $1 230 three years four months from now if the money is compounded quarterly at 4%

10 Find the compound amount of $3 265 for 20 years 4 months if money is compounded semiannually at 8%

Periodic, nominal and effective rates

When there is a choice of selecting one investment opportunity over another or one source of funds over another the ultimate decision may be determined by a comparison of the rates of interest to be received or paid Sometimes all that is necessary to reach a decision is to compare the compound amounts of each one at the end of the period Often a comparison is more difficult because the sums involved are not identical the periods different or the frequency of conversion dissimilar To simplify comparison it is customary to change all rates to a comparable basis called the *effective rate* which is defined *as the actual rate of increase during one year*

It was mentioned earlier that the stated rate per year is the nominal rate Thus if interest is at 6% converted semiannually the nominal rate is 6% In computing compound interest however the significant rate is the *periodic rate*—that is the rate per period—in this case 3% If an amount of 1 is left at interest of 3% per period for 2 periods the compound amount at the end of the year is 1 0609 The interest that has accumulated on the $1 is 6 09 cents In other words in a given period of time an amount of money left at 6 09% compounded annually accumulates to the same sum as an equal amount left at 6% compounded semiannually If the interest is converted more than once a year comparison is usually simplified by finding the effective rate

If the frequency of conversion is greater than 1 the effective rate of interest may be found by deducting 1 from the tabular value of $(1 + i)$ for the number of periods corresponding to the frequency of conversion

Illustration Find the effective rate equivalent to 4% converted quarterly

The frequency of conversion is 4 and $i = 1\%$ Since $(1 + 1\%)^4 =$ 1 04060401, the effective rate is $1\ 04060401 - 1 = 0\ 04060401 =$ 4 060401%

Effect of frequency of conversion

It has been emphasized that *the rate per period* is the basis on which compound interest is figured, or the tables are computed. The shorter the period of time—that is, the more frequent the number of conversions—the sooner interest is paid on interest. Thus the effective rate is increased by increasing the frequency of compounding.

A consideration of the illustration of 6% will show that the effective rate increases as the number of conversions per year increases. As the number of conversions is increased a progressively smaller increase in the effective rate is produced. For example, a nominal rate of 6% compounded at different frequencies produces the following effective rates:

6% Compounded	*Number of Conversions per Year*	*Effective Rate*
Annually	1	6%
Semiannually	2	6.09%
Quarterly	4	6.13636%
Monthly	12	6.16778%
Weekly	52	6.17998%
Daily	365	6.18313%
Continuously	Infinite	6.18365%

From this table it can readily be seen that there is little gained by increasing the frequency of compounding beyond the monthly limit. The concept of continuous compounding has no application in financial problems, but is important as a concept in nature. Suppose it were found that in a natural phenomenon the rate of growth were 6% at any given time. The actual annual growth would be 6.18365%—that is, the effective rate.

EXERCISE 12.6

Solve the following.

1. A $1,000 bond issued by a railroad pays interest at 4%. If half the annual interest is paid every 6 months, what is the effective rate received?

2. An investor with $1,000 has a choice of depositing his funds in a savings bank which agrees to pay 2% interest converted monthly, or buying a bond which pays $2\frac{1}{2}$% per year in 2 equal semiannual installments. Which investment furnishes the higher return?

3 An investor can buy a share of stock at $40 which pays a dividend of 30 cents every quarter He can buy a U S Savings bond for $75 which will be redeemed in 10 years at $100 Assuming no changes in either the market price or the dividend rate of the common stock during the next 10 years which furnishes the greater return?

4 What is the effective rate earned on a bond paying interest at 5% if interest payments are received quarterly?

5 Find the effective rate of interest on a share account in a savings and loan which pays 3% converted (a) monthly, (b) quarterly, (c) semiannually

6 What rate payable annually is equivalent to 7% converted quarterly?

7 What is the effective rate earned on a bond paying interest at 5% converted semiannually?

8 What is the effective rate equivalent to 8%, compounded monthly?

9. What rate converted annually is equivalent to 8% converted quarterly?

10 Find the effective rate equivalent to 4% converted monthly

Present value at compound interest

Since compound interest is commonly considered in determining a future value for a present sum it is logical that a similar method should be used to find the present value of a future sum Though one speaks of the present value as now, and though one thinks of a future date as one which has not yet occurred, mathematically it is immaterial whether the time referred to as *present* is now, in the past, or in the future It is the relationship of one date to another that is significant The difference between the amount S and the principal P at 3% compounded annually for 3 years is the same regardless of whether one date is now, in the past, or in the future If $S = P(1 + i)^n$, then $P = \dfrac{S}{(1 + i)^n}$, or, to resort to the use of negative exponents, $P = S(1 + i)^{-n}$

To find the present value of a future sum, it is necessary to discount the future value The present value P, of S, is $S(1 + i)^{-n}$ The difference between the future value and the present value is often called the *compound discount* The term $(1 + i)^{-n}$ is called the *discount factor* Since the formula for present value is derived from the formula for the compound amount, all symbols have the same meaning in both formulas i is the interest rate per period, n is the number of periods, P is the principal or present value, and S is the amount

Thus P represents the value of the obligation at one date; S represents the amount of the same obligation n periods later. Shown graphically:

$$\begin{array}{ccc} P & & S \\ \downarrow & \underline{\qquad\qquad n \text{ periods} \qquad\qquad} & \downarrow \\ \begin{pmatrix}\text{Value} \\ \text{now}\end{pmatrix} & & \begin{pmatrix}\text{Value } n \text{ periods} \\ \text{later}\end{pmatrix} \end{array}$$

It has been shown that the difference between P and the amount S is the compound interest on P for n periods. Looking at it from another point of view, we can see that the difference between S and P is the discount at a compound interest rate on S.

In calculating the present value of a future sum, compound interest tables may be used. Thus the present value of \$1,000 due in 2 years, if money is worth 4%, can be found as follows:

$$S = P(1 + i)^n; \quad P = \frac{S}{(1 + i)^n}$$

$$S = 1{,}000; \quad n = 2; \quad i = 4\%; \quad P = \text{unknown}$$

Substituting:

$$P = \frac{1{,}000}{(1 + 4\%)^2}$$

From the compound amount table, $(1 + 4\%)^2 = 1.081600$. Hence

$$P = \frac{\$1{,}000.00}{1.081600} = \$924.56$$

To facilitate calculations by avoiding division, however, tables can be constructed which show the present value of \$1 at compound discounts. From the examples given, it can be seen that the *discount factor* is the reciprocal of the accumulation factor. In constructing the table for present value, it is customary to indicate the present value of 1 by the symbol v^n, which means, therefore, that $v^n = \dfrac{1}{(1 + i)^n} = (1 + i)^{-n}$. These values are shown in the financial tables under the column headed Present Worth of 1 (What \$1 due in the future is worth today).

The table is used exactly as is the compound amount table. The value n shows the number of periods, and the body of the table shows the present value of 1 for the different rates and periods. To find the present value of any sum, the value of 1 for the time and rate is found from the table and multiplied by the stated amount.

Illustration A \$1,500 noninterest-bearing note is due in 3 years If money is worth 5% a year, what is the present value of the note?

$$S = \$1{,}500, \quad i = 5\%, \quad n = 3$$
$$P = S(1+i)^{-n}, \quad \text{or} \quad P = Sv^n$$
$$P = \$1{,}500\,(1+5\%)^{-3}, \quad \text{or} \quad P = \$1{,}500\,(v^3 \text{ at } 5\%)$$

From the table, the present worth of 1 for 3 years at 5% is 0 8638376 Therefore

$$P = \$1{,}500 \times 0\ 863838 = \$1{,}295\ 76$$

(It should be observed that if the answer is to be determined to the nearest cent, it is necessary to read the table only to as many decimal places as there are digits in S expressed in dollars and cents)

If the note is interest-bearing, two calculations must be made to find the present value It is first necessary to find the maturity value of the note (that is, the compound amount), and then the discount must be calculated on the maturity value to find the present value

Illustration A 5% note for \$3,000 is due in 4 years If the current rate charged on similar loans is only 4%, what is the present value of the note?

The maturity value of the note is

$$\$3\ 000\,(1+5\%)^4 = \$3{,}000 \times 1\ 215506 = \$3{,}646\ 52$$

The present value at 4% of \$3,646 52 due in 4 years is

$$\$3{,}646\ 52\,(1+4\%)^{-4} = \$3{,}646\ 52 \times 0\ 8548042 = \$3{,}117\ 06$$

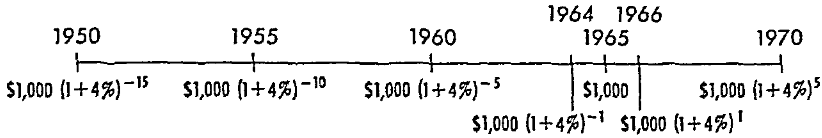
The amount of discount is the difference between the maturity value of the note and the present value In the preceding illustration, the discount is equal to

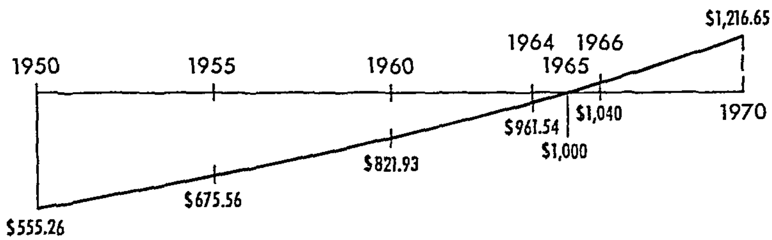
$$\$3{,}646\ 52 - \$3{,}117\ 06 = \$529\ 46$$

The observing student will realize that the term compound discount, which we have used because of its wide acceptance, is not accurate The value of $(1+i)^{-n}$ represents the present value at a *compound interest rate* The present value at a compound discount rate can be represented by $(1-d)^{-n}$ Since compound discount is not used commercially and since no compound discount tables have been published, it is unlikely that any serious misunderstanding will result from the use of the expression compound discount, rather than discount at a compound interest rate

Values at different times

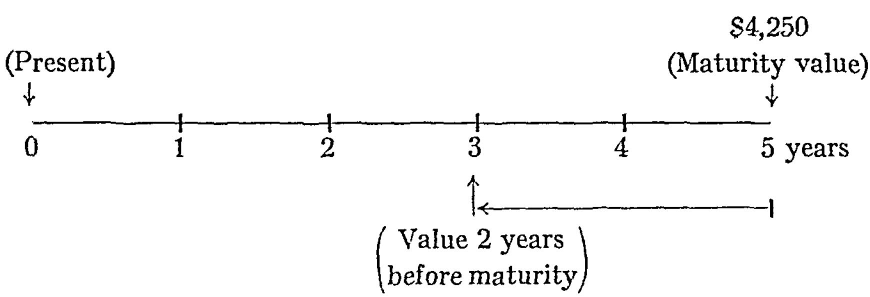
Often it is necessary to find the value of an obligation either before or after the specified date of payment. The value of a noninterest-bearing amount at different times is shown in the following illustration. Here it can be seen that an amount of $1,000 due in 1965 would be worth as little as $555.26 in 1950—that is, 15 years before the due date—and as much as $1,216.65 in 1970—5 years after the due date—if the interest rate is considered as 4% compounded annually.





The solution of such problems entails either finding the compound amount for a future value, finding the present worth by the use of the present worth tables, or a combination of the two.

Illustration: A noninterest-bearing note of $4,250 is due in 5 years. What is its value 3 years hence if money is worth 6% compounded annually?



The problem is to find the value 2 years before the due date. It is therefore necessary to discount the note for 2 years at 6%.

$$\$4,250\,(1 + 6\%)^{-2} = \$4,250 \times 0.889996 = \$3,782.48$$

EXERCISE 12.7

Solve the following

1. Find the present value of a debt of $5,000 due in 3 years if money is worth 4% converted quarterly

2. Find the compound discount on a noninterest-bearing note for $1,840 due in 3 years if money is worth 5% converted annually

3. A 1-year $3,200 note bears interest at the rate of 5% converted semiannually If money is worth 4% per year, what is the value of the note 2 years hence?

4. Compare the following

a Compound discount on $5,000 for 2 years at 4% per year

b Simple discount on $5,000 for 2 years at 4%

c Bank discount on $5,000 for 2 years at 4%

d Compound discount on $5,000 for 2 years at 4% converted semiannually

5. Find the value 1 year hence of a 6% note for $4,200 due in 5 years if money is worth 4% converted semiannually

6. An investment worth $100,000 today will increase in value at a compound rate, until 5 years from now it will be worth $130,000 What would one be justified in paying for this investment 2 years from now, if money is worth 6% converted semiannually?

7. A note for $5,000 bearing annual interest at 4% is due in 4 years What is the present value of the note if money is worth 5% payable semiannually?

8. Find the present value of a 5-year, $7,000, 6% note due in 2 years if money is worth 6% converted quarterly

9. Find the present value of $4,000 due in 3 years with interest at 5% payable semiannually, if money is worth 6% annually

10. Find the present value of a payment of $100,000 due in 20 years if money is worth 5% converted semiannually

Equation of payments at compound interest

When an estate is settled, when bankruptcy proceedings are carried on, or when property is divided among creditors, it is often necessary to find one amount which is equivalent to two or more separate obligations For example, if A died owing two separate notes to B, the first for $5,000 due in 2 years, and the second for $2,100 payable in 7 years, the person settling the estate might be anxious to get the debts paid and the remainder of the estate distributed among the heirs long before the first

note was due, and B might be willing to accept one sum of money *equivalent* to the two separate obligations due at different times.

Sometimes it becomes necessary or desirable to *commute* one set of obligations into another set—that is, to substitute one set of obligations to be paid in one manner for another set of obligations to be paid in a different manner. The value of the old set of obligations is determined, and the value of the new obligations is established as equivalent to the old on a given date, which is known as the focal date. In making such a computation in compound interest any date may be selected. Unlike the equation of payment by simple interest, at compound interest *if one set of obligations is equivalent to another on a given date, at any other date they are still equivalent.* As a matter of convenience, the focal date may logically be selected as the date on which a payment is to be made.

In the example of A's estate, if payment is to be made 1 year hence, that date would be selected as the focal date. It will be 1 year before the note for $5,000 is due, and 6 years before the $2,100 note is due. If money is worth $4\frac{1}{2}\%$ per year, the value of the two obligations on the focal date would be

$$\$5{,}000\,(1 + 4\tfrac{1}{2}\%)^{-1} + \$2{,}100\,(1 + 4\tfrac{1}{2}\%)^{-6}$$

or

$$\$5{,}000 \times 0.956938 + \$2{,}100 \times 0.767896$$

$$= \$4{,}784.69 + 1{,}612.52 = \$6{,}397.21$$

Thus a single payment of $6,397.21 at the end of 1 year would be equivalent to the value of the two debts on that date.

In solving such problems, the procedure generally adopted is as follows:

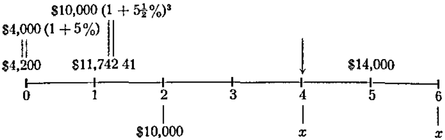
1. Select the date of the first unknown payment to be made as the focal date.
2. Find the equivalent value of each of the original sets of obligations on the focal date.
3. Find the equivalent value of each of the new obligations on the focal date.
4. Equate the sum of these two sets of equivalent values.
5. Solve the algebraic equation so found for the value of the unknown payment or each of the unknown payments.

Illustration: Because of business reverses, A has been forced during the last 5 years to borrow as much as he could under any terms his creditors cared to impose. He is now in a much better financial condition.

He has paid off all but one creditor, to whom he owes a \$4,000 note at 5% which is a year past due, a \$10,000 note for 3 years due in a year at $5\frac{1}{2}\%$ per year, a \$14,000 noninterest-bearing note due in 5 years

He would prefer to commute these debts to a note of \$10 000 due 2 years hence, and two notes for the same amount due 4 years and 6 years hence If money is worth $4\frac{1}{2}\%$ per year, how much should the face value of these two noninterest bearing notes be?

The problem may be diagrammed somewhat as follows, the arrow pointing to the focal date selected



From the figure it can be seen that the value of the original obligations on the selected focal date would be computed as follows

1 The \$4 000 debt now a year past due is worth \$4,200 today The \$4,200 would accumulate in the next 4 years until on the focal date it would amount to $\$4{,}200\,(1 + 4\frac{1}{2}\%)^4$

2 The \$10 000 note due in a year at $5\frac{1}{2}\%$ per year has drawn interest for 3 years and has a maturity value of \$11,742 41 At the focal date it would be worth $\$11{,}742\ 41\,(1 + 4\frac{1}{2}\%)^3$

3 The \$14,000 note would be discounted for 1 year to find its value on the focal date, that is, $\$14{,}000\,(1 + 4\frac{1}{2}\%)^{-1}$

The value of the new obligations on the focal date would be as follows

The \$10,000 payment due 2 years hence would be worth $\$10\ 000\,(1 + 4\frac{1}{2}\%)^2$

The first payment of x due on the focal date would be worth x

The second payment of x would be discounted for 2 years, that is, $x(1 + 4\frac{1}{2}\%)^{-2}$

Since the sum of the new obligations on the focal date must equal the sum of the present (old) obligations on the focal date, we can solve and find the value of x

Diagrams are made only to aid in the solution of such problems The user will be well rewarded if he draws diagrams to show the relation between the present obligations and the new obligations After much

practice it is possible to write the equation of value and solve without the use of a diagram.

In this illustration, the equation of value would be as follows:

$$\$4{,}200\,(1 + 4\tfrac{1}{2}\%)^4 + 11{,}742.41\,(1 + 4\tfrac{1}{2}\%)^3 + 14{,}000\,(1 + 4\tfrac{1}{2}\%)^{-1}$$
$$= \$10{,}000\,(1 + 4\tfrac{1}{2}\%)^2 + x + x\,(1 + 4\tfrac{1}{2}\%)^{-2}$$
$$\$4{,}200 \times 1.192519 + 11{,}742.41 \times 1.1411661 + 14{,}000 \times 0.9569378$$
$$= \$10{,}000 \times 1.092025 + x\,(1 + 0.915730)$$
$$1.915730x = \$20{,}875.50; \quad x = \$10{,}896.89$$

Thus the three original obligations will be cleared by making a payment 2 years hence of $10,000, a payment 4 years hence of $10,896.89, and a final payment 6 years hence of $10,896.89.

EXERCISE 12.8

Solve the following:

1. A debt of $1,000 is due in 5 years. If money is worth 4% converted annually, what is the value of the debt: (a) now; (b) 3 years hence; (c) 10 years hence?

2. A debt of $1,000 bearing interest at 5% converted annually is due in 5 years. If money is worth 4% converted annually, what is the value of the debt: (a) now; (b) 3 years hence; (c) 10 years hence?

3. Two debts of $1,000 each are due in 5 years. One is noninterest-bearing; the other bears interest at 5% a year for 5 years. If money is worth 4% converted annually, how much is required to discharge the two debts: (a) now; (b) 3 years hence; (c) 10 years hence?

4. Commute debts of $500 and $1,000 due in 2 and 3 years, respectively, into two equal payments due in 2 and 3 years, respectively. Assume that money is worth 5% compounded annually.

5. What single payment 3 years hence will discharge debts of $500 and $600 due in 2 years and 5 years, respectively, if money is worth 4% converted semiannually?

6. In settling the affairs of an uncle, a young man finds that the uncle owes $1,000 due in 1 year, $2,000 due in 2 years, and $2,500 due in 4 years. If money is worth 5%, payable quarterly, can he liquidate all the debts now with $5,000?

7. A debtor owes $5,000 due in 4 years. In his spare time he anticipates building two houses to be sold. The first will be finished and sold at the end of 2 years. If his anticipated profit of $2,500 on the first house

materializes and is used to reduce the debt, how much will he need at the end of the fourth year to pay the balance of the debt? Money is worth 5%.

8. A son agrees to assume the following three obligations of his father: (1) a \$2 000 4½% note due in 1 year; (2) a \$5 000 6% note for 3 years due in 18 months; (3) a \$6 000 note at 2% due in 2 years. The son desires to pay the debts by paying \$5 000 immediately, \$5 000 a year from today, and the remainder 2 years from today. If money is worth 5% converted semiannually, what should be the amount of the last payment?

9. A house is for sale for \$5 000 cash plus a 5-year 6% mortgage for \$5 000 and a 12-year 5% mortgage for \$8 000. If an investor has funds on which he is currently earning only 4% per year, how much should he be willing to pay as a cash price for the house?

10. A man has borrowed \$1 000 due in 4 years without interest, and \$3 000 due in 5 years with interest at 5% converted semiannually. He will pay \$2 000 in 2 years and the balance in two equal payments 3 and 5 years hence. If money is worth 4% converted annually, what should be the amount of these two payments?

13

Annuities Certain

Annuities

Compound interest, compound discount, and simple interest can be used as a basis for solving almost all business problems concerned with the evaluation of obligations at various times. If the payments extend over long periods, or if there are many payments, the solutions of the problems may be long and involved. Frequently, however, *equal payments* are made at *regular intervals.* Such a series of payments is called an *annuity.*

While the term annuity would seem to imply annual payments, in modern language it describes any series of payments made at regular time intervals whether the uniform period be weekly, monthly, quarterly, semiannually, or annually. Thus payments made on insurance policies, time payments on automobiles and houses, and furniture bought on the installment plan are everyday examples of annuities.

The length of time for which payments continue is the *term* of the annuity; the time between payments is called the *payment interval.* +' e term of the annuity is definite, such as 6 months, 5 years, or : fixed period of time, it is called an *annuity certain.*

Often contractual arrangements are made in which the p payment is not certain. For example, payments made in the f premiums on an ordinary life insurance policy are made only as l the insured is alive. Thus the number of payments, or the term annuity, is not certain. Such annuities are called *contingent annu'ti* differentiate them from annuities certain. They are discussed in chapter on life insurance.

Finding the amount of an annuity

To evaluate a series of payments, one must know the following fact (1) the amount of each payment, called the *periodic rent,* and

materializes and is used to reduce the debt, how much will he need at the end of the fourth year to pay the balance of the debt? Money is worth 5%

8. A son agrees to assume the following three obligations of his father (1) a \$2,000, $4\frac{1}{2}$% note due in 1 year, (2) a \$5,000, 6% note for 3 years due in 18 months, (3) a \$6,000 note at 2% due in 2 years The son desires to pay the debts by paying \$5 000 immediately, \$5,000 a year from today, and the remainder 2 years from today If money is worth 5% converted semiannually, what should be the amount of the last payment?

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Finding the amount of an annuity

To evaluate a series of payments, one must know the following factors: (1) the amount of each payment, called the *periodic rent*, and usually

represented by R, (2) the length of time during which the payments are to continue—that is, the term, (3) the rate of interest per period, and (4) the interval between payments

The value to which a series of payments will accumulate at a given time is called the *amount of an annuity*, or the accumulated amount of an annuity The present value of a series of future payments is called the *present value of an annuity*

The amount to which a series of deposits will accumulate is equal to the sum of the compound amounts of each payment

Illustration Assume that for the last 5 years you have invested \$100 annually with the First Federal Savings and Loan Association, receiving dividends of 3% compounded annually What is the amount to your credit immediately after the fifth payment?

Five payments have been made Immediately after the fifth payment, the first payment has accumulated interest for 4 years, the second for 3 years, the third for 2 years, and the fourth for 1 year The last payment has not had time to gather interest This series of payments is equivalent to an *ordinary annuity* since the payment may be considered as being made at the end of each period The *term* of the annuity is 5 years, since the term is measured from the beginning of the first period to the end of the last If R is used to represent the periodic payments and i the rate per period, the amount of the annuity could be shown in tabular form in any of the three following ways

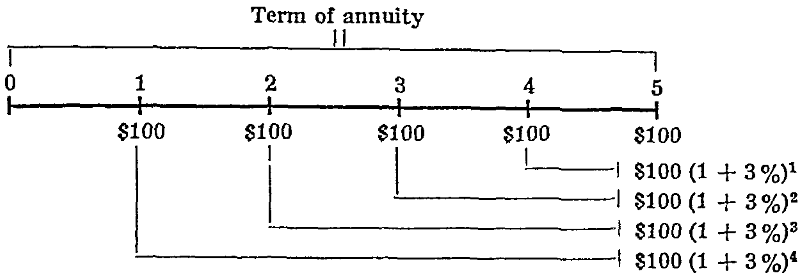
As General Numbers	*Substituting Numerical Value*	*From the Compound Amount Tables*
$R(1+i)^4$	\$100 $(1+3\%)^4$	\$112 5509
$R(1+i)^3$	\$100 $(1+3\%)^3$	109 2727
$R(1+i)^2$	\$100 $(1+3\%)^2$	106 09
$R(1+i)^1$	\$100 $(1+3\%)^1$	103 00
R	\$100	100 00
		\$530 91

To show this graphically, draw a line representing 5 periods of time and numbered from 0 to 5 as follows

0 1 2 3 4 5

The small 1 at the end of the first period indicates the time at which the first deposit of \$100 was made Since each space represents 1 year it can be seen that this payment was left at interest for 4 periods, the second

payment was left for 3 periods, the third payment for 2 periods, the fourth payment for 1 period, and that the fifth payment would draw no interest. At the time of the fifth payment the value of the first payment would be $100 $(1 + 3\%)^4$. The illustration can be drawn to show this:



If the symbol S_n (read S sub n) is used to represent the amount of an *annuity, the sum of the column to the right of the illustration can* be expressed as follows:

$$S_5 = \$100\,[1 + (1 + 3\%)^1 + (1 + 3\%)^2 + (1 + 3\%)^3 + (1 + 3\%)^4]$$

Such a series is a geometric progression. The sum of such a progression (made up of n terms) is

$$S_n = R\frac{(1 + i)^n - 1}{i}$$

Thus
$$S_5 = \$100\frac{(1 + 3\%)^5 - 1}{3\%}$$

Since an ordinary annuity is a form of geometric progression, the preceding formula is used as the formula for the amount of an annuity.

It is not necessary to know anything about geometric progressions, however, in order to work with annuities. In the preceding example, it was shown that payments of $100 each for 5 years at 3% compounded annually amount to $530.91. The amount of an annuity of 1 would be $\frac{1}{100}$ as much, or 5.30913581. The annuity is for 5 periods.

Under the terms of the formula $S_n = R\,\dfrac{(1 + i)^n - 1}{i}$, it should be a simple matter to find the amount of an annuity if the compound amount is known. The compound amount of 1 at 3% for 5 periods is shown in the compound amount table as 1.159274074. If the original principal of 1 is deducted, 0.159274074 is left as the amount of cumulation. Divided by the rate of interest, 3%, the quotient, 5.3091358, is the amount of an annuity of 1 at 3% for 5 periods. That is,

$$\frac{(1 + 3\%)^5 - 1}{3\%} = \frac{1.159274074 - 1}{0.03} = \frac{0.159274074}{0.03} = 5.3091358$$

In many accounting examinations and in accounting textbooks it is often expected that problems in annuities will be solved by reference *only* to a compound amount table If the formula for the amount of an annuity of 1 is memorized as $\frac{(1+i)^n - 1}{i}$, it should be easy to recall the following four steps necessary to find the amount of an annuity from *a compound amount table*

1 Find the compound amount of 1 for the term of the annuity at the periodic rate

2 Deduct 1 from the compound amount

3 Divide the difference by the periodic rate

4 Multiply the quotient by the periodic payment

Illustration *Using the compound amount table, find the amount of* an annuity of \$500 for 10 years at $3\frac{1}{2}\%$

1	1 at $3\frac{1}{2}\%$ for 10 years	1 41059876
2	Deduct 1	1
		0 41059876
3	Divide by $3\frac{1}{2}\%$, 0 41059876 − 0 035 =	11 73139
4	Multiply by \$500	× \$500
		\$5,865 70

The amount of an annuity table

Inasmuch as many payments, such as rent, interest, wages, pensions, installment payments, and dividends take the form of annuities, it is frequently necessary to find the amount of an annuity To facilitate calculation, annuity tables are constructed based on the amount of 1 The international symbol used to represent the amount of 1 per period is $s_{\overline{n}|i}$ *(read s angle n at i, or sometimes s sub n at i) Here s is the amount of* 1, *n* is the number of periods, and *i* is the rate per period

In the construction of the table, it is assumed, since it is generally true, that the period of the payment coincides with the period of interest conversion These tables are used in much the same way as the compound interest tables From the column headed Amount of 1 Per Period (How \$1 deposited periodically will grow) the amount of 1 for the necessary number of periods at the indicated rate *i* is found This tabular value is then multiplied by the value of the periodic payment

Illustration By moving from the city to a suburban area 10 miles away, *the buyer of a house saved \$2,500 on the original purchase price* The cost of driving back and forth to work from the new location is

equivalent to a payment of \$300 a year. If money is worth 4% a year, how much did the buyer gain or lose in 10 years?

From the Amount of 1 Per Period table, the amount of an annuity for 10 years at 4% is	12.006107
Multiply by \$300	× \$300
Amount of 10-year expenditure for driving	\$3,601.83
From the Amount of 1 table, the compound amount of 1 for 10 years at 4% is	1.480244
Multiply by \$2,500	× \$2,500
Compound amount of \$2,500 at 4% for 10 years	\$3,700.61
Net savings over 10-year period	\$98.78

Illustration: Fifteen students on graduation from college formed an investment club in which each agreed to make a monthly contribution of \$10 to the treasurer, to be invested by him. If all monthly payments are made regularly, how much should the treasurer have at the end of 10 years if the rate of earnings has been 4% converted monthly?

The total monthly payment is \$150, so $R = \$150$. The total number of periods is 120, so $n = 120$. The rate is 4% converted monthly, so $i = \frac{1}{3}\%$. From the table the amount of 1 per period for 120 periods at rate $\frac{1}{3}\%$ is 147.2498047255. Therefore the amount of the annuity is $\$150\ s_{\overline{120}|\frac{1}{3}\%} = \$150 \times 147.2498 = \$22{,}087.47$.

Amounts not included in the table

In the table the formula $s_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i}$ is shown at the bottom of the column. If this formula is known, the amount of an annuity can be computed readily for rates not in the tables and for periods greater than those shown in the tables.

Illustration: A house has been sold with the understanding that the balance of \$10,000 will be paid off in equal monthly payments of \$41.80 for the next 40 years. Assuming that money is worth 4% compounded monthly, what would be the value of the amounts paid at the time of the last payment. What would be the total cash payment made by the buyer?

In 40 years there are 480 monthly periods. The total cash payments made would be \$41.80 × 480, or \$20,064.00.

If the payments had been deposited at 4% interest payable monthly, or $\frac{1}{3}\%$ per month, the compound amount would be:

$$\$41.80\,\frac{(1+\frac{1}{3}\%)^{480} - 1}{\frac{1}{3}\%}$$

Since the table does not show the amount of an annuity of 1 for 480 periods at $\frac{1}{3}\%$, first find $(1 + \frac{1}{3}\%)^{480}$ by logarithms

$$\log (1 + \tfrac{1}{3}\%)^{480} = 480 \log 1\ 00333 = 480 \times 0\ 01444 = 0\ 693120$$

So $(1 + \frac{1}{3}\%)^{480} = 4\ 9331$, and $s_{\overline{480}|\frac{1}{3}\%}$ would have the approximate value of $\dfrac{4\ 9331 - 1}{\frac{1}{300}} = 3\ 9331 \times 300 = 1{,}179\ 93$ This figure is only approximate The figure shown *in a complete table of* $\frac{1}{3}\%$ *for 480* periods is 1,181 96315

Therefore the compound amount of monthly payments of $41 80 for 40 years if money is worth 4% compounded monthly is

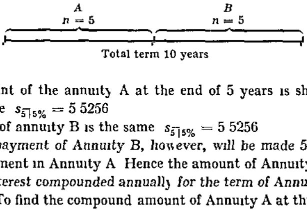
$$\$41\ 80 \times 1{,}179\ 93 = \$49{,}321\ 07$$

when worked with only 6 place logarithm

To find the amount of 1 per period for any period twice that shown in the amount of an annuity table, consider the problem to be one involving two annuities of equal periods Find the amount of the first The amount of the second will be exactly the same as the amount of the first During the period of the second annuity, however, the amount of the first will have been accumulating at compound interest To find the compound amount to which the first will have accumulated at the end of the second, multiply the amount of the first annuity by the compound amount of 1 for the period of the second annuity at the same rate per period

Illustration Find the amount of an annuity of $1 per year for 10 years at 5% using tabular values for 5 years

In effect the one annuity is separated into two annuities of equal duration (To simplify the explanation these will be referred to as annuities A and B)



The amount of the annuity A at the end of 5 years is shown by the tabular value $s_{\overline{5}|5\%} = 5\ 5256$

The term of annuity B is the same $s_{\overline{5}|5\%} = 5\ 5256$

The last payment of Annuity B, however, will be made 5 years after the last payment in Annuity A Hence the amount of Annuity A (5 5256) will draw *interest compounded annually for the term of Annuity B*—that is, 5 years To find the compound amount of Annuity A at this later date

it is necessary to multiply by the compound amount of 1 for 5 years at 5%.

The amount of Annuity A at the end of the period is thus found to be 7.0522 ($5.5256 \times 1.2763 = 7.0522$). The sum of the amount of Annuity A (7.0522) and the amount of Annuity B (5.5256) is 12.5779, which is equal to the tabular value shown for $s_{\overline{10}|5\%} = 12.577892$.

Instead of carrying out the multiplication and the addition, 1 may be added to the compound amount of $1 before multiplying to obtain the same result. Thus rather than multiplying 5.5256 by 1.2763 and later adding 5.5256 to the product, simply add 1 to the 1.2763 giving 2.2763. When multiplied by 5.5256, the result will be 12.5779.

The following rule is thus developed: *To double the term of the amount of 1 per period, multiply the tabular value by amount of 1 plus 1 at the same rate and for the same number of periods.*

In an earlier illustration the amount of a monthly annuity of $41.80 for 40 years at 4% compounded monthly was found by the use of logarithms. To solve the same problem by the use of tables, look for the value of $s_{\overline{480}|\frac{1}{3}\%}$. This value is not shown in the table, but the value for 120 periods is. When the value for 120 periods is known the value for 240 periods may be found; and when the value for 240 periods is known, the value for 480 periods may be found.

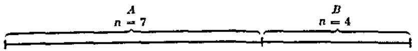
To find the tabular value for 240 periods:

1. Find the tabular value for 120 periods: $s_{\overline{120}|\frac{1}{3}\%} = 147.24989$.
2. Find the compound amount of 1 for 120 periods: $(1 + \frac{1}{3}\%)^{120} = 1.490832$.
3. Add 1 to the compound amount: $1 + 1.490832 = 2.490832$.
4. Multiply $147.24980 \times 2.490832 = 366.7745$, the value of $s_{\overline{240}|\frac{1}{3}\%}$.
5. The compound amount of 1 for 120 periods at $\frac{1}{3}\%$ is 1.490832. The compound amount of 1 for 240 periods at $\frac{1}{3}\%$ therefore is 1.490832×1.490832 or 2.222582.
6. Add 1 to the compound amount of 1 for 240 periods, giving 3.222582.
7. $s_{\overline{480}|\frac{1}{3}\%} = s_{\overline{240}|\frac{1}{3}\%} \times [(1 + \frac{1}{3}\%)^{240} + 1] = 366.7745 \times 3.222582 = 1181.96094$.

The amount of an annuity of $41.80 for 480 periods at rate $\frac{1}{3}\%$ is $\$41.80 \times 1{,}181.96094 = \$49{,}405.97$. There is a difference of 84.90 from the results found by using logarithms because of the lack of exactness in their use.

The tables may be used just as readily to find values when the period of the annuity cannot be divided into two equal periods. This method may be illustrated by a value shown in the tables. Assume that the problem is to find the amount of an annuity of $1 for 11 years at 5%

by using a table which runs only to 8 periods Consider this again as separated into two annuities, *one of 7 periods and one of 4 periods* The following diagram shows how the problem would be solved



Amount of Annuity A $= s_{\overline{7}|5\%} = 8\ 1420$

Compound amount of 1 for term of Annuity B $= (1 + 5\%)^4 = 1\ 2155$

Accumulated value of Annuity A at end of Annuity B $= 8\ 1420 \times 1\ 2155$
$= 9\ 8966$

Amount of Annuity B $= s_{\overline{4}|5\%} = 4\ 3101$

Therefore $s_{\overline{11}|5\%} = 9\ 8966 + 4\ 3101 = 14\ 2067$

EXERCISE 13.1

Solve the following problems

1. A depositor makes a deposit of $100 every 6 months to a Morris Plan Bank which pays 3% compounded semiannually Using the compound amount table, find the amount to his credit immediately after the fifth deposit

2. Every 6 months a depositor places $100 in a Morris Plan Bank which pays 3% compounded semiannually Using the annuity table, verify the amount to his credit immediately after the fifth deposit

3. Using the compound amount table, find the amount of an annuity of $200 a year for 10 years at 4% compounded annually How much of this is interest?

4. *Verify the amount of the annuity in Problem 3 by the use of the* annuity table

5. Periodic deposits of $100 a year at $2\frac{1}{2}\%$ compounded annually will furnish what sum of money immediately after the twentieth deposit?

6. A student receives the equivalent of $800 every 6 months for 4 years Assume that the money had been invested at 6% compounded semiannually instead of being given to the student What would be the amount of the annuity immediately after the eighth payment?

7. To provide for the ultimate purchase of a home, a man deposits $300 every 6 months at 3% compounded semiannually At the end of the eighth year, how much does he have to his credit?

8. By making minor changes in purchasing procedure, a firm is able to save $300 a year If money is worth 4% compounded annually, what *would the savings amount to in 4 years?*

9. Claire Beavon agreed to contribute \$20 a month for 10 years under The First Installment Investment Plan. How much will she have to her credit at the end of 10 years if the fund earns at (a) an annual rate of 3% compounded monthly? (b) 6% compounded monthly? (c) 12% compounded monthly?

10. A kind aunt gave Ellen Sterling an allowance to be paid at the rate of \$50 a month from the time of her tenth birthday through her eighteenth birthday. Ellen's father agrees to hold the money until her eighteenth birthday and give it to her then with interest at 6%. Ellen will be 18 next week. How much will her father owe her?

11. To establish a fund for expansion and improvement, a small corporation set aside \$2,500 a year at the end of each year for 5 years. If the money is invested at 5%, how much is in the fund at the end of the fifth year?

12. Mr. Vennard put \$500 in a savings and loan association each six months for 10 years. The association pays 3% compounded semiannually. Mrs. Vennard deposited \$1,000 a year at the end of each year with the Dollar Savings Plan, which paid $3\frac{1}{2}$% compounded annually. How much did each have at the end of 10 years?

13. Avril Swann buys a new car each year with an average payment of \$1,500 plus his old car. Had this money been invested at the end of each year at 4%, how much would he have had at the end of 10 years?

14. Sidney Rothe invests \$10 a month for 20 years at 4% converted monthly. How much does he have to his credit at the end of the twentieth year?

15. Robert Gilbreath saves \$250 a year, and by selecting risky investments carefully, was able to earn 15% a year. How much did he have at the end of the twelfth year?

16. Robert Waller invested \$100 a month for 5 years in the Yale Finance Club which earned $1\frac{1}{2}$% per month. How much did he have at the end of the fifth year?

17. The Grants chose to save \$25 a month and to invest their savings in a Mutual Fund. They permitted their dividends to accumulate and to be reinvested at the same rate. If the company earned at the annual rate of 9% compounded monthly, how much would they have at the end of 13 years?

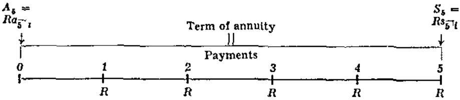
18. Robert Stanley saved \$100 at the end of each 6 months which he deposited in a savings bank at $2\frac{1}{2}$% compounded semiannually. At the end of the fifth year he discontinued making deposits for 2 years and then resumed them for 3 years. How much did he have at the time of his sixteenth deposit?

19. At their fifth annual class reunion, a group from a high school graduating class decided to make their tenth reunion a gala celebration to be held in Mexico City. To finance this adventure they agreed to each turn over $20 every 6 months to their class treasurer. If 100 of them made all payments and the treasurer was able to earn 4% compounded semiannually on the funds, how much would they have at the end of the fifth year?

20. Find the amount of an annuity of $1,200 a year for 23 years if money is worth 5% annually.

Present value

Because of the uncertainties of the world, it is often more important to be able to find the present value of a series of future payments than it is to find the amount of such payments. Again it must be reiterated that the present value and the amount are the same thing considered from different points of time. If we continue to allow S_n to represent the amount and use A_n to represent the present value, an annuity of 5 payments can be represented as follows:



Just as S_n represents the sum to which the individual payments would amount at compound interest, so A_n represents the sum of the present value of each of the future payments. In other words, the sum represented by A_n invested now at i rate per period would furnish sufficient income so that the income and the principal together would be just enough to make the future payments on the annuity.

It has been shown that

$$S_n = R\frac{(1+i)^n - 1}{i}$$

By definition,

$$A_n(1+i)^n = R\frac{(1+i)^n - 1}{i}$$

Thus $(1+i)^n$ is a common term. Divide both sides by $(1+i)^n$.

$$A_n = R\frac{(1+i)^n - 1}{i} \div (1+i)^n = R\frac{1-(1+i)^{-n}}{i}$$

The present value of an annuity of 1 per period is represented by the formula $\frac{1-(1+i)^{-n}}{i}$, or since $v^n=(1+i)^{-n}$, the formula may be written $\frac{1-v^n}{i}$. The symbol $a_{\overline{n}|i}$ (read a angle n at rate i) is used to represent the formula for the present value of an annuity of 1 at rate i. The formula for present value then is $A_n = R \cdot a_{\overline{n}|i}$.

It is possible to find the present value of an annuity from the compound discount tables without using the column headed Present Worth of 1 Per Period (What \$1 payable periodically is worth today) or the formula.

Illustration: Using only the compound discount table, find the present value of a series of payments of \$100 each for 5 years if money is worth 3%.

The present value of 1 for 5 years at 3% is 0.86260878, and $1 - 0.86260878 = 0.13739122$, is the compound discount on 1 for 5 years at 3%. Then $0.13739122 \div 3\% = 4.57970719$, present value of an annuity of \$1 for 5 years. This means that \$457.97 now is as good as 5 payments of \$100 each in the future, or as the amount of \$530.91 (the amount of the annuity) in 5 years.

This can be verified:

$$\$457.97\,(1+3\%)^5 = \$457.97 \times 1.15927407 = \$530.91$$
$$\$530.91\,(1+3\%)^{-5} = \$530.91 \times 0.86260878 = \$457.97$$

Though it is possible to find the present value of an annuity either by consulting compound discount tables or by using the formula and solving by logarithms, separate tables are usually constructed which show the present worth of an annuity of 1. These tables are used in exactly the same way as a table for the amount of an annuity.

Illustration: A house is bought with payments of \$100 at the end of each month for 5 years. If money is worth 4% converted monthly, what is the equivalent cash price?

Since $n = 60$, then $i = \frac{4}{12}\% = \frac{1}{3}\%$, and $R = \$100$

$$A_{60} = \$100 \cdot a_{\overline{60}|\frac{1}{3}\%} = \$100 \times 54.29906 = \$5{,}429.91$$

EXERCISE 13.2

Solve the following:

1. Find the present value of an annuity of \$300 a year for 8 years at 3% compounded annually, using (a) the present worth of 1 table; (b) the annuity tables.

2. In a contest the winner is allowed an option of $1,200 at the end of each year for the next 30 years, or $20,000 in cash Disregarding the possibility of the death of the winner, if money is worth 4% compounded annually, which is the more desirable?

3. What is the present value of a gravel pit which will furnish an income of $10,000 a year for the next 17 years and then be worthless, if money is worth 5% compounded annually?

4. Charles Roberts buys a house by paying $200 a month for 10 years If money is worth 6% compounded monthly, what is the cash equivalent of these payments?

5. Find the present value of a series of quarterly payments of $100 each for 5 years if money is worth 6% compounded quarterly Solve by using (a) present worth of 1 table, (b) the annuity table

6. Sven Engstrom bought a piece of property by making a cash payment of $5,000 and agreeing to pay $200 every 3 months for 8 years If money is worth 4% compounded quarterly, what was the equivalent cash value

7. Three students each agree to pay $25 a month for 1 year for a sailboat If money is worth 6% converted monthly, what was the equivalent cash value of the boat?

8. Find the present value of a series of monthly payments of $25 each for 8 years and 6 months if money is worth 3% converted monthly.

9. Paul Davis bought an abandoned quarry at a tax sale for $100 He arranged with the highway department engineers for them to dump excess dirt from a freeway development in the pit at $400 a month for 3 years He estimates that at the end of 3 years the quarry will be completely filled and that it can be sold to the city as a park site for $15,000 If his figures are correct, what is the present value of the hole, if money is worth 4% compounded monthly?

10. The author of a novel anticipates royalty payments of $1,000 at the end of each 6-month period for the next 3 years Find the present cash equivalent if money is worth 5% converted semiannually.

11. What single sum invested now at 4% is equivalent to an annuity of $250 a year for 10 years if money is worth 4%?

12. A house can be bought for $10,000 cash, or payments of $100 at the end of each month for the next 10 years If money is worth 4% converted monthly, which plan is better for the buyer, and by how much?

Determination of an unknown length of time

Often in day-to-day problems dealing with ordinary annuities it is desirable to find the period of time. For example, one who seeks to

accumulate \$5,000 may want to know approximately how long it will take him if he puts \$300 in a savings and loan association every 6 months at 3% compounded semiannually.

Substituting the known values in the formula, we have $S_n = R \cdot s_{\overline{n}|i}$, $S_n = \$5{,}000$, $R = \$300$, $i = 1\frac{1}{2}\%$. Thus $5{,}000 = 300 \cdot s_{\overline{n}|1\frac{1}{2}\%}$, or $s_{\overline{n}|1\frac{1}{2}\%} = \dfrac{5{,}000}{300} = 16.666667$.

By looking at the table for $1\frac{1}{2}\%$ it can be seen that 15 deposits of \$1 will amount to \$16.682 at the time of the fifteenth deposit. Hence the necessary sum can be accumulated in $7\frac{1}{2}$ years.

In 7 years the 14 payments of \$300 will accumulate to \$4,635.11 (\$300 × 15.45039). By the time of the final payment this sum will have accumulated interest for 1 period more and hence will amount to \$4,704.64 (\$4,635.11 + \$4,635.11 × $1\frac{1}{2}\%$).

Hence the fifteenth payment need be only \$295.36 (\$5,000 − \$4,704.64) to have a total of \$5,000 at the end of $7\frac{1}{2}$ years.

Frequently one needs to know how many payments are necessary to discharge a debt. Suppose that \$10,000 has been lent at 6% interest, to be repaid by semiannual payments of \$800 each. How many payments will be received?

Here the present value is known. Hence the formula is $A_n = R \cdot a_{\overline{n}|i} =$ \$10,000, $R = \$800$, $i = 3\%$, $n = ?$ Substituting: $\$10{,}000 = \$800 \cdot a_{\overline{n}|3\%}$; $a_{\overline{n}|3\%} = \dfrac{10{,}000}{800} = 12.50000$.

From the 3% table of the present worth of 1 per period, it is seen that for 16 periods the value is 12.561. Thus 15 full payments of \$800 and a partial payment would be received.

Under such circumstances one of two procedures is followed. The more prevalent practice is to increase the size of the last full payment. The other method is to make the last payment smaller than the others. Under the first method the amount of an annuity of 15 payments of \$800 each would be found, and deducted from the accumulated value of the debt to that time. Thus

$$\begin{aligned} &\$10{,}000\,(1 + 3\%)^{15} - 800 s_{\overline{15}|3\%} \\ &= \$10{,}000 \times 1.557967 - 800 \times 18.598913 \\ &= \$15{,}579.67 - 14{,}879.13 = \$700.54 \end{aligned}$$

If the balance is paid along with the fifteenth payment, the amount of the fifteenth payment would be increased by \$700.54.

Under the alternative method the sixteenth payment would be made as an amount sufficient to discharge the debt. If it would have taken an

additional $700 54 to discharge the debt at the time of the 15th payment, at 3% it would require $721 56 ($700 51 + 700 51 × 3%) to complete the payment at the time of the sixteenth payment

EXERCISE 13 3

Solve the following

1. To accumulate $2 500 Robert Morton puts $100 into a savings account every 3 months If money is worth 3% compounded quarterly, how many full deposits must he make? What is the amount of the last deposit?

2 To accumulate $3 200 William White puts $50 into a savings account at the end of each month If money is worth 3% compounded monthly how many full deposits must he make? What is the amount of the last deposit?

3 To pay off a debt of $1,200 Ernie Fisher agreed to make payments of $150 at the end of each 6 months If money is worth 4% compounded semiannually, how many full payments must he make? What is the size of the last partial payment if made 6 months after the last full payment?

4 To pay off a debt of $650 the Taylors agreed to make payments of $50 at the end of each month If money is worth 6% compounded monthly, how many full payments must they make? What should be the size of the last payment if no partial payment is to be made?

5 A student buys a watch that sells for $88 He pays $10 down and the balance at $2 50 a month If money is worth 1% a month, how many full payments are to be made? What additional down payment would eliminate any partial payment at the end?

6 A doctor buys some new equipment for his office for $3,650 He agrees to make a down payment of $1,000 and monthly payments of $100 until the equipment is paid for If money is worth 6% compounded monthly, how many full payments are necessary? What is the size of the final payment if made at the time of the last full payment?

Extension of the table of Present Worth of 1 per period

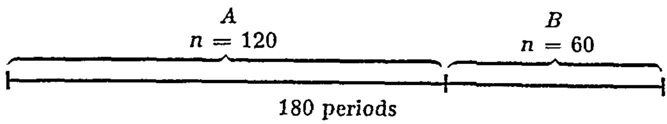
If a complete set of tables is available, it rarely becomes necessary to find a value beyond the table As an exercise in gaining an understanding of the tables, however, and for use when the complete tables are not available the method of finding values beyond the table is illustrated

To find the value of an annuity for a period beyond the tables, separate the annuity into two annuities whose values are shown, and consider each separately The value of the first will be shown in the table The present worth of the later annuity at the beginning of its life can also be

read from the table. To find the present worth of the later annuity, simply multiply its tabular value by the Present Worth of 1 for the term of the first annuity.

Illustration: What is the present worth of \$1 paid at the end of each month for 15 years at 6% compounded monthly?

Consider this as two annuities, one for 120 periods and one for 60 periods. This could be diagrammed as follows:



For clarity in explaining the illustration, the first annuity is referred to as Annuity A. The present worth of Annuity A, *in which* $n = 120$, *and* $i = \frac{1}{2}\%$, is shown in the table as $a_{\overline{120}|\frac{1}{2}\%} = 90.07345$.

The present worth of an annuity of 60 periods (taken from the table) is $a_{\overline{60}|\frac{1}{2}\%} = 51.72556$. This value is the present value of Annuity B at the *time of the end of the first annuity*, or 120 periods hence. Therefore to find its present value now, it must be discounted for 120 periods. The tabular value for $(1 + \frac{1}{2}\%)^{-120}$ is 0.549633.

The value of Annuity B at the beginning of Annuity A is 28.43008 (51.72556×0.549633). If the present worth of Annuity A (90.07345) is added to the present worth of Annuity B (28.43008), the present worth of an annuity of \$1 for 180 periods at $\frac{1}{2}\%$ per period is found to be 118.5035.

EXERCISE 13.4

Solve the following.

1. What is the present worth of an annuity of \$100 a month for $16\frac{1}{2}$ years if money is worth 8% converted monthly?

2. What is the present value of \$150 every 3 months for 35 years if money is worth 3% compounded quarterly?

3. To settle a judgment arising from an automobile accident, John Williams was required to pay \$25 a month for the next $12\frac{1}{2}$ years. If money is worth 4% converted monthly, what is the equivalent cash value of his obligation?

4. The Safeway Company agrees to pay \$300 at the end of each month for the next 20 years for the use of a site of land. If money is worth 6% payable monthly, what is the equivalent cash value of this contract?

5. Under the terms of a contract \$25 is to be received every month for the next 15 years. What is the equivalent cash value of the contract if money is worth 2% compounded monthly?

Amortization

The periodic payment R is often referred to as the *rent* of an annuity The word *rent* as used in this sense is synonymous with payment and not necessarily concerned at all with income from the use of real estate It is often desirable to find the periodic rent when the *amount* or *present value* of an annuity is known

If $$A_n = R \quad a_{\overline{n}|i}$$

then $$R = A_n \quad \frac{1}{a_{\overline{n}|i}}$$

By the simple process of division, the value of $\frac{1}{a_{\overline{n}|i}}$ can be calculated from the table showing the values for $a_{\overline{n}|i}$ Since, however, $\frac{1}{a_{\overline{n}|i}}$ is often used, tables showing these values at different rates have been prepared They are included in the appendix under the column headed Partial Payment (Annuity worth \$1 today Periodic payment necessary to pay off a loan of \$1)

Such calculations are often necessary When a debt is to be repaid in equal installments of principal and interest, it is said to be *amortized* In other words, in the course of a fixed period of time, the principal and interest are repaid at regular periods in a series of equal installments Special assessments, such as for paving a street or installing street lights, can usually be paid outright or be amortized

Illustration A street light assessment of \$65 may be spread over a 5-year period If interest is $4\frac{1}{2}\%$, what is the annual payment?

Here the present value is known

$$A_n = \$65, \quad i = 4\tfrac{1}{2}\%, \quad n = 5$$

$$R = \$65 \quad \frac{1}{a_{\overline{5}|4\frac{1}{2}\%}}$$

From the table

$$\frac{1}{a_{\overline{5}|4\frac{1}{2}\%}} = 0\ 22779164$$

$$R = \$65 \times 0\ 22779 = \$14\ 81$$

The payments necessary to discharge a debt constitute an annuity whose *present value* is equal to the *original principal* of the debt Thus the total payments made under amortization will be greater than the original amount of the debt. In the illustration, for example, the total payment under the amortization plan is \$74 05 (\$14 81 × 5) The \$9 05 excess over the \$65 (\$74 05 — 65 00) is the payment for interest If the debt had

been discharged by a single payment, only $65.00 would have been required.

More and more borrowers have come to prefer amortized loans since the payments are spread over the life of the loan and since the borrower is continuously able to appraise his current position. Lenders, too, prefer amortized loans. There has been a great increase in the volume of *term loans,* which are usually for 5 years or longer, made by commercial banks and repaid over the period of the loan. All home loans guaranteed by FHA, as well as almost all other building loans, are now financed by amortized mortgages.

Although amortized payments themselves are equal, the division between principal and interest varies from payment to payment. It is sometimes necessary to know just how the payments are apportioned. This apportionment may be shown in what is called an *amortization schedule.*

In the preceding illustration, a debt of $65 was to be discharged by 5 equal payments, with interest at $4\frac{1}{2}\%$. An amortization schedule which shows the distribution of each payment between principal and interest can be developed. During the first year, interest on $65 amounts to $2.92. When the payment of $14.81 is made at the end of the year, $2.92 goes to pay the interest, and the balance of $11.89 reduces the principal to $53.11. The next year, interest on the new principal of $53.11 amounts to $2.39. When the second payment of $14.81 is made, $2.39 goes to pay the interest, and the balance of $12.42 reduces the principal to $40.69. The interest for the third year is less than in the preceding years, only $1.83. The balance of the payment, $12.98, goes to reduce the principal to $27.71. By such calculations, the amortization schedule which follows can be constructed. Since fractional parts of cents are not considered in making payments, it is not unusual for the last payment to be a few cents more or less than the others. In the following schedule the last payment equals only $14.79, which is 2 cents less than each of the other payments.

Amortization Schedule

Year	*Outstanding Principal at the Beginning of the Year*	*Interest on the Principal at $4\frac{1}{2}\%$*	*Annual Payment*	*Principal Repaid*
1	$65.00	$2.92	$14.81	$11.89
2	53.11	2.39	14.81	12.42
3	40.69	1.83	14.81	12.98
4	27.71	1.25	14.81	13.56
5	14.15	0.64	14.79	14.15

Both debtor and creditor often need to determine what part of an amortized payment constitutes a reduction of principal and what part constitutes a payment of interest The debtor may choose to deduct such interest payments in computing his income taxes, the creditor must report such interest income on his income tax return

Finding the outstanding debt at any time

The outstanding debt may be determined in any of three ways In the first place, if an amortization schedule has been prepared as in the preceding illustration, the outstanding principal immediately after any payment can be taken from the schedule which shows the outstanding principal at the beginning of the year In the preceding illustration, the schedule shows that just after the third payment the unpaid principal is $27 71

It is not necessary, however, to construct an entire schedule to find the amount of principal unpaid at any given time According to the *prospective*—that is, looking ahead—*method,* the remaining payments form an annuity, and the unpaid principal at any time will be equal to the present value of the future remaining payments

Illustration From the information given in the preceding illustration, find the amount of principal outstanding at the beginning of the fourth year

There are 2 remaining payments The amount of principal outstanding then is the present value of the 2 remaining payments

$$A_2 = \$14\ 81 \quad a_{\overline{2}|4\frac{1}{2}\%} = \$14\ 81 \times 1\ 87267 = \$27\ 73$$

Since the periodic payments are correct only to cents, and since the last payment according to the schedule was $14 79, the amount of unpaid principal calculated by this method differs by 2 cents from the figure shown in the amortization schedule

A third method of determining the outstanding debt immediately after a payment has been made is the *retrospective method*—that is, a method which looks to the past All three methods obviously should give the same results, so that when there is a choice the most convenient method should be chosen

Under the retrospective method, the outstanding debt is equal to the original debt plus accumulated interest to date, less the amount of the annuity made up of all payments which have been made In general this method would be used only if the number of total payments is unknown, or if the size of the last payment is not the same as the others

Illustration: A debt of $10,000 at 6% is to be repaid with semiannual payments of $800 as long as necessary, with a smaller final payment. Find the outstanding debt just after the fifteenth payment.

At the time of the fifteenth payment, the outstanding principal (had no payments been made) would have been $10,000 $(1 + 3\%)^{15}$. The amount of the payments, however, was $800 $\cdot s_{\overline{15}|3\%}$. Thus the outstanding debt was

$$\begin{aligned} &\$10{,}000\,(1 + 3\%)^{15} - 800 \cdot s_{\overline{15}|3\%} \\ &= \$10{,}000 \times 1.557967 - 800 \times 18.59891 \\ &= \$15{,}579.67 - 14{,}879.13 = \$700.54 \end{aligned}$$

This is the same answer obtained to this problem in the demonstration on page 369, showing how to determine the final payment on a loan.

EXERCISE 13.5

Solve the following.

1. A bank agrees to lend $12,500 if the principal and interest are repaid in equal quarterly installments over a period of 5 years. (a) If interest is 4%, find the periodic payments. (b) The borrower chooses to repay the loan entirely at the time of the fourth quarterly installment. What is the amount of the debt immediately after the fourth quarterly installment has been paid?

2. A debt of $5,000 is now due. Find the annual payment necessary to cancel the debt in 5 years if money is worth 4%.

3. A debt of $5,000 is to be paid in equal semiannual payments. Find the semiannual payment necessary to cancel the debt in 10 years if money is worth 6% converted semiannually.

4. Payment of $3,000 is being made in 8 equal annual installments. Find the amount owed on the debt immediately after the fifth payment if money is worth $4\frac{1}{2}\%$.

5. A debt of $4,200 bearing interest at 4% is to be amortized by payments at the end of each half year for the next 5 years. Find the periodic payment and construct an amortization schedule of the debt.

6. A $2,250 loan is being paid off by making quarterly payments of $350. If money is worth 4% compounded quarterly, develop the amortization schedule, assuming that the last payment will be greater than $350 and will be made when the last full payment is due.

7. An $800 loan is being paid off with monthly payments over a period of $2\frac{1}{2}$ years. If money is worth 9% payable monthly, find the outstanding loan immediately after the twelfth payment.

8 A street improvement assessment for $450 is spread over a period of 4 years Develop an amortization schedule Money is worth 6% payable semiannually and payments are made every 6 months

9 Interest paid is deductible in computing the federal income tax If a loan of $2 000 is being paid off by equal quarterly payments over a period of 4 years at 8% payable quarterly, how much interest is paid in the last year of the loan?

10 Interest received must be reported as income A taxpayer lends $5 000 at 12% to be repaid in 18 equal monthly payments of $304 91 How much interest income does the lender receive in the first 12 payments on the loan?

Sinking fund

If a debt is not amortized the debtor (that is the one owing the money) must make provision for repaying it at maturity Often the contract between a creditor and a debtor provides that the debtor, though he need not pay the debt until maturity, must establish a fund to which he makes periodic contributions which will equal the amount of the debt at maturity Any fund established for the purpose of meeting an obligation due in the future is known as a *sinking fund*

The payments to a sinking fund need not be equal Often they are a certain percentage of earnings and vary from one period to another as earnings fluctuate *When payments to a sinking fund are equal and are made periodically, they form an annuity,* and the problem of determining the amount of each payment resolves itself into finding the *rent* of the annuity when the amount of the annuity is known The basic formula for determining the rent is the formula for the amount of an annuity $S_n = R \; s_{\overline{n}|i}$

It can be seen that

$$R = S_n \; \frac{1}{s_{\overline{n}|i}}$$

It is possible to find the rent of an annuity by determining the value of $\frac{1}{s_{n\,i}}$ from the table showing the amount of an annuity of 1, that is $s_{\overline{n}|i}$ Since such a laborious calculation would have to be made frequently, many tables of annuities show the value for $\frac{1}{s_{\overline{n}|i}}$ in a separate table or column The Financial Compound Interest and Annuity Tables reproduced in the appendix include the values for $\frac{1}{s_{\overline{n}|i}}$ under the column headed Sinking Fund (Periodic deposit that will grow to $1 at future date)

These values are used in the same manner as those previously shown in the solution of problems.

Illustration: A mortgage of \$15,000 is due in 5 years. How much should the mortgagor set aside each quarter in order to have a sum sufficient to pay the mortgage when it falls due, if he can earn 2% converted quarterly on the amount in the fund?

$$S_n = \$15{,}000; \quad n = 20; \quad i = \tfrac{1}{2}\%$$

$$R = \$15{,}000 \cdot \frac{1}{s_{\overline{20}|\frac{1}{2}\%}} = \$15{,}000 \times 0.0476664 = \$715.00$$

The problem of finding the periodic rent of an annuity when the amount is known also occurs in other business situations, including that of planning an expansion program.

Illustration: The directors plan to expand a corporation's assets by \$100,000 during the next 5 years. How much should they expand each 6 months if the increased assets bring in earnings at the rate of 12% converted semiannually?

$$S_n = \$100{,}000; \quad n = 10; \quad i = 6\%$$

$$R = \$100{,}000 \cdot \frac{1}{s_{\overline{10}|6\%}} = \$100{,}000 \times 0.07586795 = \$7{,}586.80$$

the semiannual expansion necessary.

When a debt contract contains a provision for a sinking fund, the payments to the fund are separate and distinct from the periodic payments of interest which must also be made on the debt. *The sum of the periodic contributions to the sinking fund and the periodic interest payments to the creditor is called the total periodic charge.* It is found by adding the periodic interest and the periodic payment made to the sinking fund.

Illustration: A corporation has a debt of \$1,000,000 due in 10 years, to be discharged by a sinking fund. The debtor is to make semiannual contributions to the sinking fund, which will be invested at 3% converted semiannually. Find the total periodic charge if the debt bears: (a) no interest; (b) interest at 4% payable semiannually.

$$\text{(a)} \quad S_n = \$1{,}000{,}000; \quad n = 20; \quad i = 1\tfrac{1}{2}\%$$

$$R = \$1{,}000{,}000 \cdot \frac{1}{s_{\overline{20}|1\frac{1}{2}\%}} = \$1{,}000{,}000 \times 0.043245736 = \$43{,}245.74$$

the semiannual contribution to the sinking fund.

(b) The semiannual contribution to the sinking fund would still b $13,215.74 In addition, interest at 2% must be paid every 6 months on the $1,000,000, namely, $20,000.00 Therefore the total periodic charge is $13,215.71 + 20,000.00 = $63,215.71

EXERCISE 13 6

Solve the following

1. A corporation has a bond issue of $150,000 to be repaid 25 years hence If money is worth 3% payable semiannually, how much must the corporation set aside each 6 months to accumulate the desired amount If the debt bears interest at 4% payable semiannually, what is the total periodic charge?

2 What payment, made at the end of every 3 months, will accumulate to $2,000 at the end of 5 years at 4% compounded quarterly?

3 Richard Ackerman wants to have enough money at the end of 3 years to buy an airplane for $2,800 If he can obtain 6% on his money compounded monthly, how much should he save per month?

4 If $20,000 is needed 5 years hence, how much should be saved quarterly if money will earn 2% converted quarterly?

5 Payments of $832.27 are being put into a sinking fund every 6 months at 4% converted semiannually How much should be in the fund just after the twentieth payment?

6 Annual payments of $325 are invested at 3% for 10 years Just after the 10th payment the interest rate is raised to $3\frac{1}{2}\%$ The annual payments of $325 are continued for 5 more years What is the amount of the fund?

7 The University Housing Association borrowed $250,000 for 5 years at 4% Under the terms of the loan the borrower was required to make 5 equal annual contributions to a sinking fund to be invested in government bonds at 3% to provide for the repayment of the debt at maturity Find the total periodic charge

8 The Pleasant Ridge School District borrowed $200,000 with the provision that a sinking fund would be built up by 10 equal annual payments The sinking fund is to be invested in United States government bonds paying $2\frac{1}{2}\%$ interest annually What is the total periodic charge if the District bonds issued bear interest at the rate of 4% payable annually?

9 A city borrows $100,000, and agrees to pay interest at $3\frac{1}{2}\%$ They establish a sinking fund to repay the principal at the end of 12 years If the payments to the sinking fund can be invested at 3%, what will be the total annual payments?

10. Lyon City borrows $750,000 for 10 years at 4% for building a public auditorium. They establish a sinking fund to repay the principal. What will be the annual periodic charge if the sinking fund is invested at 3%?

Finding the book value of a debt

An accountant must be able at any time to determine the amount in a sinking fund. If the payments into the fund have been irregular, the amount may be found by computing the compound amount of each payment from the time it was made, and finding the sum of these values.

This discussion of sinking funds, however, is limited to those in which an equal periodic payment is made. Immediately after any payment, the amount in the sinking fund may be found by determining the amount of an annuity of the periodic payments made to the sinking fund.

If a debtor chooses to discharge the principal of the debt completely at any given time, he must pay the difference between the face amount of the debt and the amount in the sinking fund. The difference between the face amount of the debt and the amount in the sinking fund is called the *book value of the debt.*

Illustration: A debt of $20,000 due in 5 years is to be paid by the sinking fund method. If the fund earns interest at the rate of 5%, find the periodic payments and the amount in the sinking fund immediately after the fourth payment.

$$R = S_n \cdot \frac{1}{s_{\overline{n}|i}}; \quad S_n = \$20{,}000; \quad n = 5; \quad i = 5\%$$

$$R = \$20{,}000 \cdot \frac{1}{s_{\overline{5}|5\%}} = \$20{,}000 \times 0.180975 = \$3{,}619.50$$
$$= \text{the periodic payment}$$

To find the amount in the sinking fund immediately after the fourth payment,

$$R = \$3{,}619.50; \quad n = 4; \quad i = 5\%$$

$$S_4 = \$3{,}619.50 \cdot s_{\overline{4}|5\%} = \$3{,}619.50 \times 4.310125 = \$15{,}600.50$$

The book value of the debt immediately after the fourth payment is $20,000.00 – 15,600.50 = $4,399.50.

These figures can also be found by constructing a schedule of the payments into the sinking fund. At the end of the first year, $3,619.50 would be contributed to the fund. The balance of the debt, $16,380.50, would be the book value at the end of the year. During the second year,

the $3 619 50 in the sinking fund would earn $180 98 of interest at 5% At the end of the year, the fund is increased by another contribution of $3,619 50 and by the interest earnings of $180 98

The book value of the debt at the end of the second year is $12,580 02 The amount in the sinking fund would earn interest at 5% during the third year, and at the end of the year the fund would be increased by the contribution of $3,619 50 augmented by the interest earned of $371 00 The fund would contain $11,410 48 at the end of the third year, and the book value of the debt would be $8 589 52 The income during the fourth and fifth years, along with a tabular summary of the first three years, is shown in the following schedule Because fractional parts of cents have been considered as a whole, the last payment of $3,619 48 is 2 cents less than each of the others

Sinking Fund Schedule

End of Year	*Periodic Payment to Fund*	*Interest Income on Funds at 5%*	*Periodic Increase in Sinking Fund*	*Amount of Sinking Fund*	*Book Value of Debt*
0					$20,000 00
1	$3 619 50		$3,619 50	$3,619 50	16,380 50
2	3,619 50	$180 98	3,800 48	7,419 98	12,580 02
3	3,619 50	371 00	3,990 50	11,410 48	8,589 52
4	3,619 50	570 50	4,190 00	15,600 48	4,399 52
5	3,619 48	780 02	4,399 52	20 000 00	0,000 00

Comparison of the amortization method and sinking fund method of debt retirement

One of the principles of good financial planning, whether for a consumption loan or a business loan, is planning the method of repayment Whether you are the borrower or the lender, you will want a plan of repayment most satisfactory to you Though you may not always be able to carry out the plan you favor, you should at least be able to make the comparison

A person seeking to borrow $10,000 at 6% for 5 years has the choice of (1) repaying the principal and interest in equal annual installments at the end of each year, or (2) of paying the interest annually and setting aside a fund in a savings bank at 4% interest sufficient to discharge the debt at the end of 5 years Which plan is less expensive?

The best method of comparison is to find the annual charges under the different plans Under the sinking fund plan the annual interest on

\$10,000 at 6% is \$600, and the annual deposit which must be made in the savings bank to accumulate to \$10,000 at 4% is the rent of the annuity, $\$10{,}000 \cdot \frac{1}{s_{\overline{5}|4\%}} = \$10{,}000 \times 1.84627 = \$1{,}846.27$. The total charge under the sinking fund method is $\$1{,}846.27 + 600.00 = \$2{,}446.27$.

Under the amortization plan the annual charge would be the partial payment necessary to discharge a debt of \$10,000 at the end of 5 years, or

$$R = \$10{,}000 \cdot \frac{1}{a_{\overline{5}|6\%}} = \$10{,}000 \times 0.237396 = \$2{,}373.96$$

Since the sinking fund bears interest at a rate less than the rate on the debt, here the annual cost under the sinking fund plan is higher. Had the rate on the sinking fund been the same as the interest rate on the debt, the annual charge would be \$1,773.96 ($\$10{,}000 \cdot \frac{1}{s_{\overline{5}|6\%}} = \$10{,}000 \times 0.177396$) plus the annual interest charge of \$600, or a total annual charge of \$2,373.96. Thus the annual charge under the two plans will be the same when the rates are the same.

EXERCISE 13.7

Solve the following.

1. A fund of \$8,500 is needed 4 years hence. If money is worth 7% payable semiannually, find the semiannual payments and develop a sinking fund schedule.

2. A fund of \$20,000 is needed 5 years hence. Money is worth 6% annually. Find the annual payment to the sinking fund and develop a schedule.

3. A watch is purchased for \$75. The down payment is \$15. The balance of \$60 is to be paid in 12 equal monthly installments. If money is worth 12% converted monthly, what is the monthly payment?

4. Acme Cleaners borrow \$4,000. The debt is due in 5 years and is to be repaid by the sinking fund method. If the fund earns interest at the rate of $4\frac{1}{2}\%$, find the annual payment and the amount in the sinking fund immediately after the third payment.

5. A sanitary district votes \$100,000 in bonds to construct a sewage disposal plant. The bonds bear interest at 4% payable semiannually over a period of 12 years. The district is legally required to establish a sinking fund to provide for the retirement of the bonds at maturity. Find the semiannual contribution to the sinking fund if interest at 3% converted semiannually is to be earned. Find the periodic charge. What is the book value of the debt at the end of the fifth year?

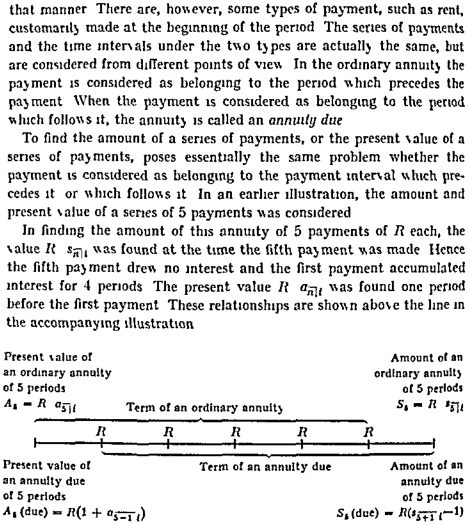
6. A state sold \$500,000 of bonds to finance a bridge Since the interest on the bonds was exempt from federal income taxes, the annual rate was very low—1½% The law required that a sinking fund of equal annual installments be established to retire the debt at the end of 10 years The money in the sinking fund could be invested at 3½% Find the periodic payments Find the book value of the debt at the end of the seventh year

Annuities due

An annuity has been defined as a series of equal payments made at regular intervals The payments discussed so far have been considered as having been made at the *end* of the payment period since many types of payments, such as bond interest and installment payments are made in that manner There are, however, some types of payment, such as rent, customarily made at the beginning of the period The series of payments and the time intervals under the two types are actually the same, but are considered from different points of view In the ordinary annuity the payment is considered as belonging to the period which precedes the payment When the payment is considered as belonging to the period which follows it, the annuity is called an *annuity due*

To find the amount of a series of payments, or the present value of a series of payments, poses essentially the same problem whether the payment is considered as belonging to the payment interval which precedes it or which follows it In an earlier illustration, the amount and present value of a series of 5 payments was considered

In finding the amount of this annuity of 5 payments of R each, the value $R \; s_{\overline{n}|i}$ was found at the time the fifth payment was made Hence the fifth payment drew no interest and the first payment accumulated interest for 4 periods The present value $R \; a_{\overline{n}|i}$ was found one period before the first payment These relationships are shown above the line in the accompanying illustration



From this illustration it can be seen that the present value of an annuity [A_n (due)] of 5 payments would be the value at the time the first payment (R) is made. The present value of the four future payments would be $R \cdot a_{\overline{4}|i}$. Since the time of valuation is simultaneous with the first payment, the present value of the first payment of R is R. Hence this amount, R, added to the present value of the remaining four payments, gives the present value of the annuity due. The rule is thus developed that *to find the present value of an annuity due, find the tabular value for the present worth of 1 per period for one less than the number of periods and add 1 to this tabular value.* The formula is

$$A_n \text{ (due)} = R(a_{\overline{n-1}|i} + 1)$$

Illustration: Instead of paying \$600 a year rent at the beginning of each year for the next 20 years, a renter decides to buy a house. If money is worth 5% a year, what would be the cash value equivalent to 20 years' rent?

$$R = \$600; \quad n = 20; \quad i = 5\%$$

$$A_{20} \text{ (due)} = \$600\,(a_{\overline{19}|5\%} + 1) = \$600\,(12.08532 + 1) = \$600 \times 13.08532$$
$$= \$7,851.19$$

Referring again to the section of the illustration under the line it is seen that the amount of an annuity of 5 payments, in which the payments are made at the beginning of the period, would be equivalent to an ordinary annuity of 6 payments, immediately before the sixth payment. That is, the first payment would have accumulated interest for 5 periods, the second for 4, and the last for 1, whereas in the ordinary annuity the last payment would have drawn no interest and the first would have drawn interest for only 4 periods. When this relationship is understood it is seen that the amount of such an annuity can be found by looking in the table for the amount of an annuity of *1 period more than the term of the annuity due, and then deducting the last payment from this tabular value.* Since the table is based on a payment of 1, if 1 is deducted from the tabular value, the amount of an annuity due of 1 is found. That is, in solving problems dealing with annuities due, the same tables are used as for ordinary annuities, but the amount of an annuity due is found by looking up the value of $s_{\overline{n+1}|i}$ (not $s_{\overline{n}|i}$) and deducting 1 from the tabular value. The formula for the amount of an annuity due is

$$S_n \text{ (due)} = R\,(s_{\overline{n+1}|i} - 1)$$

Illustration: A housewife subscribes to an encyclopedia which is given to her free with the provision that she is to pay \$5 at the beginning of

each 6-month period for the next 10 years for the annual supplements If money is worth 6% converted semiannually, what is the total amount of her payments?

$$R = \$5, \quad n = 20, \quad i = 3\%$$

$$S_{20}(\text{due}) = \$5\,(s_{\overline{21}|3\%} - 1) = \$5\,(28\ 6765 - 1) = \$5 \times 27\ 6765 = \$138\ 38$$

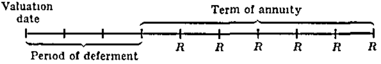
The periodic payments made for the free encyclopedia would amount to \$138 38 during the 10-year period

Deferred annuities

In an ordinary annuity the present value is found one payment interval before the first payment When the valuation date of a series of payments precedes the first payment date by more than one payment interval, the series of payments is called a *deferred annuity* The present value of such an annuity is equal to the present value of the annuity at the beginning of the annuity that is $R\ a_{\overline{n}|i}$, discounted to the valuation date

Illustration What is the value of a series of \$1,000 payments at the end of each year for 6 years if the first payment is to be received 4 years from today if money is worth 5%?

If this is to be considered an ordinary annuity, the valuation date is 3 years before the date of the annuity This can be diagrammed as shown



Value of annuity at the beginning of the annuity

$$\$1{,}000\ a_{\overline{6}|5\%} = \$1{,}000 \times 5\ 07569 = \$5{,}075\ 69$$

Since this amount, \$5 075 69, must be discounted 3 years,

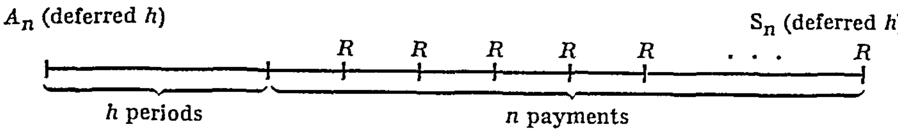
$$\$5\ 075\ 69\ (1 + 5\%)^{-3} = \$5{,}075\ 69 \times 0\ 863838 = \$4\ 384\ 57$$

If in the illustration payments had been made for 9 periods instead of 6, the present value at the beginning of the ordinary annuity would have been \$1,000 $a_{\overline{9}|5\%} = \$7{,}107\ 82$

The present value of the first 3 payments (which were never made) would have been \$1,000 $a_{\overline{3}|5\%} = \$2{,}723\ 25$ If the value of this annuity of 3 payments is deducted from the value of an annuity for the total period, the value of the deferred annuity should be obtained, that is A_6 (deferred 3) $= \$7{,}107\ 82 - \$2{,}723\ 25 = \$4{,}384\ 57$

Hence it can be seen that the formula for the present value of an annuity of n payments of R at the end of each of the n periods deferred for h periods is

$$A_n \text{ (deferred } h) = R\,(a_{\overline{n+h}|i} - a_{\overline{h}|i})$$



It should perhaps be noted that the period of deferment does not enter into the calculation of the amount of an annuity. That is

$$S_n \text{ (deferred } h) = R \cdot s_{\overline{n}|i}$$

Under certain circumstances it may be necessary to find the periodic payment of a deferred annuity. The solution of such a problem is easier if the formula used is A_n (deferred h) $= R \cdot a_{\overline{n}|i} \cdot (1 + i)^{-h}$, since division can be avoided. Thus

$$R = A_n \text{ (deferred } h) \cdot (1 + i)^h \cdot \frac{1}{a_{\overline{n}|i}}$$

Illustration: A grower makes a \$20,000 loan to purchase orange tree seedlings. He plans to pay off the loan in 10 equal annual payments from the sale of the oranges, the first payment to be made 7 years hence. If money is worth 5%, what will be his annual payments?

$$A_n \text{ (deferred } h) = \$20{,}000; \quad h = 6; \quad n = 10; \quad i = 5\%$$

$$R = \$20{,}000 \times (1 + 5\%)^6 \times \frac{1}{a_{\overline{10}|5\%}}$$

$$= \$20{,}000 \times 1.34009564 \times 0.129504575 = \$3{,}470.97$$

EXERCISE 13.8

Solve the following.

1. Find the amount of an annuity due of \$300 every 6 months for 10 years at 4% converted semiannually.

2. Under the Universal Thrift Plan \$400 is invested at the beginning of each quarter for 10 years. If money is worth 4% converted quarterly, what should be the amount of savings at the end of 10 years?

3. A corporation sets aside \$5,000 at the beginning of each year for the ultimate repayment of a mortgage. If money is worth 5%, how much is in the fund at the end of 8 years?

4. A wife is to receive $100 at the time of the death of her husband and $100 a month for 9 years 11 months What is the present worth of these payments at the time the first payment is received if money is worth 3% converted monthly?

5 In making plans to go abroad, a college professor leases his house for $300 a month for 2 years If the rent is to be paid monthly in advance, what is the equivalent cash payment if money is worth 6% converted monthly?

6 For arranging a financial transaction in which an investment banker profited handsomely, Dave Beavon was offered a fee of $100,000 He requested that in place of this fee the corporation guarantee him an annual retainer of $15,240 to be paid in equal monthly installments at the beginning of each month for 8 years If money is worth 4% converted monthly what is the present value of such future payments?

7. In evaluating an estate, there is a trust deed on which monthly payments of $100 are received at the beginning of each month The trust deed has 5 years to run If money is worth 6% converted monthly, what is the value of this income?

8. A grandfather wants to be sure that his grandson has adequate spending money when the boy enters college How much should he set aside on the boy s twelfth birthday to assure him an income of $200 a month from the time of his eighteenth birthday to his twenty second birthday? Money is worth 4% converted monthly

9. In developing some farm land, a farmer received a loan of $6,000 which he agreed to repay in 20 equal semiannual payments The first payment is to be made 5 years hence If money is worth 7% compounded semiannually, what is the size of each payment?

10. Lois Paul inherited $25,000 on her eighteenth birthday The money is to be given to her in 10 equal annual payments beginning on her twenty first birthday What is the size of each payment if the money is invested at 5%?

11 Waste land left by a strip mine in Illinois is considered an investment It is estimated that the land will produce a net income of $20,000 a year for 20 years The first income is to be received 25 years hence If such money is worth 7%, what is the value of the land?

12 A company is to be formed by 3 men for the purpose of making investments It has been agreed that all profits will be retained in the business and that 10 years hence and every 6 months thereafter for 10 years each will receive $2,000 If they are able to earn 6% compounded semiannually on the funds, what equal contribution should the 3 make now?

13. A farmer bought a tractor on February 1 with the agreement that he would make equal monthly payments of \$250 for 18 months beginning September 30. If money is worth 9% compounded monthly, what would be an equivalent cash price?

14. An underground sprinkler system was installed for a farmer, with the understanding that at the end of a year he would make the first payment of \$5,000, and monthly payments thereafter of \$500 each for 5 years. If money is worth 6% compounded monthly, what was an equivalent cash price?

15. An inventor has developed a new process on which he has been granted a patent. A company seeking to gain control of the patent estimates that it would not be profitable before 3 years. Then they anticipate a monthly income from the process of \$10,000 for 7 years. What is the value of the patent if money is worth 18% compounded monthly?

16. It is estimated that a date ranch will come into full yield in another 5 years. If it is expected to yield an income of \$50,000 a year for 25 years, at which time the land may be sold for an estimated \$1,000,000, what is the present value of the ranch if money is worth 6% compounded annually?

Summary of tabular relationships

For virtually all calculations tabular values should be available, but their absence presents no insurmountable difficulty if the relationships are understood. From the formulas at the bottom of the table the relationships between one column and the next should be readily seen. Thus, given $s = (1 + i)^n$, the compound amount of 1, all the other tabular values can be found by one or more of the fundamental arithmetic operations.

1. The amount of an annuity is $s_{\overline{n}|i} = \dfrac{(1 + i)^n - 1}{i}$.

2. The periodic payment, given the amount, is $\dfrac{1}{s_{\overline{n}|i}} = \dfrac{i}{(1 + i)^n - 1}$.

3. The present worth of 1 is $v^n = \dfrac{1}{(1 + i)^n} = (1 + i)^{-n}$.

4. The present value of an annuity is $a_{\overline{n}|i} = \dfrac{1 - v^n}{i}$.

5. The periodic payment, given the present value, is $\dfrac{1}{a_{\overline{n}|i}} = \dfrac{i}{1 - v^n}$.

REVIEW PROBLEMS

Chapters 12 and 13

1. A man deposited $720 in a mutual savings bank which pays 3% converted annually Find the amount in his account at the end of 5 years

2. Robert Brice borrowed $800 from an insurance company against the cash surrender value of his policy The rate of interest was 5% converted annually If he made no payment to the insurance company, how much would he owe at the end of 6 years?

3. An area of land is available today at $1,000 There is sufficient income from the land to pay the taxes on the property If a speculator who buys the property today is able to sell it at $1,480 25 5 years hence, what rate converted semiannually has he made on his funds?

4 A savings and loan association increased its dividend rate from 3% payable annually to $3\frac{1}{2}$% payable semiannually If payments at these rates were made on accounts totaling $10,000,000, find how much additional dividends they must pay the year after the increase

5. If $500 was deposited in a savings bank at 4% converted semiannually 9 years ago, what was the amount in the account 2 years later?

6. Suppose that 9 years ago $500 was deposited in a savings bank at 4% payable semiannually At the end of the second year the rate paid by the bank was reduced to 3% payable semiannually Find the amount when the rate was changed, and the amount in the account today

7. Five years ago Roy Olson deposited $515 in the bank, today he has $602 02 to his credit The bank has paid interest semiannually Although the rate received may not have been uniform, find the average rate of interest received converted semiannually

8. Seven years ago a man borrowed $600 from an insurance company at 5% converted semiannually Two years later he paid $150 to the insurance company Find the amount he owes the insurance company today

9. An investor has an opportunity to buy a noninterest bearing note due in 5 years If he anticipates a return on his money of 6% converted semiannually, what is the maximum amount he may pay for a $10,000 note?

10. An investor buys some nondividend paying stock for a net price of $200 He expects to sell it at a $50 profit Find the maximum time he may wait before selling it if he is to receive the equivalent of 6% converted annually on his investment

11. A house is purchased for $18,000 If the income received from the house is just equal to the amount expended for taxes and upkeep, what

is the annual rate of appreciation if the property is sold for $20,000 at the end of 5 years?

12. A $400 loan on an insurance policy at 6% converted semiannually is to be repaid at the end of 2 years 3 months. Find the amount of the debt.

13. For $150 an investor can buy a share of stock which pays quarterly dividends of $2.25; or for $500 he can buy a bond on which no interest is paid, but which will be redeemed in 12 years at $1,000. Assuming that in 12 years the share of stock can be sold for $150, which investment would furnish the greater rate of return?

14. An investor is offered a return of $2,000 in 12 years for every $1,000 invested today. If interest is converted quarterly, what rate is the investor offered?

15. During a 10-year period the population of a midwestern city increased from 22,500 to 33,500. Assuming that the increase was according to the compound interest table, find the annual rate of growth during the decade.

16. A trust officer in a bank is called upon to divide $20,000 so that, if it is invested at 3% payable annually, two children who are now 12 and 17 will each receive the same amount on their twenty-first birthdays. How much will each receive on his twenty-first birthday?

17. A is to receive an inheritance of $1,000,000 in 10 years. If money is worth 4% converted quarterly, what is the value of the legacy: (a) today; (b) in 3 years; (c) in 10 years?

18. Two debts of $4,000 each are owed. One is a $4\frac{1}{2}$% mortgage for 10 years with 5 years to run to maturity; the other is a 5-year note due in 1 year with interest at 6% converted semiannually. What payment must be made today to pay off both the note and the mortgage if money is worth 5% converted annually?

19. In borrowing money, a merchant pledged with the bank 2 paid-up endowment policies, one for $10,000 due in 2 years, and the other for $5,000 due in 4 years. The bank lent the maturity value of these policies, discounted at 6% payable annually. How much did the bank lend?

20. The principal on a $1,000 bond is due in 30 years. If money is worth 5% converted semiannually, what is the value of the principal today?

21. A manufacturer estimates a demand for 1,000,000 units of an item which he has just patented. The first year he plans to manufacture and market 100,000. In order to exploit his patent to the full extent, he plans to expand at a uniform rate until at the end of the fifth year his annual production is 1,000,000 items. What rate of expansion per year is necessary?

22. A deposit of $100 a month invested at 2% converted monthly will amount to how much at the end of 5 years?

23. A man invests $500 in a savings and loan at the end of each half year for 10 years The savings and loan pays a dividend of 3% payable semiannually Find the total of the account at the end of the tenth year

24. In selling 2 houses a builder received a $2,500 note for 5 years at 7%, and a 3 year 6% note for $3,000 An investor offers to buy these notes at a price which will give him a return of 8% payable semiannually on his investment He agrees to give the builder $4,000 now and the balance at the end of 2 years How much should the second payment be?

25. In organizing his business a man borrowed a sum of money, agreeing to pay $5,000 in 5 years and $7,500 in 3 years Neither note is interest bearing When can he equitably pay these 2 debts for $12,500 if money is worth 6% converted semiannually?

26. After the death of a wealthy man it was found that he had borrowed the following from a single lender, giving a pledge of some valuable land as security $5,000 at 6%, $10,000 on a noninterest-bearing note, and $4,000 at 8% converted semiannually In order to give a clear title to the land it is necessary to discharge these debts now If the courts hold that 6% is the rate at which the obligations should be discharged, what single payment should be made to the creditor if the $5,000 note has 5 years to run to maturity, the $10,000 note has 3 years to run, and the $4,000 note has 6 years to run

27. A $10,000 savings bond which pays no annual interest and which may not be redeemed before maturity is offered by a probate court 3 years before maturity Three potential buyers expecting returns of 6%, 5%, and 4%, respectively, bid for the bond What should each offer?

28. Find the present value of $2,500 due in 5 years without interest if money is worth 5% converted semiannually

29. Find the present value of $5,000 due in 3 years 3 months if money is worth 4% converted quarterly

30. A debt of $15,000 is due in 2 years If the borrower finds that he can borrow money at 2% converted quarterly, how much should he be willing to pay to settle the debt now?

31. An investment certificate is offered for sale at $800 with the understanding that at the end of 10 years it will have a value of $1,000 What is the effective rate?

32. A savings bond is offered for sale for $750 with the understanding that it will be redeemed for $1,000 at the end of 9 years What is the effective rate?

33. Periodic deposits of \$10 a month at 6% converted monthly will furnish what sum of money immediately after the last payment at the end of 10 years?

34. To provide for the ultimate repayment of a debt, a corporation deposits \$600 with a trustee at the end of each month. At the end of the third year, how much does the corporation have with the trustee if the money has earned at the rate of 4% converted monthly?

35. The owner of a parking lot leased the lot on a 10-year basis at 50% of total sales. It is estimated that at present rates and use the lot will have an average income of \$80,000. If such income is discounted at 16% converted quarterly, what is the value of the lease 5 years before its expiration?

36. Richard Ackerman was injured in an industrial accident. The insurance company offers to pay him compensation of \$100 a month for 2 years, but he prefers a cash settlement. If money is worth 6% to him payable monthly, what cash settlement should he be willing to accept?

37. Assuming that money is worth 4%, what cash amount today is equivalent to payments of \$200 to be received at the end of each year for the next 5 years?

38. If money is worth 4%, what will be the value 5 years hence of \$200 deposited at the end of each year for 5 years?

39. A father promised his son on his twelfth birthday that he would give him on his twenty-first birthday a penny for every minute from the time he is 12 until he is 21. If money is worth 4% converted semiannually, how much should the father set aside on the twelfth birthday to assure the payment on the twenty-first? (Assume each year is 365 days.)

40. The sum of \$890.34 is deposited at 4% converted annually. If \$200 is withdrawn at the end of each year for 5 years, what is the balance of the account?

41. To settle the balance of \$1,800 due on an automobile, the buyer agrees to pay \$125 a month. If money is worth 6% converted monthly, how many full payments are to be made? What should be the size of the last payment if no partial payment is to be made?

42. A lot is sold for \$9,000 with a down payment equal to $\frac{1}{3}$ the purchase price. The balance is to be paid off in monthly payments of \$200 each, with interest at 9% converted monthly. How many full payments should be made? What is the size of the last payment if no partial payment is to be made?

43. What is the present worth of a 25-year lease which pays \$100 a month, if money is worth 6% converted monthly?

44. In the sale of a business Samuel Clements was to receive $10,000 at the end of each year for 6 years If money is worth 4%, what is the equivalent cash value of these payments?

45. How much should be deposited today at 3% converted annually to furnish 4 consecutive annual tuition payments of $400 each, the first payable 10 years from today?

46. Interest on a $1,000 bond will be paid at the rate of $15 every 6 months for the next 30 years If money is worth 5% payable semiannually, what is the present value of these payments?

47. In order to create a fund of $50,000 at the end of 15 years, what equal semiannual payments should be made at the end of each 6 months to a savings and loan association which pays 3% converted semiannually?

48. Which has the greater value today, an annuity of $1,200 payable at the end of each year for 10 years at 3% converted annually, or payments of $100 at the end of each month for 10 years at 3% converted monthly?

49. Compare the amount of an annuity of $1,200 payable at the end of each year for 5 years at 3% converted annually, with the amount of an annuity of $100 payable at the end of each month for 5 years at 3% converted monthly

50. Find the payment necessary at the end of each month to accumulate $20,000 in 5 years if money is worth 4% converted monthly

51. How much insurance should a man carry if he plans to leave a sum sufficient to pay $2,400 at the beginning of each year for the next 5 years, $1,800 at the beginning of each year for the next 10 years, and $1,000 for the next 20 years, if money is worth 3% converted annually?

52. A special assessment of $100 may be paid over the next 6 years If interest is 5%, what is the annual payment?

53. If a special assessment of $100 is being paid over a 6-year period with interest at 5%, what is the amount of principal outstanding at the beginning of the fifth year?

54. A debt of $50,000, bearing interest at 6%, payable semiannually, is to be amortized by payments at the end of each half year for the next 10 years Find the periodic payment, and the amount of principal outstanding just after the fifth payment

55. A man whose annual income is $6,000 plans to amortize a $15,000 mortgage by semiannual payments over the next 5 years Interest is 4% converted semiannually Construct an amortization schedule

56. A debt of $12,000 is to be amortized by payments at the beginning of each month for the next 5 years If the interest rate is 4% converted monthly, find the periodic payment and the amount owed on the debt immediately after the twenty-fourth payment has been made

57. A corporation plans to retire a bond issue of $5,000,000 in 10 years. How much should the corporation set aside each quarter in order to have a sum sufficient to pay off the bonds when they fall due, if it can earn 3% converted quarterly on the amount in the fund?

58. A corporation plans to issue $10,000,000 in $3\frac{1}{2}$% bonds due in 20 years to be discharged by a sinking fund. The corporation is to make semiannual contributions to the fund, which will be invested at 3% converted semiannually. If bond interest is payable semiannually, find the total periodic charge. Find the amount in the sinking fund immediately after the tenth payment.

59. A debt of $12,000,000 due in 4 years is to be paid by the sinking fund method. Assuming that the fund draws interest at $2\frac{1}{2}$%, construct a sinking fund schedule.

60. A corporation sells $100,000 worth of 6% bonds. The company is required to provide for the retirement of the bonds within 10 years by creating a sinking fund. If the sinking fund is to be invested at 5%, find the book value of the debt at the end of the fifth year.

61. A loan of $6,000 at 4% payable quarterly is to be paid off over 15 years in equal quarterly installments. Find the quarterly installment. How much of the loan of $6,000 would be paid off after 12 years?

62. A corporation has borrowed $600,000 at 4%. It plans to reduce the debt by equal payments until the debt is reduced to $150,000 at the end of 5 years. Find the amount of each annual payment.

63. In financing a $90,000 expansion program, a small corporation borrowed at 6% converted monthly. It plans to repay this amount at the rate of $1,500 a month for 5 years, and the balance at $1,000 a month. When will the debt be repaid?

64. A gravel pit yields $5,000 a year. When the gravel is exhausted in 5 years, the pit will be worthless. If money is worth 5%, what is the value of the pit?

65. An investor has $100,000 to invest. He has a choice of investing it in 30-year government bonds at $2\frac{1}{2}$% interest payable semiannually, with a return of $100,000 at the end of the period, or of investing it in property which will return $3,000 every 6 months and be worthless at the end of 30 years. Which has the greater present value if money is worth 3% converted semiannually?

14

The Application of Annuity Principles

Bonds

Small amounts can usually be borrowed from one lender, but exceedingly large amounts must usually come from many lenders Most successful investors believe that there is less risk if their investments are spread among several companies rather than concentrated with any one company Therefore when a corporation, governmental, eleemosynary, or business wants to borrow a relatively large amount for a long period, it usually seeks the funds from many different lenders

To facilitate such lending operations the borrower issues certificates known as *bonds* which state the terms under which it borrows money No doubt you are familiar with one type of bond, Series E bonds issued by the federal government and known as Savings Bonds This is a special type of bond which bears little if any similarity to a typical corporate bond Other bonds issued by the federal government, however, follow more conventional patterns

Although bonds differ in type, most outstanding bonds conform to a well defined pattern The discussion which follows is limited to existing types of bonds with particular attention to the most common type, the *standard bond pattern* One common characteristic is that each certificate indicates a specified date, called the *maturity date*, on which the corporation will pay to the holder of the bond a designated amount of money known as the *redemption value* of the bond Ordinarily the redemption value on the maturity date equals the *par*, or the *face* value, which is the amount stated on the face of the certificate Par value is generally $1,000 (or multiples of $1,000) since most investors find $1,000 a desirable minimum in which to deal Although occasionally bonds with a face

value of \$100 or \$500 are issued, they cause unnecessary work for the issuer and carry no offsetting advantages.

The coupon rate

Another common characteristic of standard-type bonds is that the bond certificates state the rate of interest to be paid and the frequency of payments. In the standard bond pattern, interest is paid semiannually on dates specifically stated in the bond. While the number of payments made annually is standard, the total number of payments varies widely. It is not expected that the rate of interest, usually referred to as the *coupon rate*, will change or be modified during the period of the bond. While every bond contains a coupon rate, the rate is not uniform among bonds. Bonds are spoken of as 4% bonds, $3\frac{1}{2}$% bonds, 6% bonds, and so on, according to the stated rate, even though the bond interest paid twice each year is equal to only half of the stated rate.

To facilitate the collection of interest, many bonds include coupons, which are drawn in an amount equal to semiannual interest payments. Each coupon is dated with the day on which the interest payment it represents falls due. The holder of the bond simply detaches the coupon on the indicated date and deposits it in his bank for collection much as he would a check. But like a postdated check, he does not present it until the due date. Bonds which bear such coupons are known as *coupon bonds*. It is up to the holder of the bond to submit the coupon in order to collect his interest. Many bonds make it possible for the owner to register the bonds in his name with the issuing corporation. The corporation then mails a check on each interest payment date for the amount of interest to which the bondholder is entitled. When the bond is sold the new owner has it registered in his name in order to receive the future interest payments.

Since many investors are interested in bonds, leading newspapers carry daily price quotations. Available at all brokerage houses and at most banks and public libraries are small books, issued monthly, known as *Bond Guides*. Usually only one line across the book is devoted to each bond. Thus a bond issue may be listed as

Southern Counties Gas Company 3's'77 Ms.

This means that the bonds were issued by the Southern Counties Gas Company, that the interest rate is 3%, that interest is paid semiannually in March and September. The capital M indicates that the bond was issued in March instead of in September, which is represented by a small letter. The maturity date is March, 1977.

The yield rate

Once a corporation has sold bonds, it is under no obligation to redeem them that is to return the principal, until the maturity date The original purchaser on the other hand, might prefer to have cash again rather than to hold the bond to maturity Since in the standard bond contract the *corporation is under no obligation to redeem* the bond early the holder of the bond can get his money before maturity only by selling the bond to some other investor When bonds are issued it is usually expected that they will be freely bought and sold among investors Such bonds are called *marketable bonds* to distinguish them from certain types of government bonds which are nontransferable, such as the Savings Bonds

It is to be expected that anything which can be bought and sold will be subject to some degree of price fluctuation Bonds are no exception A bond bought at a price equivalent to its face amount is said to be *bought at par* A bond bought for more than its face value is said to be bought at a *premium*, and a bond bought for less than par is said to be bought at a *discount*

When a bond is bought at par, the rate of return received by the investor on his investment, known as the *yield* or *yield rate*, is equal to the coupon rate When an investor buys a bond at a price above the redemption price, the yield rate is less than the coupon rate It can easily be seen, however, why an investor might pay more than the face value for a bond

If a 6% bond is bought for $1,000 one year before maturity, the buyer expects to receive $60 in interest, and on the maturity date to be repaid the principal of $1,000 Under such conditions the investor at the end of the year has $1,060, or $60 more than he had at the beginning If conditions in the bond market are such that companies issuing new bonds are able to sell as many as they wish by paying only 3% interest, there is no reason for them to pay more The investor with money to invest has a choice of buying bonds currently being issued at 3% or buying bonds issued in the past at different rates of interest It is not difficult to understand why under such circumstances an investor would be willing to pay, say, $1,020 for a 6% bond 1 year before its maturity On this investment he would receive $60 interest, and at maturity he would receive the face amount of the bond, $1,000 Thus at the end of the year he would receive $1,060, or $40 more than he originally had invested in the bond—$1,020 Thus on the investment of $1,020 he would receive income of $40, a higher return ($40 \div \$1,020 = 3.92\%$) than he would receive on a 3% bond bought at par.

Similarly if new bonds are being sold to pay 6% interest, an investor who had a 3% bond should not expect to sell it at par value one year before maturity. The buyer of such a bond would expect to buy it at a price sufficiently below par to give him a return of 6% on the amount invested.

In financial mathematics two principal types of problems arise relative to bonds. One is to determine the actual yield which an investor receives on a bond which he has bought at a premium or discount. The other is to determine what price to pay for a bond under given conditions to obtain a desired yield.

The value of a bond to be redeemed at par

To determine the present value of a bond, an investor must consider both the present value of the periodic interest payments and the present value of the principal which will be received on the maturity date. If the interest is payable semiannually at the rate of 3% a year, the buyer of a $1,000 bond will receive $15 every six months until the maturity of the bond. At the maturity date, the holder expects to receive the face value, or redemption price, of the bond. The value of a bond is therefore equal to the sum of the present value of the redemption price and the present value of an annuity made up of the future interest payments, both evaluated at the desired yield rate.

Illustration: A $1,000, 3% bond with interest payable semiannually matures in exactly 5 years. Find the value of the bond to yield 4% payable semiannually.

The present value of the principal is

$$\$1{,}000\,(1 + 2\%)^{-10} = \$1{,}000 \times 0.8203483 = \$820.35$$

The present value of the interest income is

$$A_{10} = \$15 \times a_{\overline{10}|2\%} = \$15 \times 8.9826 = \$134.74$$

The present value of the bond is $820.35 + $134.74 = $955.09.

The following symbols are often used to express these relationships in a formula:

F = the face value of the bond, usually $1,000.

V = the value of the bond which will furnish the desired yield to maturity. The value of the bond and the price of the bond are not necessarily the same, since the practice in the bond market is to quote the price of the bond as a percentage of the face amount, and to show mini-

mum variations in price of $\frac{1}{8}\%$, equivalent to \$1 25 on a \$1,000 bond Thus a bond whose value is anything between \$950 625 and \$951 875 is priced at $95\frac{1}{8}$

r = *the coupon* rate per period

R = the amount of each periodic interest payment If the bond is a coupon bond this is the value of each coupon, an amount equal to Fr

i = the yield rate per period on the present value

n = the number of interest periods to maturity If the bond is a coupon bond, n is the number of coupons still attached to the bond

The present value of the principal could be expressed as

$$F(1 + i)^{-n}$$

The present value of the income R could be expressed as

$$Ra_{\overline{n}|i}$$

The sum of the two is the present value of the bond to furnish the desired yield,

$$V = F(1 + i)^{-n} + Ra_{\overline{n}|i}$$

To find the value of a bond using this formula it is necessary to obtain values from both the Present Worth of 1 table and the Present Worth of an Annuity table The formula can be expressed as the sum of the present value of the face amount and the value of an annuity of the difference *between the coupon value and the yield rate on the maturity value*

From the previous chapter it is known that

$$a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i}$$

Solving this for $(1 + i)^{-n}$ gives

$$(1 + i)^{-n} = 1 - i \times a_{\overline{n}|i}$$

Substituting this value for $(1 + i)^{-n}$ in the preceding formula,

$$V = F(1 + i)^{-n} + Ra_{\overline{n}|i}, \text{ gives}$$
$$V = F(1 - i \times a_{\overline{n}|i}) + Ra_{\overline{n}|i} = F - Fi \times a_{\overline{n}|i} + Ra_{\overline{n}|i}$$
$$= F + (R - Fi)\,a_{\overline{n}|i} \quad \text{or} \quad V = F + F(r - i)\,a_{\overline{n}|i}$$

since $R = Fr$

This formula may be used more advantageously than the preceding formula, since it requires finding only one tabular value instead of two

Illustration Find the price of a \$1,000, 3% bond with interest payable semiannually, which matures in exactly 5 years, if bought to yield 4% payable semiannually

Here $F = \$1{,}000$; $R = \$15$, since $r = 1\frac{1}{2}\%$; $i = 2\%$; $n = 10$; and $Fi = \$20$. Substituting these values in the formula $V = F + F(r - i)\, a_{\overline{n}|i}$ gives

$$V = \$1{,}000 + (\$15 - \$20)\, a_{\overline{10}|2\%} = \$1{,}000 - \$5 \times 8.9826$$
$$= \$1{,}000 - \$44.91 = \$955.09$$

the value of the bond. (The quoted price of the bond would be $95\frac{1}{2}$.)

Premium and discount

The difference between the par value of the bond and the value of the bond is either a premium or a discount. The amount of the premium or discount can easily be determined by finding the difference between F and V. If F is deducted from both sides in the formula, $V = F + F(r - i)a_{\overline{n}|i}$, the resulting formula is $V - F = F(r - i)\, a_{\overline{n}|i}$.

If the bond sells for a yield rate greater than the coupon rate, the difference between the value and the face of the bond will be a negative value. Rather than express this as a negative premium, it is shown as a positive value and called *discount*. If the yield is less than the coupon rate, the value is positive and is called a *premium*.

Accounting for bond discount

In an earlier illustration it was assumed that a 3% bond had been bought 5 years before maturity at \$955.09 to yield 4%. The periodic income of \$15 received every 6 months does not alone represent the annual income arising from the bond; as the date approaches at which the entire principal is to be repaid, the value of the bond rises to par. It may be desirable to think of a bond as having a book value which increases periodically until at maturity it equals the redemption value. If a bond is purchased for less than the redemption value, the actual income received over the life of a bond is greater than the interest received. The increase each year in book value, along with the interest received, can be considered income. The problem of determining how much income is received each year as a result of the discount can be readily solved, if one considers the difference between the redemption price and the purchase price (this difference is the discount) as the *amount of an annuity*. The periodic payment necessary with each interest payment to accumulate by the maturity date a fictitious fund equal to the discount, can readily be considered as the periodic income received on the bond in addition to the interest received.

Given a periodic payment, and considering it as an annuity which is creating a "discount fund," one can at any time determine the amount in

the fund as the *amount of an annuity*, or one can construct a schedule to show the periodic increase in the book value of the bond

Illustration A \$1,000, $3\frac{1}{2}\%$ bond due in 4 years is bought to yield 4% to maturity If interest is paid semiannually, find the price of the bond, and set up a schedule for the accumulation of the discount Without using the schedule, find the book value immediately after the third semiannual payment of interest has been received

Here $r = 1\frac{3}{4}\%$, $i = 2\%$, and $n = 8$ The discount on the bond is

$F(r - i)\, a_{\overline{n}|i} = \$1{,}000\,(2\% - 1\tfrac{3}{4}\%)\, a_{\overline{8}|2\%} = \$2\ 50 \times 7\ 325 = \$18\ 31$

The value of the bond is $V = \$1{,}000\ 00 - 18\ 31 = \$981\ 69$

If there are to be 8 payments of interest, the discount should be considered as accumulated in the same number of payments and at the same per cent, that is, 2% per period The periodic accumulation to the fund would be

$$\$18\ 31 \times \frac{1}{s_{\overline{8}|2\%}} = \$18\ 31 \times 0\ 1165 = \$2\ 13$$

A schedule showing the book value of the bond, the periodic accumulation of discount, and the interest income can be constructed as follows

Schedule Showing the Accumulation of Bond Discount

At End of Period	*Interest Received*	*Accumulation Payment*	*2% Interest on Accumulation Fund*	*Increase in Fund*	*Size of Fund*	*Book Value of Bond*
0	——	——	——	——	——	\$981 69
1	\$17 50	\$ 2 13	——	\$ 2 13	\$ 2 13	983 82
2	17 50	2 13	\$0 04	2 17	4 30	985 99
3	17 50	2 13	0 09	2 22	6 52	988 21
4	17 50	2 13	0 13	2 26	8 78	990 47
5	17 50	2 13	0 18	2 31	11 09	992 78
6	17 50	2 13	0 22	2 35	13 44	995 13
7	17 50	2 13	0 27	2 40	15 84	997 53
8	17 50	2 13	0 32	2 45	18 29	999 98
Total		\$17 04	\$1 25	\$18 29		

Like any other annuity, the fund will be augmented by the interest received on the amount already in the fund, plus the periodic addition, in this case, \$2 13 Thus in the second period, the fund is increased by a

payment of \$2.13 and interest of \$0.04; in the third period, by a payment of \$2.13 and interest of \$0.09. Adding the amount in the accumulation fund to the original book value gives a book value of \$988.21 just after the third payment.

The amount in the fund just after the third payment can also be found without the schedule by using the formula for finding the amount of an annuity. Here $R = \$2.13$, $i = 2\%$, $n = 3$.

$$S_3 = \$2.13 \times s_{\overline{3}|2\%} = \$2.13 \times 3.0604 = \$6.52$$

The increase in the accumulation fund can be found without the use of an annuity table. In the preceding illustration, the bond was bought at a price of \$981.69 to yield 4% payable semiannually. The buyer therefore expects a periodic income equal to 2% on the purchase price, or \$19.63 (\$981.69 × 2%). Since the periodic interest received is only \$17.50, the income of \$2.13 (\$19.63 − \$17.50) can be considered as delayed until the maturity of the bond, and hence it goes to increase the book value of the bond to \$983.82 (\$981.69 + \$2.13). The next period he will expect an income of 2% (\$19.67) on the book value of \$983.82; but again he receives only \$17.50, and the book value is further increased to \$985.99 (\$983.82 + \$2.17). A schedule of the increase in the fund and of the book value is given herewith. It will be observed that it differs

Schedule of Increases in Book Value of Bond Bought at a Discount

At the End of Period	*Investment Rate Times Book Value*	*Interest Received*	*Amount Added to Book Value*	*Book Value*
0	——	——	——	\$ 981.69
1	\$ 19.63	\$ 17.50	\$ 2.13	983.82
2	19.68	17.50	2.18	986.00
3	19.72	17.50	2.22	988.22
4	19.76	17.50	2.26	990.48
5	19.81	17.50	2.31	992.79
6	19.86	17.50	2.36	995.15
7	19.90	17.50	2.40	997.55
8	19.95	17.50	2.45	1,000.00
Total	\$158.31	\$140.00	\$18.31	

slightly from the preceding schedule. Since the accumulation is calculated accurately only in cents, neither schedule is exactly accurate. In each

case the last payment may be a few cents larger or smaller to compensate for the cumulative errors For accounting purposes, however, either method is completely satisfactory

Accounting for bond premium

If the coupon rate is above the yield rate, a bond is bought at a price *above its redemption value* A bond bought at a *premium is considered* an asset by the person or company acquiring it, but it is an asset which will decrease in value until the maturity date This fact is true because on that date its value is equal to the redemption price

Since it is necessary over the life of the bond to create a fund to amortize, or write off the premium paid, the actual income from the investment is less than the periodic income received as interest payments The investor should consider the periodic income as the difference between the interest received and the payment necessary to create a fund which at the maturity of the bond would amount to the premium paid Thus the periodic income received should be considered as made up of two parts (1) the income from interest, and (2) the payment necessary to create a fund to amortize, or write off the premium paid over the life of the bond The premium paid may be considered as the *amount of an annuity* made up of as many payments as there are interest payments yet to be received By finding the periodic payments of such an annuity, it is easy to find how much should be considered as making up the amortization payments

An amortization schedule can be constructed to show the accumulation and the book value of the bond Once the periodic payment is known the amount in the amortization fund at any given time can be found as the amount of an annuity of an equal number of payments for the period covered It is not necessary to construct a schedule

Illustration Four years before maturity, a \$1 000, $4\frac{1}{2}\%$ bond with interest payable semiannually is bought to yield 4% payable semiannually. Find the value of the bond and construct an amortization schedule

Here $r = 2\frac{1}{4}\%$, $i = 2\%$, $n = 8$ The premium on the bond is

$$F(r - i)\,a_{\overline{n}|i} = \$1{,}000\,(2\tfrac{1}{4}\% - 2\%)\,a_{\overline{8}|2\%} = \$2\ 50 \times 7\ 325 = \$18\ 31$$

The price of the bond is $V = \$1{,}000\ 00 + 18\ 31 = \$1{,}018\ 31$ The periodic amortization is $\$18\ 31 \times \dfrac{1}{s_{\overline{8}|2\%}} = \$2\ 13$

The schedule for the amortization of a bond premium is constructed in much the same way as the schedule for the accumulation of bond discount. The amount in the fictitious fund draws interest like any other annuity, and the next period it is augmented not only by the periodic payment but also by the interest on the amount already in the fund. To find the amount in the fund at any time without constructing a schedule, it is necessary only to find the *amount of an annuity* made up of the periodic payments at the given rate for the number of periods. This amount deducted from the original value gives the book value immediately following the interest payment date.

Schedule Showing the Amortization of Bond Premium

At End of Period	*Interest Received*	*Amortization Payment*	*2% Interest on Amortization Fund*	*Increase in Fund*	*Size of Fund*	*Book Value of Bond*
0	——	——	——	——	——	$1,018.31
1	$22.50	$ 2.13	——	$ 2.13	$ 2.13	1,016.18
2	22.50	2.13	$0.04	2.17	4.30	1,014.01
3	22.50	2.13	0.09	2.22	6.52	1,011.79
4	22.50	2.13	0.13	2.26	8.78	1,009.53
5	22.50	2.13	0.18	2.31	11.09	1,007.22
6	22.50	2.13	0.22	2.35	13.44	1,004.87
7	22.50	2.13	0.27	2.40	15.84	1,002.47
8	22.50	2.13	0.32	2.45	18.29	1,000.02
Total		$17.04	$1.25	$18.29		

The amortization of a bond premium can be determined from a schedule without using an annuity table. If a bond is bought to furnish a yield of less than the coupon rate, the difference between the actual interest received and the amount expected as a return on the investment can be considered as going to reduce the book value. In the next period, since the book value is lower, having been reduced the preceding period, less can be expected on the amount invested, though the interest payment received is uniform. The reduction in book value therefore is slightly greater than it was in the preceding period. The accompanying schedule is prepared by this method.

Schedule for the Reduction in the Book Value of a Bond Bought at a Premium

At the End of Period	*Investment Rate Times Book Value*	*Interest Received*	*Amount Subtracted from Book Value*	*Book Value*
0	——	——	——	\$1,018 31
1	\$ 20 37	\$ 22 50	\$ 2 13	1,016 18
2	20 32	22 50	2 18	1,014 00
3	20 28	22 50	2 22	1,011 78
4	20 24	22 50	2 26	1,009 52
5	20 19	22 50	2 31	1,007 21
6	20 14	22 50	2 36	1,004 85
7	20 10	22 50	2 40	1,002 45
8	20 05	22 50	2 45	1,000 00
Total	\$161 69	\$180 00	\$18 31	

EXERCISE 14.1

Find the premium or discount on each of the following \$1,000 bonds Interest is payable semiannually

	Coupon Rate	*Yield Rate*	*Years to Maturity*		*Coupon Rate*	*Yield Rate*	*Years to Maturity*
1.	4%	5%	23	6.	$2\frac{1}{4}$%	$2\frac{1}{2}$%	6
2.	3%	$2\frac{1}{2}$%	10	7.	$1\frac{7}{8}$%	$1\frac{1}{2}$%	$3\frac{1}{2}$
3.	5%	4%	18	8.	$2\frac{7}{8}$%	2%	$1\frac{1}{2}$
4	3%	$3\frac{1}{2}$%	7	9.	$3\frac{3}{8}$%	4%	2
5.	$2\frac{1}{2}$%	2%	12	10.	2%	3%	3

11. A \$1,000, 4% bond with interest payable semiannually matures in $4\frac{1}{2}$ years Find the value of the bond to yield 5% payable semiannually

12. Interest on a \$1,000, 6% bond is payable semiannually It is exactly 8 years to maturity Find the value of the bond to yield 5% payable semiannually

13. Interest on a \$1,000 bond in the amount of \$50 is paid once a year Find the price of the bond to yield 4% exactly 10 years before maturity

14. Interest on a \$1,000 bond in the amount of \$50 is paid annually Find the price of the bond to yield 6% exactly 10 years before maturity

15. Find the value of a \$1,000, 3% bond with interest payable semiannually 10 years before maturity if bought to yield (a) 3%, (b) 4%, (c) 2% payable semiannually

16. A \$1,000, 3% bond due in 3 years is bought to yield 4% to maturity. If interest is paid semiannually, find the value of the bond and construct a schedule for the accumulation of the discount.

17. The interest on a $4\frac{1}{2}\%$ bond for \$1,000, due in 12 years, is paid semiannually. If the bond is bought to yield 5% payable semiannually, find the book value of the bond immediately after the sixth interest payment has been received.

18. Semiannual interest payments of \$12.50 are received on a \$1,000 bond. Find the value of this bond to yield 3%, exactly $3\frac{1}{2}$ years before maturity. Without using an annuity table, construct a schedule showing the increase in the book value of the bond.

19. Interest is received every 6 months on a \$1,000, $2\frac{1}{2}\%$ bond. If bought to yield 4% payable semiannually, find the value of the bond exactly 5 years before maturity. Without the use of an annuity table, find the book value of the bond immediately after the third interest payment has been received.

20. Using the annuity tables, find the book value of the bond in Problem 19 immediately after the fifth interest payment is received.

21. Find the value of a \$1,000 bond due in 7 years paying $3\frac{1}{4}\%$ annually if bought to yield 3%. Construct an amortization schedule using annuity tables.

22. Nine years before maturity a \$1,000, $3\frac{1}{2}\%$ bond with interest payable semiannually is bought to yield 3% payable semiannually. Find the value of the bond, and without using an annuity table construct an amortization schedule.

23. Find the value 8 years before maturity of a \$1,000, $3\frac{1}{2}\%$ bond with interest payable semiannually priced to yield 3% payable semiannually. Using the annuity tables, find the book value of the bond immediately after the tenth interest payment has been received.

24. Determine the value of a \$10,000 bond paying 4% payable semiannually if bought $5\frac{1}{2}$ years before maturity to yield $3\frac{1}{2}\%$ payable semiannually. Find the book value of the bond after the fifth interest payment.

25. Find the value 12 years before maturity of a \$1,000, 4% bond with interest payable annually priced to yield 3%. Find the book value of the bond just after the sixth interest payment has been received.

Bond tables and their uses

Before an accurate amortization schedule of bond premium or discount can be set up, the investor must know the yield he is to receive on his bond. Yet as a bond buyer he may be forced in many instances to buy at the market price or not at all. To be sure of obtaining a satisfactory

rate of return, he must be able to compare yields on different bonds with different coupon rates and different maturities at varying prices To make a satisfactory comparison between the yields on various bonds he must know the yield rates Thus the determination of the yield is one of his most important mathematical problems

Because many persons are interested in bonds and the yields on bonds tables called *bond tables* have been constructed They are sold at many bookshops, or they can be consulted at many libraries, brokerage houses, and banks Bond tables are essential for anyone who deals extensively in bonds There are many types available, from small pocket editions to tables so large and complete that they show values accurately to cents for $1,000,000 at yields varying by months from one to 40 years and rates varying by $\frac{1}{8}\%$ The Financial Publishing Company, which publishes the most complete line of bond tables in the world, has one called *Monthly Bond Values* which contains 1,156 pages Such tables are necessary for dealers and investors who frequently buy and sell government bonds

For many purposes a table which shows variations in prices at half-year intervals correct to 2 decimal places is satisfactory Two pages from such a table are reproduced on pp 408–409 These pages show the yield if held to maturity for a 3% bond maturing in from 5 years to 8 years The value of the bond is shown under the separate columns which show years to maturity, and the yield is shown in the column along the left-hand margin of the page The table may be used to find the yield if the value is known, or the value if the yield is known Interpolation may be made if exact values or the exact yield are not found in the table Rather than make frequent interpolations, however, it is wise to acquire a more comprehensive bond table

Since it is necessary to know the yield on a bond in order to account properly for bond premium or discount, the average accountant who deals with clients holding bonds should have ready access to a bond table Once the yield rate is known, the future book value of the bond can be determined as previously explained

Callable bonds

Thus far in the discussion of bond values it has been assumed that the bond was valued on an interest payment date, and that it was to be redeemed on maturity at par Both of these conditions are not always met

When corporations issue bonds they usually include a provision permitting them to pay the bond before maturity if they so desire Since

the prepayment is completely at the option of the corporation, an investor ordinarily does not agree to having his principal returned early unless he is in some small measure compensated for the additional trouble of finding another outlet for his funds. The compensation usually takes the form of additional income if the bond is called early by providing for a series of *call prices*—that is, the redemption value before maturity—at a small premium above par value. For example, a 20-year bond may be callable at 105—105% of face value—the first 5 years of its life, at 103 the next 5 years, and at 101 thereafter, until maturity when it may be paid at 100.

Bonds issued by the government may have two dates such as $3\frac{1}{4}$%'s 81–78. The later date shows the date the bonds mature, in this case, 1981; the earlier date indicates the earliest year in which they may be called, in this case, 1978. In most instances they may be called at par any time after the first call date. Since the yield is generally increased if a bond is called at a premium before maturity, the yield on such bonds is usually computed to the maturity date. When a bond may be called at par before maturity, the rule is to compute the lower of the two yields, that is, the yield to the earliest call date or the yield to maturity. Thus if a bond has been bought at a premium, the yield would be computed to the nearest call date, since if the bond is paid off at par on that date the premium would have to be written off more rapidly than if the bond is not paid until maturity. On the other hand, if the bond has been bought at a discount the yield would be computed to the maturity date, since calling the bond at an earlier date would result in a higher rather than a lower yield. The conservative approach is always to assume the lower yield.

The investor has no way of knowing that a corporate bond is to be called at a premium before maturity until the corporation announces that such is to be the case. He does know what the contractual provisions are under which the corporation may call the bonds. Thus the yield on corporate bonds is almost always figured on a redemption value equal to par value. If the price of the bond is at or near the call price, the cautious buyer may hesitate to assume a position in which if the bond is called in the near future his yield will be low or even negative. Under such circumstances he may in arriving at an investment decision compute yields under varying assumptions.

If corporate bonds are called after they have been acquired by an investor, it may easily turn out that the yield he actually has received on the bond has differed materially from the yield he anticipated. The principal point to remember, however, is that a bond is valued,

3%

Mat Yield	5 YRS	5½ YRS	6 YRS	6½ YRS	7 YRS	7½ YRS	8 YRS
.50	112.33	113.55	114.76	115.97	117.18	118.38	119.58
.75	111.02	112.10	113.18	114.25	115.32	116.38	117.44
.80	110.76	111.81	112.86	113.91	114.95	115.98	117.02
.85	110.50	111.53	112.55	113.57	114.58	115.59	116.59
.90	110.24	111.24	112.24	113.23	114.22	115.20	116.17
.95	109.99	110.96	111.93	112.89	113.85	114.81	115.76
1.00	109.73	110.68	111.62	112.56	113.49	114.42	115.34
1.05	109.47	110.39	111.31	112.22	113.13	114.03	114.93
1.10	109.22	110.11	111.00	111.89	112.77	113.64	114.51
1.15	108.96	109.83	110.70	111.55	112.41	113.26	114.10
1.20	108.71	109.55	110.39	111.22	112.05	112.87	113.69
1.25	108.46	109.27	110.09	110.89	111.69	112.49	113.28
1.30	108.20	109.00	109.78	110.56	111.34	112.11	112.88
1.35	107.95	108.72	109.48	110.23	110.99	111.73	112.47
1.40	107.70	108.44	109.18	109.91	110.63	111.35	112.07
1.45	107.45	108.17	108.88	109.58	110.28	110.98	111.67
1.50	107.20	107.89	108.58	109.26	109.93	110.60	111.27
1.55	106.95	.07.62	108.28	108.93	109.58	110.23	110.87
1.60	106.70	107.34	107.98	108.61	109.24	109.86	110.47
1.65	106.45	107.07	107.68	108.29	108.89	109.49	110.08
1.70	106.21	106.80	107.39	107.97	108.55	109.12	109.69
1.75	105.96	106.53	107.09	107.65	108.20	108.75	109.29
1.80	105.71	106.26	106.80	107.33	107.86	108.38	108.90
1.85	105.47	105.99	106.50	107.01	107.52	108.02	108.52
1.90	105.22	105.72	106.21	106.70	107.18	107.66	108.13
1.95	104.98	105.45	105.92	106.38	106.84	107.29	107.74
2.00	104.74	105.18	105.63	106.07	106.50	106.93	107.36
2.05	104.49	104.92	105.34	105.75	106.17	106.57	106.98
2.10	104.25	104.65	105.05	105.44	105.83	106.22	106.60
2.15	104.01	104.39	104.76	105.13	105.50	105.86	106.22
2.20	103.77	104.12	104.47	104.82	105.16	105.50	105.84
2.25	103.53	103.86	104.19	104.51	104.83	105.15	105.46
2.30	103.29	103.60	103.90	104.20	104.50	104.80	105.09
2.35	103.05	103.34	103.62	103.90	104.17	104.45	104.72
2.40	102.81	103.07	103.33	103.59	103.85	104.10	104.34
2.45	102.57	102.81	103.05	103.29	103.52	103.75	103.97
2.50	102.34	102.55	102.77	102.98	103.19	103.40	103.61
2.55	102.10	102.30	102.49	102.68	102.87	103.05	103.24
2.60	101.86	102.04	102.21	102.38	102.54	102.71	102.87
2.65	101.63	101.78	101.93	102.08	102.22	102.37	102.51
2.70	101.39	101.52	101.65	101.78	101.90	102.02	102.15
2.75	101.16	101.27	101.37	101.48	101.58	101.68	101.78
2.80	100.93	101.01	101.10	101.16	101.26	101.34	101.42
2.85	100.69	100.76	100.82	100.88	100.95	101.01	101.07
2.90	100.46	100.51	100.55	100.59	100.63	100.67	100.71
2.95	100.23	100.25	100.27	100.29	100.31	100.33	100.35
3.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
3.05	99.77	99.75	99.73	99.71	99.69	99.67	99.65
3.10	99.54	99.50	99.46	99.42	99.38	99.34	99.30
3.15	99.31	99.25	99.19	99.12	99.06	99.00	98.95
3.20	99.08	99.00	98.92	98.83	98.75	98.68	98.60
3.25	98.85	98.75	98.65	98.55	98.45	98.35	98.25
3.30	98.63	98.50	98.38	98.26	98.14	98.02	97.91
3.35	98.40	98.26	98.11	97.97	97.83	97.70	97.56
3.40	98.17	98.01	97.85	97.68	97.53	97.37	97.22
3.45	97.95	97.76	97.58	97.40	97.22	97.05	96.88
3.50	97.72	97.52	97.32	97.12	96.92	96.73	96.54
3.55	97.50	97.27	97.05	96.83	96.62	96.41	96.20
3.60	97.28	97.03	96.79	96.55	96.32	96.09	95.86
3.65	97.05	96.79	96.53	96.27	96.02	95.77	95.53

3%

Mat. / Yield	5 YRS	5½ YRS	6 YRS	6½ YRS	7 YRS	7½ YRS	8 YRS
3.70	96.83	96.55	96.26	95.99	95.72	95.45	95.19
3.75	96.61	96.30	96.00	95.71	95.42	95.14	94.86
3.80	96.39	96.06	95.74	95.43	95.12	94.82	94.53
3.85	96.17	95.82	95.48	95.15	94.83	94.51	94.20
3.90	95.95	95.58	95.23	94.88	94.53	94.20	93.87
3.95	95.73	95.34	94.97	94.60	94.24	93.89	93.54
4.00	95.51	95.11	94.71	94.33	93.95	93.58	93.21
4.05	95.29	94.87	94.46	94.05	93.66	93.27	92.89
4.10	95.07	94.63	94.20	93.78	93.36	92.96	92.56
4.15	94.86	94.40	93.95	93.51	93.08	92.65	92.24
4.20	94.64	94.16	93.69	93.24	92.79	92.35	91.92
4.25	94.42	93.93	93.44	92.97	92.50	92.04	91.60
4.30	94.21	93.69	93.19	92.70	92.21	91.74	91.28
4.35	93.99	93.46	92.94	92.43	91.93	91.44	90.96
4.40	93.78	93.23	92.69	92.16	91.64	91.14	90.64
4.45	93.56	92.99	92.44	91.89	91.36	90.84	90.33
4.50	93.35	92.76	92.19	91.63	91.08	90.54	90.02
4.55	93.14	92.53	91.94	91.36	90.80	90.24	89.70
4.60	92.93	92.30	91.69	91.10	90.52	89.95	89.39
4.65	92.71	92.07	91.45	90.84	90.24	89.65	89.08
4.70	92.50	91.84	91.20	90.57	89.96	89.36	88.77
4.75	92.29	91.62	90.96	90.31	89.68	89.07	88.46
4.80	92.08	91.39	90.71	90.05	89.40	88.77	88.16
4.85	91.87	91.16	90.47	89.79	89.13	88.48	87.85
4.90	91.66	90.94	90.23	89.53	88.86	88.19	87.55
4.95	91.46	90.71	89.98	89.27	88.58	87.91	87.25
5.00	91.25	90.49	89.74	89.02	88.31	87.62	86.94
5.05	91.04	90.26	89.50	88.76	88.04	87.33	86.64
5.10	90.83	90.04	89.26	88.51	87.77	87.05	86.35
5.15	90.63	89.82	89.02	88.25	87.50	86.76	86.05
5.20	90.42	89.59	88.78	88.00	87.23	86.48	85.75
5.25	90.22	89.37	88.55	87.74	86.96	86.20	85.45
5.30	90.01	89.15	88.31	87.49	86.69	85.92	85.16
5.35	89.81	88.93	88.07	87.24	86.43	85.64	84.87
5.40	89.61	88.71	87.84	86.99	86.16	85.36	84.58
5.45	89.40	88.49	87.60	86.74	85.90	85.08	84.28
5.50	89.20	88.27	87.37	86.49	85.64	84.80	83.99
5.55	89.00	88.05	87.14	86.24	85.37	84.53	83.71
5.60	88.80	87.84	86.90	86.00	85.11	84.25	83.42
5.65	88.60	87.62	86.67	85.75	84.85	83.98	83.13
5.70	88.40	87.40	86.44	85.50	84.59	83.71	82.85
5.75	88.20	87.19	86.21	85.26	84.33	83.44	82.56
5.80	88.00	86.97	85.98	85.02	84.08	83.17	82.28
5.85	87.80	86.76	85.75	84.77	83.82	82.90	82.00
5.90	87.60	86.55	85.52	84.53	83.56	82.63	81.72
5.95	87.40	86.33	85.30	84.29	83.31	82.36	81.44
6.00	87.20	86.12	85.07	84.05	83.06	82.09	81.16
6.10	86.81	85.70	84.62	83.57	82.55	81.56	80.60
6.20	86.42	85.28	84.17	83.09	82.05	81.04	80.06
6.25	86.23	85.07	83.94	82.86	81.80	80.78	79.78
6.30	86.03	84.86	83.72	82.62	81.55	80.51	79.51
6.40	85.65	84.44	83.28	82.15	81.06	80.00	78.97
6.50	85.26	84.03	82.84	81.68	80.56	79.48	78.43
6.60	84.88	83.62	82.40	81.22	80.08	78.97	77.90
6.70	84.50	83.21	81.96	80.76	79.59	78.46	77.37
6.75	84.31	83.01	81.75	80.53	79.35	78.21	77.11
6.80	84.12	82.80	81.53	80.30	79.11	77.96	76.85
6.90	83.74	82.40	81.10	79.85	78.63	77.46	76.33
7.00	83.37	82.00	80.67	79.39	78.16	76.97	75.81
7.50	81.52	80.02	78.57	77.18	75.84	74.54	73.29

or the yield is computed, on the assumptions which furnish the lower yield

Suppose, for instance, that the United States Treasury $3\frac{1}{4}$'s of 1978–83 are quoted at 112, 25 years before maturity By looking at a bond table, it can be seen that the yield on a 20-year $3\frac{1}{4}\%$ bond at 112 is 2 48%, *the yield on a 25-year $3\frac{1}{4}\%$ bond at the same price is 2 56%* The more conservative method is to assume that the bond will be called in 1978 Therefore the lower yield would prevail On the other hand, if the United States Treasury $2\frac{1}{2}$ s 1967–72 are priced at 96 just 15 years before maturity, the yield would be 2 83% if the bond is not paid until maturity, but 2 97% if it is paid at the first call date In determining the yield the assumption would be made that the bond would not be paid until maturity Hence the lower yield would be anticipated

Valuation of bonds between coupon dates

Bonds may be bought in one of two ways One method used in the purchase of bonds traded on the stock exchange is to buy the bond at the market price If such a buyer chose to buy a 3% bond due in about 10 years at a quoted price of 95, and found that the bond he bought actually had 10 years 1 month before maturity, how much should he pay for the bond?

The question might well have been raised from the point of view of the seller as well as the buyer The buyer knows that in 1 month he will receive an interest payment of $15 Yet he will have earned only 1/6 of the $15, or $2 50 The seller, however, who held the bond 5 months since the last coupon date feels entitled to the interest for the period he held it, or $12 50 The customary practice is for the buyer to pay the quoted price, here $950, plus the accrued interest figured as the portion of the coupon elapsed since the last coupon date, here $12 50 Thus he will pay $950 plus $12 50 plus the broker's commission The market price is often called the *and interest* price, or the *quoted price* Persons accustomed to buying and selling bonds know that in buying a bond at the market price the buyer expects to pay and the seller expects to receive the quoted price plus any accrued interest computed as simple interest since the last coupon date In addition there is a broker's commission added to the cost the buyer pays, and a broker's commission deducted from the proceeds which the seller receives That is, both buyer and seller expect to pay a brokerage commission to their own brokers on each transaction

The accrued interest is determined like ordinary interest That is, each month is considered as 30 days and the approximate number of days

between the previous coupon date and the date of purchase is found. If interest is paid semiannually, as it usually is, the accrued interest is

$$\text{Accrued interest} = \text{Value of coupon} \times \frac{\text{Approximate number of days}}{180}$$

The amount of money actually received, or paid, ignoring commissions and incidental expenses, is referred to as the *flat price of a bond.* The flat price and the *and interest price* of a bond bought on a coupon date are the same. When a bond is purchased between coupon dates, however, the flat price is greater than the quoted price.

Often bonds are bought not on the basis of a quoted market price but rather on the basis of a price to furnish a designated yield. In this case the price paid between coupon dates must be one which will furnish the desired yield if the bond is held to redemption. Thus if the yield stays the same, the value of the bond will change daily during its entire life.

In a preceding illustration it was shown that a $4\frac{1}{2}\%$ *bond bought 4* years before maturity to yield 4% would have a value of $1,018.31. Although a schedule was established showing the value on each coupon date, the problem still arises of how much should be paid between coupon dates by a prospective buyer to achieve the desired yield. Such a value can be found by the use of the basic formula previously presented. In business practice, however, another method of valuing such a bond *between coupon dates is well established in which emphasis is on simplicity* rather than exactness.

In valuing the bond between coupon dates it must be recalled that in making such a schedule it is assumed that the interest is constantly being earned on the book value at the yield rate, and on the face value at the coupon rate. The book value of the bond at the previous coupon date shown was $1,018.31. During the next 6 months, interest was accrued at the rate of $3.75 ($22.50 $\div$ 6) per month. If the purchaser chose to sell the bond to another buyer on a 4% yield 4 months after buying it for $1,018.31, how much should the seller receive? He has invested $1,018.31 on which he hopes to earn 4%. At 4% simple interest then he expects to earn $13.57 ($1,018.31 $\times \frac{1}{3} \times$ 4%) in 4 months. To achieve this income the selling price, or the flat price, of the bond will be $1,031.88 ($1,018.31 + $13.57).

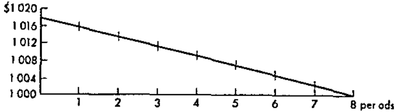
In summary, the flat price of a bond bought on a yield basis between coupon dates may be found as follows:

1. Determine the value of the bond on the preceding coupon date to furnish the specified yield.
2. Compute the simple ordinary interest on this value, at the yield

rate, for the period which has lapsed since the last coupon date Add this interest to the value found in the first step

Book value of bonds bought between coupon dates

In explaining how to account for a bond premium an example was given of a 4½% bond bought 4 years before maturity to yield 4% and a schedule of book values on each coupon date was developed In the following figure the values are shown on the solid black line as decreasing from $1 018 31 to $1,000 on maturity date



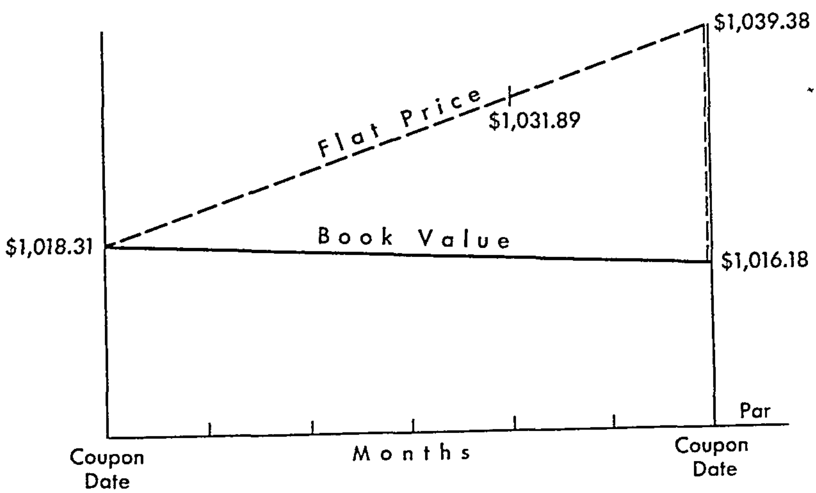
The question of determining the book value of a bond bought between coupon dates was not considered In the preceding illustration it was shown that the flat price of a 4½% bond bought 3 years 8 months before *maturity would have by current financial practices, a flat price of* $1 031 88 but the book value on this date was not determined

The purchaser of the bond is to receive 4% on his money Had he waited 2 months and bought the bond on the coupon date at a cost of $1 016 18 he would have received a 4% yield to maturity, since he would have bought the bond at its book value which on the coupon date is equal to the *and interest* price By buying the bond now at $1,031 88 he must hold it 2 months before he may cash the coupon for $22 50 The buyer has in effect reimbursed the seller for holding the bond 4 months by paying the accrued interest of $15 00 ($22 50 × $\frac{2}{3}$) That is, *the quoted price (book value) of a bond, or the and interest price on the date of purchase,* (here $1 016 18) *is the flat price of the bond* (here $1,031 88) *less the interest accrued on the face value of the bond at the coupon rate since the last interest payment date* (here $15) The and interest price and the book value of a bond correspond on all dates when an exact method of computing the book value is used Under present commercial practices they may vary when computed between coupon dates by a few cents, but the advantages gained by the simplicity of the methods used probably offset any inaccuracies

In the bond under discussion the purchaser expected to receive 4% on his money He paid a total of $1,031 88 for the bond but $15 00 of this purchase price was to reimburse the seller for the accrued interest That

is, the book value of the bond, or the *and interest* price, was \$1,016.88 (\$1,031.88 — \$15.00) on the date acquired. The buyer expects to earn 4% on the \$1,016.88, that is, in the next 2 months he expects an income of \$6.78 (\$1,016.88 $\times \frac{1}{6} \times$ 4%). Since he will be able to cash a coupon for \$22.50 on that date, $\frac{2}{3}$ of which goes to the previous owner, he will gain \$7.50, or 72 cents (\$7.50 — \$6.78) more than he expected in interest income. This extra 72 cents represents the decrease in the book value of his asset, that is, the amount of reduction in the book value. According to the schedule shown on page 404 the book value should be \$1,016.18 on the next coupon date. By the method just used to compute it, the book value will be \$1,016.16 (\$1,016.88 — \$0.72). The small difference of 2 cents arises from two facts: one, that interest on the \$15 was not computed for the 2 months, that is, simple interest only is used; and, second, ordinary inaccuracies arising from rounding off the figures to only 2 decimal places.

The black line on page 412 shows the book value at all times, with the qualification that there may be small deviations as pointed out. The flat price of the bond tends to rise from one coupon date to the next and then drop abruptly to the book value on the coupon date. The dotted line in the following figure shows the nature of the variation in the flat prices for one coupon period.



The steps necessary to find the book value of a bond purchased on a yield basis between coupon dates may be summarized as follows:

1. Find the flat price of the bond on the purchase date.
2. Deduct from the flat price of the bond the interest that has accrued on the face value of the bond at the coupon rate since the last coupon date.

EXERCISE 14.2

Solve the following

Find the flat price and the book value of the following bonds whose face value is $1,000 Use the bond table when possible

	Interest Dates	*Coupon Rate*	*Yield Rate*	*Maturity Date*	*Purchase Date*
1.	Jj	3%	$3\frac{1}{2}$%	1/1/84	4/24/56
2.	fA	$3\frac{1}{2}$%	3%	8/1/72	10/18/56
3	Ms	$3\frac{1}{2}$%	4%	3/1/80	3/20/57
4.	mS	$2\frac{1}{2}$%	3%	9/1/82	12/18/57
5.	jJ	4%	$3\frac{1}{2}$%	7/1/75	5/12/56
6	Jd	3%	$3\frac{1}{2}$%	6/1/65	9/15/58
7.	Fa	3%	4%	2/1/62	2/1/57
8.	Ms	3%	$2\frac{1}{2}$%	3/1/65	9/1/57
9.	Ao	3%	1 30%	1/1/63	8/25/57
10.	aO	3%	6%	10/1/66	1/1/58

11. A $1,000 3% jD bond due 12/1/66 is purchased on 9/1/59 to yield $3\frac{1}{2}$% Find the *and interest* price

12. A $1,000 3% Fa bond due 2/15/64 is callable at 103 If the price on 8/15/58 is $105\frac{1}{2}$, what is the yield? Is it a reasonable investment?

13. A $10,000 $2\frac{1}{2}$% Jj bond due 1/1/76 is callable at $102\frac{1}{2}$ If the approximate yield is 2% on 4/1/56, find the flat price and the quoted price Is it a reasonable investment?

14. A $100,000, 3% Jj bond due 1/8/78 is purchased on 3/17/59 to yield 4% Find the quoted price

15. A $10,000, 6% mS bond due 9/15/80 is purchased on 11/15/59 to yield 5% Find the quoted price and the flat price

Finding the approximate yield on a bond

The investor interested in buying a bond has an almost unlimited number to choose from, varying as to type, maturity, coupon rate, and price In order to compare various investment outlets as well as to furnish facts for accounting and income tax purposes, an investor may want to know the approximate, if not the accurate, yield Usually the yield can be found from a bond table, a bond guide, or from one of the more complete financial periodicals, which include both bond prices and yields

It should be understood that approximate bond yields are usually of theoretical value only Although any serious investor will consult a bond table, he usually turns to one of three other methods when seeking an approximate yield One method, called for no specific reason *the bond*

salesman's method, is reasonably accurate for short- to medium-term bonds selling near par. The basic assumption on which this method rests is that the approximate yield is equal to the average amount of income divided by the average book value. There are thus three steps involved in application of this method.

1. Find the average book value by averaging the price and the maturity value of the bond.

2. Find the average income. If a bond is priced at a premium, the periodic income is decreased by the amount of the premium written off each coupon period. Thus if the bond is priced at a premium, divide the premium by the number of interest payments to maturity. Deduct this quotient from the coupon payment to find the average income.

If the bond is priced at a discount, the periodic income is increased by the amount of the discount recovered each period. Divide the discount by the number of interest payments to maturity. Add this quotient to the coupon payment to find the average income.

3. Divide the average periodic income by the average book value. This furnishes the yield per period. In the standard bond pattern, this quotient will have to be multiplied by 2 to get an annual yield. The same results would be obtained if the interest is considered as being paid annually, and the premium or discount adjustment made on an annual basis.

Illustrations:

a. Ten years before maturity a \$1,000, $4\frac{1}{2}\%$ bond is offered at \$1,075. Find the approximate yield if the interest is paid semiannually.

1. The average book value is $\dfrac{\$1{,}075 + 1{,}000}{2} = \$1{,}037.50$.

2. The \$75 premium is to be written off in 20 periods, or the equivalent of \$3.75 (\$75 ÷ 20) per period. Since the periodic coupon is \$22.50, the average income per period is thus \$22.50 − 3.75 = \$18.75.

3. The approximate yield is 1.807% (\$18.75 ÷ \$1,037.50) per period, or 3.61% per year.

b. A buyer pays \$947 for a \$1,000, $3\frac{1}{2}\%$ bond $8\frac{1}{2}$ years before maturity. Find the approximate yield if interest is paid semiannually.

1. The average book value is $\dfrac{\$947 + 1{,}000}{2} = \$973.50$.

2. There are 17 coupons to be paid on the bond. The total discount is \$53. The discount to be added to each coupon is \$3.12 (\$53 ÷ 17). Since the periodic coupon is \$17.50, the average income per period is thus \$17.50 + 3.12 = \$20.62.

3 The approximate yield is 2 119% ($20 62 − $973 50) per period, or 4 24% per year

An adaptation of this method is often used in financial circles to find the approximate yield if a bond has less than 50 years to run and is bought at a premium The formula is

$$\frac{\text{Annual interest payment} - \dfrac{\text{Premium}}{\text{Years to maturity}}}{\text{Redemption value} + 0\ 6 \times \text{Premium}} = \text{Yield}$$

This formula, frequently used to compare bonds, has been devised through trial and error Though it gives only a scientific estimate of the yield, it is sufficiently accurate for most purposes The factor in the denominator of 0 6 of the premium is a constant factor, that is, it does not change so long as the maturity date is less than 50 years

Illustration Ten years before maturity a $1,000 $4\frac{1}{2}$% bond is offered at $1,075 Find the approximate yield

Annual interest payment is $45, premium is $75, years to maturity is 10, redemption value is $1,000 Therefore

$$\frac{\$45 - \dfrac{75}{10}}{\$1{,}000 + 0\ 6 \times 75} = \frac{\$37\ 50}{\$1{,}045} = 3\ 59\%$$

When a bond with less than 50 years to run to maturity is bought at a discount, a close approximation of the yield can be found by substituting values in the following formula

$$\frac{\text{Annual interest payment} + \dfrac{\text{Discount}}{\text{Years to maturity}}}{\text{Redemption value} - 0\ 6 \times \text{Discount}} = \textit{Yield}$$

(It should be observed again that the factor of 0 6 is a constant which has been found to furnish fairly accurate results)

Illustration A buyer pays $947 for a $1,000, $3\frac{1}{2}$% bond $8\frac{1}{2}$ years before maturity Find the approximate yield

Annual interest payment is $35, discount is $53, years to maturity is 8 5, redemption value is $1,000 Therefore

$$\frac{\$35 + \dfrac{53}{8\ 5}}{\$1{,}000 - 0\ 6 \times 53} = \frac{\$41\ 23}{\$968\ 20} = 4\ 26\%$$

A third method, both more accurate and more laborious than the other two, entails the use of the annuity tables Called the *interpolation method,*

the approximate yield is first found by one of the methods already discussed. Using this approximate yield as a basis for interpolation, the formula $V = F + (R - Fi)\, a_{\overline{n}|i}$ for the price of a bond is used with rates as nearly above and below the approximate yield as the tables show. When the price of the bond is found by this method the difference between the prices found at the known yields and the price at the unknown yield are compared. Since the known price is between the two prices used, the unknown yield is between the known yields, and the approximate yield can be found by interpolation.

Illustration: Twelve years before maturity a 5% bond with interest payable annually is priced at $93\frac{5}{8}$. Find the yield by interpolation.

First find the approximate yield.

1. Average book value $= \dfrac{\$1,000 + \$936.25}{2} = \$968.125$
2. Discount $= \$1,000 - \$936.25 = \$63.75$

   Discount to be written off each year $= \dfrac{\$63.75}{12} = \$5.31$

   Average annual income $= \$50 + \$5.31 = \$55.31$
3. Approximate yield $= 55.31 \div 968.125 = 5.71\%$

With this approximate yield, it appears that the actual annual yield is probably between $5\frac{1}{2}\%$ and 6%. Using the formula, $V = F + (R - Fi)a_{\overline{n}|i}$, we find first the price of a bond to yield $5\frac{1}{2}\%$ and then the price to yield 6%.

$F = \$1,000$; $R = \$50$; $n = 12$

If $i = 5\frac{1}{2}\%$, then

$$\begin{aligned} V &= \$1,000 + (\$50 - 55)\, a_{\overline{12}|5\frac{1}{2}\%} \\ &= \$1,000 - (\$5 \times 8.6185) = \$1,000 - 43.09 = \$956.91 \end{aligned}$$

If $i = 6\%$, then

$$\begin{aligned} V &= \$1,000 + (\$50 - 60)\, a_{\overline{12}|6\%} \\ &= \$1,000 - (\$10 \times 8.3838 = \$1,000 - 83.84 = \$916.16 \end{aligned}$$

The price of the bond was $93\frac{5}{8}$, or \$936.25. Knowing three prices and the two yields, we see that

Price Differences	*Prices*	*Yields*	*Yield Differences*
\$20.66	\$956.91	$5\frac{1}{2}\%$	x
	936.25	?	
\$40.75	916.16	6%	$\frac{1}{2}\%$

From the preceding diagram it is seen that a price difference of \$10 75 results in a yield difference of $\frac{1}{2}$% Hence it can be reasoned that a price difference of \$20 66 should account for a yield difference of approximately 0 253% That is,

$$\frac{x}{0\ 5\%} = \frac{20\ 66}{40\ 75} \quad \text{or} \quad x = \frac{2{,}066}{4{,}075} \times 0\ 5\% = 0\ 253\%$$

The yield on the bond priced at \$936 25 then is approximately 5 5% + 0 253%, or 5 753%

EXERCISE 14.3

Solve the following

1. Find the approximate yield on a \$1,000, 4% bond with interest payable semiannually, bought 8 years before maturity at 108

2 An investor has a choice of buying a 5% bond of the Granite City Generating Company 4 years before maturity at $105\frac{1}{2}$, or a 4% bond of the Gilchrist Company $5\frac{1}{2}$ years before maturity at $103\frac{3}{4}$ If the interest on each bond is paid semiannually, what is the approximate yield on each?

3 Find the approximate yield on a \$1,000, 3% bond with interest payable annually, bought 15 years before maturity at 90

4. If the Washington Water and Power $3\frac{1}{2}$% bonds can be purchased $7\frac{1}{2}$ years before maturity at 94, what is the approximate yield?

5. An investor bought a 3% bond of the Missouri Power Company at $89\frac{1}{2}$ He held it $10\frac{1}{2}$ years and sold it at par What was the approximate yield received?

6. Baltimore and Ohio Railroad has outstanding some 4% bonds What would be the approximate yield on such a bond purchased 20 years before maturity at 111?

7. Find the approximate yield on the Gatineau Power Company $3\frac{1}{4}$'s of 1970 bought at $105\frac{3}{4}$ exactly $22\frac{1}{2}$ years before maturity?

8. If the Great Lakes Power $4\frac{1}{2}$'s of 1969 are priced $102\frac{1}{2}$ exactly $11\frac{1}{2}$ years before maturity, what is their approximate yield?

9. By interpolation find the approximate yield on 4% bonds of the Carolina, Clinchfield, and Ohio Railroad purchased 10 years before maturity at 102

10. By interpolation find the approximate yield 13 years before maturity of the El Paso and Southwestern Railroad 5's purchased at $107\frac{3}{8}$

11. By interpolation find the yield $8\frac{1}{2}$ years before maturity of Boston Edison $2\frac{3}{4}\%$ bonds priced at $87\frac{3}{4}$.

12. Four years before maturity a 3% bond is priced at $90\frac{1}{2}$. Find the approximate yield by interpolation.

13. A $3\frac{1}{2}\%$ bond with interest payable semiannually is bought $4\frac{1}{2}$ years before maturity at 105. Find the approximate yield.

14. Find the approximate yield on a $1,000 Jj 4% bond bought 10 years before maturity at 108.

15. Find the approximate yield on a $10,000 Fa 3% bond bought 5 years before maturity at 95.

Perpetuities

Some bonds without maturity dates have been issued by governments and corporations. In effect, therefore, the interest payments are to continue forever. Any series of payments which is to continue for an unlimited period is referred to as continuing in *perpetuity.*

The amount of bond interest on a perpetual bond does not change. If the rate on the bond is 4% and the bond has a par value of $1,000, the value of the bond as a perpetuity will be determined by the minimum rate of income which an investor is willing to accept. If the investor is willing to accept a return of 5%, the value of the bond can be determined as follows:

Let x = value of the bond. Yield wanted = 5%; periodic payment received = $40. Since the yield times the value of the bond is equal to the periodic income,

$$5\% \times x = \$40; \quad x = \frac{\$40}{5\%} = \$800$$

If R = periodic payment received, and i = yield wanted, the present value of a perpetuity having payments of R therefore is equal to $\frac{R}{i}$, and is often spoken of as the *capitalized value of R.*

Institutions often have problems dealing with perpetuities. For instance, colleges and universities are often given money, the income from which is to provide permanently enough money to pay a professor of a given subject. Thus a wealthy man is said to endow a chair in economics or mathematics, and so on. Another example of a perpetuity is the establishment of a fund for the perpetual care of a cemetery.

Investors often attempt to appraise what they hope will be a perpetuity. For example, a preferred stock which has no maturity date and on which dividends will be paid regularly is in effect a perpetuity.

Capitalized cost

Engineers as well as financiers are often confronted with special kinds of perpetuities When a bridge is built, a library established, a factory erected, or a house painted, provision may also be made for periodic replacement or maintenance The owner or donor may want to create a fund S to provide for the periodic demand every k years A somewhat similar problem arises when it is necessary to compare or contrast the total future expense of one method, plan, machine, or piece of equipment to that of others

The *capitalized cost* K is defined as the original cost plus the present value of an unlimited number of future renewals of an asset Such a fund must be sufficiently large (1) to meet the original cost C, and (2) to provide S dollars every k periods for renewals It has been shown that the present value of the periodic income of R dollars is equal to $\frac{R}{i}$ If S dollars were needed every k periods for renewal, the size of the fund would be the present value of the periodic payment of S dollars, namely, $\frac{S}{i}$ But since the payment is not needed every period, but only every k periods, it becomes necessary to find the value of an annuity which amounts to $\frac{S}{i}$ at the end of k periods Using formulas with which we are already familiar for the periodic deposit that will grow to \$1 at a future date, we can readily see that the present value of the amounts needed for periodic replacement every k periods is $\frac{S}{i} \times \frac{1}{s_{\overline{k}|i}}$ Thus the formula for the capitalized cost is

$$K = C + \frac{S}{i} \times \frac{1}{s_{\overline{k}|i}}$$

Illustrations

a Find the capitalized cost of a swimming pool to be built at a cost of \$2,500 if major repairs amounting to \$500 must be made every 10 years Money is worth 4%

Here $C = \$2{,}500$, $S = \$500$, $k = 10$, $i = 4\%$ Therefore

$$K = \$2{,}500 + \frac{\$500}{4\%} \times \frac{1}{s_{\overline{10}|4\%}} = \$2{,}500 + \$12{,}500 \times 0\ 0832909$$

$$= \$3{,}541\ 14$$

b What would be the capitalized cost in the preceding illustration if the annual expense of operating the pool is \$100 in addition to the cost of construction and the \$500 expenditure every 10 years?

The capitalized cost will be increased by the present value of a perpetuity of \$100 at 4%, namely, $\frac{\$100}{4\%} = \$2{,}500$.

$$K = \$3{,}541.14 + \$2{,}500.00 = \$6{,}041.14$$

If the amount S is equal to the original cost—that is, if the assets must be replaced in entirety every k periods—then, since $C = S$, the formula for the capitalized cost will be

$$K = S + \frac{S}{i} \times \frac{1}{s_{\overline{k}|i}}$$

Although the proof is omitted here, it can be shown that

$$\frac{1}{s_{\overline{n}|i}} = \frac{1}{a_{\overline{n}|i}} - i$$

Substituting the value of $\frac{1}{a_{\overline{k}|i}} - i$ for $\frac{1}{s_{\overline{k}|i}}$, we can see that

$$K = S + \frac{S}{i}\left(\frac{1}{a_{\overline{k}|i}} - i\right) = S + \frac{S}{i} \times \frac{1}{a_{\overline{k}|i}} - S = \frac{S}{i} \times \frac{1}{a_{\overline{k}|i}}$$

The original formula on capitalized cost, $K = C + \frac{S}{i} \times \frac{1}{s_{\overline{k}|i}}$, can be used for any problem dealing with capitalized costs. The new formula, $K = \frac{S}{i} \times \frac{1}{a_{\overline{k}|i}}$, which can be used only when the total assets must be replaced periodically, is much more convenient for problems to which it is applicable. It has many applications. Repainting a building periodically is equivalent to replacing an asset, the paint, every k periods. To compare one type of paint with another then becomes the problem of finding the present value of future replacements when cost and average life of various paints are known. The same problem applies to choosing one machine over another when the original costs and average lengths of use differ, or to the selection of one automobile rather than another, or the choice of one kind of roofing rather than another.

Illustration: A red cedar roof costing \$600 will last for 8 years. A composition roof costing \$400 will last for 5 years. Determine which roof is cheaper on the basis of their capitalized costs. Money is worth 5%.

For the shingle roof $S = \$600$; $k = 8$; $i = 5\%$.

$$K = \frac{\$600}{5\%} \times \frac{1}{a_{\overline{8}|5\%}} = \$12{,}000 \times 0.1547218 = \$1{,}856.66$$

For the composition roof $S = \$400$, $k = 5$, $i = 5\%$

$$K = \frac{\$400}{5\%} \times \frac{1}{a_{\overline{5}|5\%}} = \$8,000 \times 0\ 2309748 = \$1,847\ 80$$

Since $1,856 66 − $1,847 80 = $8 86, the composition roof is cheaper by $8 86

In the preceding illustrations it has been assumed that either asset during the period of its estimated life will give satisfactory service A similar assumption would be made in any problem in which one is trying to determine how much may be expended to extend the life of an asset, or how much could be expended on another type which differs in cost and life span

Illustration If money is worth $4\frac{1}{2}\%$, how much could one afford to spend for treating a shingle roof, originally costing $600, to extend its life from 8 years to 20 years?

If we assume that one roof is as economical as the other, then capitalized cost should be equal

For the untreated shingle roof $S = \$600$, $k = 8$, $i = 4\frac{1}{2}\%$

$$K = \frac{\$600}{0\ 015} \times \frac{1}{a_{\overline{8}|4\frac{1}{2}\%}} = \$13,333\ 33 \times 0\ 1516096 = \$2\ 021\ 46$$

For the treated shingle roof $S = ?$, $k = 20$, $i = 4\frac{1}{2}\%$

$$K = \frac{S}{0\ 045} \times \frac{1}{a_{\overline{20}|4\frac{1}{2}\%}}$$

These two values for K can be set equal to each other

$$\frac{S}{0\ 045} \times \frac{1}{a_{\overline{20}|4\frac{1}{2}\%}} = \$2,021\ 46$$

$$S = \$2,021\ 46 \times 0\ 045 \times a_{\overline{20}|4\frac{1}{2}\%} = \$90\ 966 \times 13\ 007936 = \$1,183\ 28$$

It would be just as economical to spend $1,183 28 for a roof lasting 20 *years as to spend $600 for a roof lasting 8 years Therefore if the life* of a roof can be extended to 20 years for anything less than $583 28 ($1,183 28 − $600) it would be economically feasible

EXERCISE 14 4

Solve the following

1. Eastman Kodak has outstanding some preferred stock on which dividends of $1 50 are paid quarterly Assuming the annual return of $6 will continue in perpetuity, what is the value of a share of this stock when investors expect an annual return of (a) 3%, (b) 4%, (c) 5%?

2. J.N. Wright wanted to establish an award of \$800 to be given each year to the graduating student with the best record in the local high school. The cost of administrating the award averages \$50 a year. What sum of money should he give to assure the annual payment in perpetuity if interest rate is assumed to be $3\frac{1}{2}\%$?

3. A piece of land is leased in perpetuity at \$2,500 a year. If money is worth 5%, what is the value of the lease?

4. One section of an irrigation ditch requires an annual expenditure of \$2,000 for maintenance. It is believed that by a special treatment the annual expenditure of maintenance can be reduced to \$800. If money costs 4%, how much can the company afford to spend to reduce the annual upkeep by \$1,200?

5. Wisconsin Electric and Power Company preferred pays \$6 annually. If money is worth $5\frac{1}{2}\%$, what is the value of this stock?

6. How much would need to be invested at 5% to assure an income of 1 cent per minute in perpetuity?

7. How much is needed to endow a hospital bed at an annual cost of \$1,800, if money is worth $2\frac{1}{2}\%$?

8. A philanthropist leaves his art collection to a public museum. It is estimated that \$8,400 will be needed annually to provide adequate care for the collection. If money is worth 3%, what sum should the donor leave in perpetual trust to provide the necessary income?

9. A memorial fountain is built in a park. What amount should be put in a perpetuity to assure water and lights for the fountain at an annual cost of \$1,500, if money is worth 3%?

10. How much should be established in a fund for keeping the grass in a cemetery mowed, if the annual charge is estimated at \$2,400 a year? Money is worth $2\frac{1}{2}\%$.

11. If money is worth 5%, find the capitalized cost of a machine which costs \$1,000 and which must be replaced at the same cost every 5 years.

12. If money is worth 5%, find the capitalized cost of a machine which costs \$1,500 and which must be replaced at the same cost every 8 years.

13. What is the value of a perpetuity of \$400 a year if money is worth 4% payable annually?

14. Find the capitalized cost of a driveway which costs \$2,000 to build and which has an annual upkeep expense of \$200. Money is worth 4%.

15. What amount of endowment is necessary to construct and maintain a library building if the original cost is \$1,500,000 and the annual upkeep is \$150,000 when it is estimated that the entire building must be replaced at the same cost every 50 years? Money is worth 4%.

16. A stucco house can be painted for \$60 with a paint which lasts 2 years If money is worth 4%, how much can one afford to spend on a better grade of paint which will last 4 years?

17. In remodeling an office building, the engineer estimates one type of pipe would cost \$30,000 and would last 10 years Another type of pipe less subject to corrosion would cost \$50,000 and would last 25 years If money is worth 5%, which is cheaper?

18. The city manager of Fulton has a choice of paying \$30,000 for fireplugs with an estimated life of 10 years, or \$50,000 for a better grade valve with a life of 25 years He is interested in making the more economical purchase If the city borrows at $3\frac{1}{2}$%, which should he select? How much difference is there in the capitalized costs?

19. A school playground can be graveled for \$20,000, the annual upkeep will amount to \$1,000 If money is worth 5%, how much can the school board afford to spend in covering the playground with asphalt if it needs to be repaired every 5 years at a cost of \$2,500?

20. One type of irrigation system, with an estimated life of 12 years, can be installed at a cost of \$12,500 What is the maximum amount that could be spent economically on a system which would last 20 years if money can be borrowed at 5%?

21. For \$25,000 the City Sanitary System can build one type of tank which will last 10 years and have an annual upkeep of \$1,000 How much can it afford to pay for another type tank with a life of 25 years and an annual upkeep of \$500, if money is worth 4%?

22. A cost study reveals that a city is spending \$1,200 annually for traffic policemen to direct traffic at one intersection If the city can borrow money at 3% per year, how much can it profitably spend on a traffic light which will cost \$100 a year to operate, and which must be renewed every 12 years?

23. It now costs \$40,000 a year to move the mail from the post office to the railroad station, and from the railroad station to the post office If money is worth 3% per year, how much can the post office profitably spend on a conveyor system which requires an annual expenditure of \$1,200, which must be renewed every 15 years, and which needs the services of 2 employees at \$2,400 each per year?

24. The Middle West Cement Company, facing a substantial increase in electric power rates, finds that by spending \$500,000 for equipment it could generate its own power, and after paying all expenses of operation, increase its annual income after taxes by \$60,000 To keep the equipment in perfect working condition, \$75,000 would have to be spent every other year If the company can borrow at 4%, would it pay to make the change?

25. An isolated filling station in the desert has an annual net income of $21,000. A prospective purchaser estimates it will produce this income for 5 years. At the end of 5 years it will have a nominal value of $5,000, since at that time a new highway will be completed which will carry a large percentage of the traffic which now passes the station. What can he afford to pay if he expects a return of 6% on his investment, and can invest his replacement fund at the same rate?

Depreciation

It is difficult in any course or textbook to keep within the bounds of the field being studied. Excursions into closely related fields may be justified if the student learns something in the process. While the principles of mathematics of finance have many applications in investments, insurance, and accounting, the fundamental purpose of the text should be to explain mathematical principles and to provide adequate illustrations of their practical application. This is not intended as a text in accounting, insurance, or merchandising.

Accountants have found it necessary to distinguish between costs which are written off as expenses in one accounting period and those which may spread over more than one period. In the operation of an automobile in a business, plainly the cost of gasoline may be allocated to the period in which it is purchased—that is, it is considered an expense. The purchase price of the automobile, $3,500, is not considered an expense incurred in one period if the automobile has a useful life of 3 years. *Depreciation is the process of spreading the cost of the asset over the period of its useful life.*

For illustrative purposes consider an automobile that costs $3,500, has a useful life of 3 years, and an estimated trade-in, or salvage value, of $500 at the end of the 3-year period. The problem is how to spread ' cost of $3,000 over the 3-year period.

The customary accounting practice calls (1) for showing as an exp that part to be allocated to the particular period; and (2) for establish an offsetting account, called a *reserve for depreciation,* which is incre each year during the life of the asset. The problem in depreciation is determine how much should be charged as an expense each year. Th are about ten different methods which, theoretically at least, could be used.

Straight-line depreciation

Probably the simplest—at any rate, by far the most widely used—method of computing depreciation expense is the *straight-line method.* It

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Straight-line depreciation

Probably the simplest—at any rate, by far the most widely used—method of computing depreciation expense is the *straight-line method.* It

is so called because it is based on the assumption that the net cost—that is, the difference between the cost and the estimated salvage value of the asset—should be spread uniformly over its estimated life The computation of the depreciation by such a method requires a knowledge of only elementary arithmetic *Three things must be known C, the original cost, S,* the salvage value, n, the estimated life in years

To find R, the annual depreciation expense, divide the difference between the cost and the salvage value by the estimated life in years Thus

$$R = \frac{C - S}{n}$$

In the illustration of the automobile, $C = \$3{,}500$, $S = \$500$, $n = 3$ Therefore

$$R = \frac{\$3{,}500 - 500}{3} = \$1{,}000$$

Depreciation is not affected by how the car was bought and paid for It might have been bought for cash, or it might have been paid for on the installment plan over a period of years If it was paid for in one year, the buyer was out \$3,500 but the purchase price was not an expense From the above calculations only \$1,000 was an expense the first year, *the second year no cash was involved, but \$1,000 was counted as an* expense, and at the end of the third year another \$1,000 was counted as an expense By this method the cost was spread over 3 years, but in no sense was there a fund built up, or an offsetting investment made Thus for all intents and purposes the equal cost each year was reasonable

When there are many assets it may be necessary to know the book value of any one of them The book value is defined as the cost less any depreciation that has been taken The amount of depreciation taken is generally shown in an account called the Reserve for Depreciation In the preceding illustration the credits made to the account showed suc*ceeding balances of \$1,000 at the end of the first year, \$2,000 the end of* the second year, and \$3 000 at the end of the third That is, the balance in the account was equal to the annual depreciation charge times the number of years

When the concept of the book value of an asset is introduced, the student is likely to think that the depreciation truly represents a decrease in value It may, but such an occurrence is merely fortuitous The book value is intended to show how much of the cost has not yet been written off as an expense Thus the book value is the difference between the cost and the reserve for depreciation A depreciation schedule of an asset shows the Cost, the Annual Depreciation, the Amount in Depreciation Reserve, and the Book Value

If R is the annual depreciation, at the end of k periods the amount in the depreciation reserve will be kR. The book value $B.V._k$ will be the difference between the cost C and the amount of depreciation previously taken, kR. That is, $B.V._k = C - kR$

Illustration: A machine costing \$950 has an estimated life of 6 years and a scrap value of \$50. Find the annual depreciation and the book value at the end of 4 years. Develop a depreciation schedule.

Here $C = \$950$; $S = \$50$; $n = 6$. Therefore $R = \dfrac{\$950 - \$50}{6} = \$150$, the annual depreciation expense. Since $k = 4$,

$$B.V._4 = \$950 - 4 \times \$150 = \$950 - \$600 = \$350$$

End of Year	*Annual Depreciation*	*Reserve for Depreciation*	*Book Value*
0	—	—	\$950
1	\$150	\$150	800
2	150	300	650
3	150	450	500
4	150	600	350
5	150	750	200
6	150	900	50
Total	\$900		

It should be observed that the sum of the Reserve for Depreciation and the Book Value is equal to the original cost.

Sinking fund method

It is often assumed, theoretically, that a depreciation reserve is in the nature of a fund created to replace the asset. Under such an assumption it is logical to consider such a fund as analogous to a sinking fund on which interest is earned at a constant rate. Under such conditions the periodic contribution to the fund would come from two sources: first, from the level annual periodic charge; second, from the interest accumulating on the amount previously put in the fund. Under such circumstances the value of the annual contribution would tend to increase each year, since the interest income from the fund would grow larger and larger as the size of the fund increased.

The sinking fund method is often used for analytical reasoning in arriving at estimations of the value of an asset, but it is rarely used in accounting procedures. Though it was used for a time by public utilities and is still a legally approved method, now few major utility companies

use this method The annual charge under this method is simply the *partial payment necessary to accumulate to the value of the asset at maturity* Using the same letters as before, $\frac{1}{s_{\overline{n}|i}}$ represents the amount of periodic deposit that will grow to 1 at a future date Thus under the sinking fund method the periodic charge R would be

$$R = (C - S)\frac{1}{s_{\overline{n}|i}}$$

The amount in the Reserve for Depreciation, which can be here considered a sinking fund would be $R \times s_{\overline{k}|i}$ at the end of k periods The book value at the same time, the end of k periods, would be equal to the cost less the amount in the fund, or

$$B\,V_k = C - R \times s_{\overline{k}|i}$$

Illustration A machine has an original cost of $950, and a scrap value of $50 after 6 years of use If the interest rate for the sinking fund is 5%, find the annual depreciation and the book value after 4 years Develop a depreciation schedule

Here $C = \$950$, $S = \$50$, $n = 6$ Therefore $R = (\$950 - \$50)\frac{1}{s_{\overline{6}|5\%}}$ $= \$132\ 32$ Since $k = 4$,

$$B\,V_4 = \$950 - 132\ 32 \times s_{\overline{4}|5\%} = \$379\ 69$$

Depreciation Schedule

End of Year	*Annual Depreciation*	*Interest Earned*	*Amount added to Sinking Fund*	*Amount in Sinking Fund*	*Book Value*
0	—	—	—	—	$950 00
1	$132 32	$ 0 00	$132 32	$132 32	817 68
2	132 32	6 61	138 93	271 25	678 75
3	132 32	13 56	145 88	417 13	532 86
4	132 32	20 86	153 18	570 31	379 69
5	132 32	28 52	160 84	731 15	218 85
6	132 30	36 56	168 85	900 00	50 00

Constant-percentage or the declining balance method

If depreciation is thought of as representing the decrease in value of an asset it is readily apparent that the straight-line method of depreciation is imperfect Consider again the case of the automobile which cost $3,500, had a salvage value of $500, and a life of 3 years Under the straight line method this car was depreciated $1,000 a year for 3 years Probably in the first 6 months of its life the car depreciated as much, *measured by its resale value, as it did in the next $1\frac{1}{2}$ years*

The *constant-percentage method* of computing depreciation is based on the assumption that depreciation can best be measured as a fixed rate of the book value. As the book value decreases throughout the life of the asset, the amount of depreciation also decreases. Thus if it were assumed that the value of the automobile decreased 48% a year, the $3,500 car would depreciate $1,680 the first year. The book value at the start of the second year would be $1,820 ($3,500 — $1,680), and the depreciation the second year would be $873.60 ($1,820 × 48%). The book value at the beginning of the third year would be $946.40 ($1,820 — $873.60). At 48% the depreciation that year would amount to $454.27 and the book value would be $492.13.

If we continue to use the same letters as before with C to represent the cost, S to represent the salvage value at the end of n years, r may be used to represent the rate of depreciation per period. Since the depreciation is computed on the basis of the book value, it can be seen that the depreciation the first year is Cr. The book value at the end of the first year will be $C - Cr$ or $C(1 - r)$. The depreciation the second year would be r times $C(1 - r)$, and the book value at the end of the second year would be $C(1 - r) - Cr(1 - r)$, or $C(1 - r)(1 - r)$, which is equal to $C(1 - r)^2$. Indeed it can be seen that the book value at the end of k years is $C(1 - r)^k$. That is,

$$B.V._k = C(1 - r)^k$$

The mathematical problem for the accountant is to determine the rate r which applied to the book value each year will depreciate the asset during its useful life to its estimated salvage value. Thus at the end of its useful life of n years, $B.V._n = C(1 - r)^n$. Since by definition this is equal to the value S, we have

$$S = C(1 - r)^n$$

Obviously S cannot be 0, for it is impossible to depreciate an asset to a 0 value by taking a constant percentage deduction of book value for depreciation each year.

With this equation, given the value of any three of the variables, the value of the fourth variable can be found. The cost is usually known, the value for S and n are carefully estimated, so the problem is to find the value for r. This can be done by logarithms.

$$\begin{aligned} C(1 - r)^n &= S, \\ (1 - r)^n &= \frac{S}{C} \\ n \times \log(1 - r) &= \log S - \log C \\ \log(1 - r) &= \frac{\log S - \log C}{n} \end{aligned}$$

$$1 - r = \text{antilog}\left(\frac{\log S - \log C}{n}\right)$$

$$r = 1 - \text{antilog}\left(\frac{\log S - \log C}{n}\right)$$

Illustration A machine has an original cost of \$950, and a scrap value of \$50 after 6 years of use Find the uniform rate of depreciation What is the book value after 4 years? Develop a depreciation schedule

Now $C = \$950$, $S = \$50$, $n = 6$ Therefore $\$950\,(1 - r)^6 = \$50$, and

$$r = 1 - \text{antilog}\left(\frac{\log 50 - \log 950}{6}\right)$$

$$= 1 - \text{antilog}\left(\frac{1\ 698970 - 2\ 977724 + 60 - 60}{6}\right)$$

(That is, since log C is always greater than log S, add and subtract 10 times n, here $10 \times 6 = 60$, to the dividend and simplify)

$$= 1 - \text{antilog}\left(\frac{58\ 721246 - 60}{6}\right) = 1 - \text{antilog}\,(9\ 786874 - 10)$$

$= 1 - 0\ 61215 = 0\ 38785 = 38\ 785\%$, the annual rate of depreciation

The book value at the end of 4 years is $B\ V_4 = \$950\,(1 - 38\ 785\%)^4$

Tables showing the value for $(1 - r)^n$, that is, compound discount tables, are not readily available, since they have limited use Therefore to solve for the book value at the end of the fourth year it is necessary to employ logarithms Then $\log B\ V_4 = \log 950 + 4 \times \log\,(1 - r)$, where $\log\,(1 - r) = 9\ 786874 - 10$ (see calculation just made to find r)

$$\log B\ V_4 = 2\ 977724 + 4 \times (9\ 786874 - 10) = 2\ 977724 + 39\ 147496 - 40$$
$$= 2\ 125220$$
$$B\ V_4 = \$133\ 41$$

The depreciation schedule shows the annual depreciation, the total depreciation taken, and the book value

Depreciation Schedule

End of Year	*Annual Depreciation*	*Total Depreciation Taken*	*Book Value*
0	—	—	\$950 00
1	\$368 46	\$368 46	581 54
2	225 55	594 01	355 99
3	138 07	732 08	217 92
4	84 51	816 59	133 41
5	51 74	868 33	81 67
6	31 67	900 00	50 00

Sum of the digits method

The sum of the digits method of computing depreciation is somewhat similar to the constant-percentage method in that it results in a higher depreciation provision during the early years. Thus it may more nearly represent the decrease in value. The sum of the digits is determined by adding the figures representing the successive years of estimated life. Thus, using the same figures as before, an automobile costing \$3,500 has an estimated life of 3 years and a salvage value of \$500. The sum of the digits of the estimated life is $1 + 2 + 3 = 6$. This sum becomes the denominator in computing the fractional part of the value to write off each year. The numerator of the fraction used in determining the amount of depreciation each year is the number which represents the remaining life in years. Thus the first year the depreciation would be $\frac{3}{6}$ of \$3,000, or \$1,500; the second year it would be $\frac{2}{6}$ of \$3,000, or \$1,000; and the third year it would be $\frac{1}{6}$ of \$3,000, or \$500.

Illustration: A machine has an original cost of \$950, and a scrap value of \$50 after 6 years of use. Using the sum of the digits method, develop a depreciation schedule.

The sum of the digits is $1 + 2 + 3 + 4 + 5 + 6 = 21$. Depreciable value is \$900. The depreciation by years is $\frac{6}{21}$; $\frac{5}{21}$; $\frac{4}{21}$; $\frac{3}{21}$; $\frac{2}{21}$; $\frac{1}{21}$, respectively.

Depreciation Schedule

End of Year	*Annual Depreciation*	*Total Depreciation Taken*	*Book Value*
0	—	—	\$950.00
1	\$257.14	\$257.14	692.87
2	214.28	471.42	478.58
3	171.43	642.85	307.15
4	128.57	771.42	178.58
5	85.72	857.14	92.86
6	42.86	900.00	50.00

EXERCISE 14.5

Solve the following:

1. An asset which cost \$20,000 has an estimated life of 8 years and a scrap value of \$1,500. What is the annual depreciation under the straight-line method? Prepare a depreciation schedule.

2. A machine which cost \$1,850 has an estimated life of 6 years and a scrap value of \$200 Find the annual depreciation under the straight-line method and prepare a depreciation schedule

3. An asset which cost \$725 has an estimated life of 7 years and a scrap value of \$65 Find the annual depreciation under the straight line method, and the book value at the end of 3 years

4. A machine purchased for \$1,640 has an estimated life of 6 years and a trade in value of \$440 Find the annual depreciation under the straight-line method, and the book value at the end of 4 years

5. What must have been the estimated service life of an asset which cost \$4,800 which had an estimated salvage value of \$400 and an annual depreciation charge of \$550 under the straight line method?

6. What must have been the estimated service life of an asset which cost \$840, had an estimated salvage value of \$120 and an annual depreciation charge of \$72?

7. The steel tipple of a mine cost \$120,000 It is estimated that at the end of 12 years the ore in the mine will be exhausted, and that the tipple will have no salvage value If interest is assumed at 5% and the sinking fund method of apportioning depreciation is used, find the charge for depreciation the third year, and the book value at the end of 3 years

8. A public utility has built a power line at a cost of \$100,000 It is estimated that at the end of 20 years when the line is to be replaced by a much more expensive type of construction, the present line will have no salvage value In establishing the annual charge for depreciation, the sinking fund method is used with an assumed interest rate of 6% Find the depreciation charge the fifth year and determine the book value at the end of 5 years

9. An asset which cost \$1,850 has an estimated life of 5 years and trade in value of \$850 Using the sinking fund method with an interest rate of 6%, develop a schedule of depreciation

10. An asset which cost \$1,200 has a salvage value of \$200 at the end of 4 years Computing interest at 5%, develop a schedule of depreciation, using the sinking fund method

11. A piece of machinery costing \$12,500 has an estimated life of 5 years and a scrap value of \$1,500 Using the constant-percentage method of depreciation, construct a schedule of depreciation

12. A machine that cost \$750 has an estimated life of 4 years and a salvage value of \$150 Using the constant-percentage method of depreciation, construct a schedule of depreciation

13. A machine costing \$20,000 has an estimated life of 5 years and a trade in value of \$5,000. Construct a schedule showing the book value of the asset by straight-line depreciation; sinking fund depreciation, interest rate of 5%; and constant-percentage method.

14. A truck costing \$16,000 has an estimated life of 5 years and a salvage value of \$4,000. Construct a schedule showing the book value of the asset by straight-line depreciation; sinking fund depreciation, interest rate of 4%; and constant-percentage method.

15. Find the rate of depreciation under the constant-percentage method for an asset which cost \$1,200, has an estimated life of 10 years, and salvage value of \$200. Find the book value at the end of 7 years.

16. Under the constant-percentage method, find the rate of depreciation of an asset which cost \$180, has an estimated life of 6 years, and a salvage value of \$20. Find the book value at the end of 3 years.

17. An asset costing \$100 has no salvage value after an estimated life of 10 years. Prepare a depreciation schedule using the sum of the digits method.

18. A machine that cost \$1,200 has an estimated life of 5 years and a salvage value of \$200. Make a depreciation schedule showing the annual depreciation under both the straight-line method and the sum of the digits method.

19. The owner of a hotel adds \$10,000 worth of furniture with an estimated life of 8 years. Prepare a depreciation schedule for him using the sum of the digits method.

20. An investor has an opportunity to buy either of two apartment buildings, each with an estimated remaining life of 20 years. One is new; on this he may take depreciation, using the sum of the digits method. The other is more than 3 years old and must be depreciated on the straight-line basis. If the apartments cost \$100,000, and are to be fully depreciated at the end of 20 years how much more depreciation may be taken during the first 5 years on the new apartment than on the old?

Life Annuities and Life Insurance

Introduction

The purposes of this section are first, to introduce you to the theory of probability, second to discuss the mortality tables as an application of the probability theory, third, to consider the method of calculating elementary life insurance functions, using a mortality table and the mathematical theories previously presented

From this study you should improve your understanding of the purpose function, and operation of life insurance and life annuities For some students it will open a new vista for further study and employment Life insurance has been one of the most rapidly growing businesses in the American economy With the growth of pension plans and the growing complexity of life insurance, there is sure to be a great increase in the need for trained actuaries One result of these changes is that the future executives of any business must have more than a superficial understanding of the problems involved in pension planning, life insurance, and life annuities

Probability

In discussing the theory of probability, one term which appears frequently is *event* As a term it is difficult to define precisely, since it may refer to the throw of a die, the drawing of a card, the continuance of life, or the occurrence of death The term refers to the particular happening under discussion If an event can happen in h different ways and can fail in f different ways and all are equally likely to occur, the probability of its happening is shown by the formula

$$p = \frac{h}{h+f}$$

and the probability of its failing is

$$q = \frac{f}{h+f}$$

If, for example, a die is thrown, the probability of throwing a 3 is $\frac{1}{6}$ because there are 6 faces to the die and only 1 is numbered 3. Hence the probability that the event would happen is; $p = \frac{1}{1+5} = \frac{1}{6}$, since h is 1 and f is 5. The probability that it would fail is $q = \frac{5}{1+5} = \frac{5}{6}$.

Frequently when a comparison is made between the probability that an event will occur or will not occur, we speak of the *odds* in favor of or the odds against an event occurring. The relationship between p and q shows the odds in favor of an event and the relationship between q and p shows the odds against it. Thus the odds against throwing the 3 by one throw of a die are shown by the relationship between q and p or $\frac{5}{6} : \frac{1}{6}$, that is, the odds are 5 to 1 against throwing a 3 and p to q in favor of, or $\frac{1}{6} : \frac{5}{6}$, that is 1 in 5 of throwing a 3.

A review of these facts shows first that the sum of the probabilities of an event happening or failing is 1. That is,

$$p + q = \frac{h}{h+f} + \frac{f}{h+f} = \frac{h+f}{h+f} = 1$$

Hence $p + q = 1$.

From this fact, it follows that if p, the probability of an event occurring, is known, then the probability of its failing is also known, since $p + q = 1$, $q = 1 - p$, and $p = 1 - q$.

In the third place, it can be seen that the value of neither p nor q can be greater than 1 or less than 0. That is, if an event is certain to happen, $p = 1$ and $q = 0$. If it is certain to fail, $p = 0$ and $q = 1$.

Empirical probability

Empirical probability is distinguished from mathematical probability in that in mathematical probability the number of ways in which an event can occur, is either known or can be computed. In many problems of human affairs it is impossible to detail the ways in which an event can happen or fail. Under such circumstances the probability is determined by empirical methods of observing past experience. The greater the number of observations, the more likely the conclusions are to be correct.

Suppose, for example, that a careful analysis of the data from the United States Census from one decade to another showed that for every 100,000 young men 21 years of age in 1 year, there are only 99,770 young men 22 years old 1 year later. In other words, 230 out of every 100,000 men aged 21 have died. It would then appear reasonable to assume that, on the basis of past experience, a man aged 21 has 99,770 chances out of

100,000 of living to be 22 The probability of survival thus is 0 998 This is based on the assumption that nothing is known about the man except that he is 21 years of age and lives in this country

Let us suppose that a further study of the vital statistics were made and it was found that out of the group studied 10 000 either rode motorcycles, raced hotrods, practiced skindiving or carried on chemical experiments at home as a hobby Suppose also that this purely hypothetical study disclosed that 32 of the 10 000 failed to reach the age of 22 Knowing nothing further about a man aged 21 than the fact that he carried on one of these activities one could conclude that the probability of his surviving one year would tend to be reduced to 9,780/10,000 or 0 978 It should be considered that a prediction based on the smaller sample of 10,000 might be less dependable than one based on the study of 100,000

In empirical or statistical probability, if an event occurs h times in n trials, the relative frequency of the occurrence is designated as h/n, and the value is taken as the value of p, if based on a sufficient number of observations

Mathematical expectation

It frequently happens that the income which a person is to receive is contingent upon the occurrence of a specific event The probability then arises of the value of such an expectation If p represents the probability of the event and M represents the sum of money which is certain if the event occurs, then Mp is said to be the mathematical expectation If, for example, a man is to receive \$10 if a tossed coin comes up heads, the probability of heads coming up is $p = \frac{1}{2}$, and his mathematical expectation is $\$10 \times \frac{1}{2}$ or \$5

This concept of mathematical expectation plays an important role in much thinking relative to business and investment decisions Although there is often little attempt at an exact expression of the relationship between risk and income, the astute business man in making his decisions must keep a constant balance between the chance of gain and risk of loss, both applications of the principles of mathematical expectation

Independent events

The discussion so far has been concerned with the happening of single events If the occurrence of one event has no effect upon the occurrence of another, the events are said to be *independent* The man throwing a single die once has one chance of throwing a 3, and 5 chances of throwing other than a 3, $p = \frac{1}{6}$ The number that he throws the first time has no

influence at all on the number he will throw the next time. On the second throw of the die his probability of a 3 is still $\frac{1}{6}$.

What, however, is his probability of throwing a 3 twice in a row? The probability that two independent events will succeed is equal to the probability of success for the first, times the probability of success for the second. Thus his chance of throwing a 3 twice in succession is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$. Although the probability of throwing a 3 after having thrown it once is just the same regardless of the number of throws, since it is a completely independent event, the probability of two 3's in a row is $\frac{1}{36}$.

Dependent events

Two events are said to be dependent if the occurrence of one event affects the probability of the occurrence of the second event. For example, in drawing a card from a deck of 52 cards, the probability of drawing an ace on the first draw is $\frac{4}{52}$. If an ace is drawn and not replaced in the deck, the probability of drawing a second ace is $\frac{3}{51}$. If a second ace is drawn and not replaced, the chance of drawing a third ace is $\frac{2}{50}$. The conclusion is reached that the probability that two or more dependent events will succeed is equal to the product of the probability of the first, p_1; and after it is obtained, the probability of the second, p_2; and after it has been attained, the probability of the third, p_3. That is, the probability that these events will occur in the prescribed order is

$$p_1 \times p_2 \times p_3 \cdots \times p_n.$$

Mutually exclusive events

When the occurrence of one event excludes the occurrence of another, they are said to be *mutually exclusive.* Thus if a bowl contains 8 white, 7 black, and 5 red balls, what is the probability that a single ball drawn from the bowl will be black or red. The probability that the ball will be black is $\frac{7}{20}$, and that it will be red is $\frac{5}{20}$. Hence the probability that the ball which is drawn will be black or red is: $\frac{7}{20} + \frac{5}{20} = \frac{12}{20}$, or $\frac{3}{5}$. Thus it is said that if n mutually exclusive events have the separate probabilities of $p_1, p_2, p_3, \ldots, p_n$, then the probability that one of the events will occur is

$$p_1 + p_2 + p_3 + \cdots + p_n$$

Illustration: Your state has two United States senators. If the probability that the senior senator will live to the expiration of his term is 0.8, and the probability that the junior senator will live to the expiration of

his term is 0.75, (a) what is the probability that they will both serve out their terms, (b) what is the probability that at least one will die during his term?

a *If it is assumed that the death or survival of one in no way depends* upon the death or survival of the other, the two events would be classed as independent events. Thus $p = p_1p_2$ where $p_1 = 0.8$ and $p_2 = 0.75$. Therefore $p = 0.8 \times 0.75 = 0.6$, which is the probability that both will live out their term in the Senate.

	Living	*Dying*
b. Probability of the senior senator	0.8	0.2
Probability of the junior senator	0.75	0.25

There are four possibilities:

That both will live	$0.8 \times 0.75 = 0.60$
That both will die	$0.2 \times 0.25 = 0.05$
That the senior senator will live and the junior senator will die	$0.8 \times 0.25 = 0.20$
That the senior senator will die *and the junior senator will live*	$0.2 \times 0.75 = 0.15$
	1.00

The probability that only one will die (two mutually exclusive events) is $0.20 + 0.15 = 0.35$.

EXERCISE 15.1

1. The probability that a person aged 21 will live to be 75 is 0.33290. What is the probability he will not live to be 75?

2. The probability that a man aged 50 will die between the ages of 65 and 70 is 0.15210. What is the probability that he will not die between the ages of 65 and 70?

3. The probability *that a man aged 20 will live at least 25 years is* 0.93. The probability that a man aged 50 will live 25 years is 0.38967. What is the probability that a father aged 50 and a son aged 20 will (a) both live 25 years, (b) both die within 25 years, (c) that the father will live at least 25 years and the son not, (d) that the son will live at least 25 years and the father not?

4. The probability that a child aged 10 will die before reaching age 11 is 0.00197. The probability that a man aged 45 will die before age 46 is 0.00861. What is the probability that a father aged 45 and a son aged 10 will (a) both die within a year, (b) both live for a year?

5. The probability that a woman aged 21 will live to reach age 75 is 0.479. The probability that a man aged 26 will reach age 75 is 0.338. What is the probability that a young married couple aged 26 and 21 will both be alive at age 75?

6. Of 955,942 students graduating from high school at age 18, a total of 946,789 were alive 4 years later. What was the probability of death between ages 18 and 22?

7. The probability that a person aged 20 will live at least 25 years more is 0.9317. Assume that the average age of a graduating class was 20. How many persons out of a high school graduating class of 100 (average age 20) would you expect to be alive for the 25th reunion?

8. The probabilities that three men aged 25, 30, and 35 will live to be 50 years of age are 0.863, 0.877, and 0.894, respectively. What is the probability that: (a) all three will survive to age 50; (b) the younger two will survive and the eldest die; (c) the oldest and youngest will survive and the middle one die?

9. The probability that a man aged 25 will live at least 25 years is 0.863. The probability that a man aged 50 will live to be 75 is 0.390. What is the probability that a man aged 25 will live to be 75?

10. The probability that a child aged 10 will live to be 35 is 0.933. The probability that a man aged 35 will live to be 70 is 0.50104. What is the probability that a child aged 10 will live to be 70?

Mortality tables

If it were possible to predict when the average persons was going to die, there would be no life insurance business, for life insurance is based on the principle that what is an unknown risk to each member of a group becomes a known risk for the group.

The shift from the unknown to the known is based on the assumption that a study of the length of life of a large group will furnish information sufficiently accurate to predict the probable number in the stated group who will die within any specific year. Thus no prediction is made of when a given person will die, but the probability of death at a given age can be computed.

There have been several tables of mortality developed and used by insurance companies in this country. In most states the tables which may be used are defined by law. Most insurance currently issued in this country is based on a table known as the *Commissioners 1941 Standard Ordinary Mortality Table*, generally referred to as the *CSO Table.*

This table was developed from the experience of life insurance companies in the period 1930–1940. The observed rates of mortality were somewhat arbitrarily raised to provide a factor of safety for determining the probability of death. It is interesting to note that an arbitrary increase in the actual mortality rate results in a conservative mortality table. If the actual rate of mortality decreases over a period of years, the table becomes more conservative.

On the other hand, it should be considered that the number of pension funds is growing. If distribution made to retired workers is to be determined on the basis of a conservative mortality table, the probable span of life will be underestimated, the size of the annual payment will be overestimated, and the fund will be depleted. Similarly insurance companies selling annuity contracts will lose. Thus a conservative mortality table would be a disastrous basis for issuing annuities.

These facts are recognized and the rates of annuities sold by insurance companies are based on the *1937 Standard Annuity Table*. It appears reasonable to assume that another table for computing annuities will supplement the present table.

How to use the mortality table

In constructing a mortality table the members of the group are classified by age. Thus the assumption is that 1,023,102 are selected at birth. Of the number 1,000,000 live to the age of 1. Thus for all intents and purposes we may say that the table begins with 1,000,000 persons aged 1. Of this group 994,230 live to age 2. The probability that a person aged 1 will die within a year is 0.00577. Only 990,114 of the 994,230 persons aged 2 live to be 3. In other words, 4,116 die between their second and third birthdays. The probability of death then between ages 2 and 3 is 4,116 divided by the number living at age 2, here 994,230. The probability of death is 0.00414.

In the CSO table the limit of the table is age 99. That is, the assumption is made that for statistical purposes the last of the original 1,000,000 persons at age 1 will succumb by age 100. Although it is known that some may live beyond 100, the percentage is too small to be significant in determining insurance rates.

To utilize time and space, actuaries use standard symbols to represent the facts presented in a mortality table. The age of a person is indicated by x. Thus the column headed l_x shows the number of persons from the original group who have survived to a given age of x. If the age shown in the left hand column is 20, the figure in the column l_x of 951,483 indicates the number of the original group who lived to age 20.

At the end of a year a person aged x will be $x + 1$ years old. Of the 951,483 persons aged 20, not all will reach age 21. Indeed the table shows that at age 21 there were only 949,171 alive. The difference of 2,312 shows the number dying after reaching age 20, but before reaching age 21. The symbol used to represent the number dying at age x is d_x. That is, $d_{20} = 2{,}312$.

If $l_{20} = 951{,}483$ and $l_{21} = 949{,}171$, the probability of a person aged 20 living to age 21 is the quotient $l_{21} \div l_{20}$, or $949{,}171 \div 951{,}483$. This relationship is represented by p_{20}. That is, $p_x = \frac{l_{x+1}}{l_x}$.

The number of deaths, d_x, divided by l_x, the number reaching age x, is represented by the symbol q_x. It shows the probability at age x of dying before attaining the age of $x + 1$.

$$q_x = \frac{d_x}{l_x}$$

The symbol l_x is also used to represent a single person aged x, and is read a life aged x. All those alive at age x will either die before reaching age $x + 1$, and be included in the d_x column, or live and be included in the number l_{x+1}.

Thus $p_x + q_x = 1$, since $l_{x+1} + d_x = l_x$.

The probability that a life aged x will live at least n years from the time he attains the precise age of x, is represented as

$${}_np_x = \frac{l_{x+n}}{l_x}$$

The probability that a person will die during the period from age x to age $x + n$, is represented by the symbol ${}_nq_x$. Since ${}_np_x + {}_nq_x = 1$,

$${}_nq_x = 1 - {}_np_x = 1 - \frac{l_{x+n}}{l_x} = \frac{l_x - l_{x+n}}{l_x}$$

The symbol ${}_{m|n}q_x$ denotes the probability that a life aged x will live m years but die in the next n years, that is, the probability that a person aged x will live until age $x + m$, and will die between ages $x + m$ and $x + m + n$. From the study of probability we know this is the product of the probability of living m years:

$${}_mp_x = \frac{l_{x+m}}{l_x}$$

and the probability of dying between ages $x + m$ and $x + m + n$:

$${}_nq_{x+m} = \frac{l_{x+m} - l_{x+m+n}}{l_{x+m}}$$

Thus $${}_{m|n}q_x = {}_mp_x \cdot {}_nq_{x+m} = \frac{l_{x+m}}{l_x} \times \frac{l_{x+m} - l_{x+m+n}}{l_{x+m}} = \frac{l_{x+m} - l_{x+m+n}}{l_x}$$

These notations are sufficiently important that they must be studied until they can be read with ease You should remember that

1 *The right hand subscript represents the age of the life under consideration*

2 The left-hand subscript represents the duration of time in years in which the event is to take place

3 The letter to the left of the bar shows the period of deferment

Here p is used to represent the probability of living, and q is used to represent the probability of dying

Illustrations

a The symbol p_{18} is read as the probability that a life aged 18 will live to be 19

b q_{20} represents the probability that a life aged 20 will die before reaching age 21

c ${}_5p_{18}$ represents the probability that a life aged 18 will live 5 years, that is, to 23

d ${}_5q_{20}$ represents the probability that a life aged 20 will die before reaching age 25

e ${}_{5|}q_{20}$ represents the probability that a life aged 20 will die between the ages of 25 and 26

f ${}_{5|10}q_{20}$ represents the probability that a life aged 20 will die between the ages of 25 and 35

EXERCISE 15.2

State in words the probabilities represented by the following symbols

1. p_{40}	**6.** ${}_5q_{18}$	**11.** ${}_{6|}q_{18}$
2. p_{20}	**7.** ${}_{10}q_{21}$	**12.** ${}_{4|}q_{46}$
3. ${}_{10}p_{21}$	**8.** q_{20}	**13.** ${}_{5|10}q_{35}$
4. ${}_5p_{55}$	**9.** q_{45}	**14.** ${}_{5|10}q_{60}$
5. ${}_5p_{65}$	**10.** ${}_{5|}q_{30}$	**15.** ${}_{20|10}q_{40}$

Show in symbols the following

16. The probability that a life aged x will live 1 year

17. The probability that a life aged x will die within 1 year

18. The probability that a woman aged 35 will live to be 40 and die between the ages of 40 and 45

19. The probability that a man aged 24 will die between the ages of 30 and 31

20. The probability that a man aged 40 will die between the ages of 49 and 50.

The preceding drill on the symbols and their interpretation should furnish sufficient acquaintance with their meaning to permit you to solve many problems from the tables.

Illustrations:

a. Express but do not solve the probability that a life aged 16 will live to age 21.

Of the l_{16} alive at age 16, l_{21} will be alive at age 21. ${}_5p_{16} = \frac{l_{21}}{l_{16}} = \frac{949{,}171}{960{,}201}$, from the table.

b. What is the probability that a person aged 18 will be alive at age 19?

Of the l_{18} persons alive at age 18, l_{19} will still be alive the next year. $p_{18} = \frac{l_{19}}{l_{18}} = \frac{953{,}743}{955{,}942}$, from the table.

c. What is the probability that a person aged 20 will die within 1 year?

Of the l_{20} persons alive at age 20, d_{20} will not survive to age 21. $q_{20} = \frac{d_{20}}{l_{20}} = \frac{2{,}312}{951{,}483}$, from the table.

d. What is the probability that a person aged 22 will die between the ages of 55 and 65?

Of the l_{22} persons alive aged 22, l_{55} will be alive at age 55, and l_{65} at age 65. Hence of the l_{22} persons, $l_{55} - l_{65}$ will die between ages of 55 and 65. The required probability is 33 ${}_{10}q_{22} = \frac{l_{55} - l_{65}}{l_{22}} = \frac{754{,}191 - 577{,}882}{946{,}789}$, from the table.

Expectation of life

There is a natural curiosity on the part of a person seeing a mortality table for the first time to question what the table shows about his own expectation of life. The person who understands a mortality table and its uses, is fully aware that the table tells nothing definite about any single person. It does show, however, the number of persons who have attained the age x—that is, the age of the questioner—and it also shows that the complete number of years this group of persons age x will probably live will be the sum $l_{x+1} + l_{x+2} + l_{x+3} + \ldots$ to the end of the table. When this total number of years is equally divided among those living at age x, it shows the average number of complete years that will be

lived by each member of the group The expectation of life of a person aged x measured in complete years is represented by the symbol e_x

If it is assumed that the number of deaths is evenly distributed throughout each year, it can be expected that the average person will have lived $\frac{1}{2}$ year beyond his birthday at the time of death Therefore his life expectancy is approximately $\frac{1}{2}$ year more than e_x The average number of years including fractions that a person aged x can be expected to live in the future is called the *complete expectancy of life* and is denoted by the symbol $\mathring{e}_x$ Since the average person will probably live approximately $\frac{1}{2}$ year beyond his birthday we can say that

$$\mathring{e}_x = e_x + \tfrac{1}{2} \text{ (approximately)}$$

The column in the CSO Table headed $\mathring{e}_x$ shows the complete expectation of life for a life aged x This column is not used directly in determining the cost of insurance or annuities

EXERCISE 15 3

1. What is the probability that a boy of 15 will live to age 16?
2. What is the probability that a boy of 15 will die between the ages of (a) 25 and 30, (b) 45 and 50?
3. What is the probability that a man aged 21 will live 5 years?
4. What is the probability that a girl of 18 will live to age 23?
5. What is the probability that a student aged 19 will live 5 years and die within the next 5 years?
6. Compare the probability that a boy aged 15 will live to be 70, with the probability that a man aged 50 will live to be 75
7. From the CSO Table make a chart which shows the probability of death every 10 years from age 10 to age 90
8. What is the probability that Charley s rich aunt, aged 65, will (a) live to be 100, (b) die before she is 75, (c) live to be 75, but die before she is 80?
9. What is the probability that Charley, aged 21, will (a) live to be 46, (b) live to be 65, (c) live to be 65, but die before he is 66?
10. What is the probability that Charley, aged 21, and his aunt, aged 65, will both (a) be alive 20 years hence, (b) be dead in 20 years?

Life annuities

To offset the financial loss resulting from death, a life insurance company issues a contract, known as a *policy*, under which it agrees to pay a stipulated sum at the death of the person named in the policy When

an insurance company issues an ordinary whole life policy it knows that at some time in the future it will have to pay the face amount of the policy, since the death of every person it insures is bound to occur sooner or later.

Frequently people are faced with the problem of managing large sums of money. Perhaps it is the proceeds from an insurance policy, or perhaps it is an amount accumulated over a long period of years. A person seeking financial security in his old age may be forced to use not only the income he receives but his principal as well. If his future income must come from the principal he now possesses, he must determine how much he may reasonably spend each year in order to meet his needs during his lifetime. Even if he does not care about leaving an estate but plans to use all his principal and income in meeting his expenses, he may find either that by spending too much he has used up his fund and left himself destitute, or that by spending too little he has denied himself many comforts which he could well have afforded.

Insurance companies help to solve such problems. Legally a *life annuity* is a contract with an insurance company under which the company agrees, in exchange for a given sum of money, to pay periodically a fixed amount of money to a person designated as the *annuitant*, so long as he may live.

A person interested in buying an annuity will usually pay particular attention to his own family history of longevity and consider seriously his chance of gain through the purchase of the annuity. Suppose, for example, that a man aged 60 decides to retire. He has some savings, an income from a pension fund, and he believes that additional income of $100 a month will make it possible for him to maintain an adequate standard of living. At a cost of $18,000 he buys from an insurance company an annuity which will pay him $100 a month for the rest of his life.

With the $18,000 invested at $2\frac{1}{2}\%$ the company could pay monthly installments of $100 for 18 years 9 months and just break even. Even if an annuitant lives beyond the estimated number of years, payments to him under the life annuity continue. The insurance company making the payments draws funds from three sources: first, the interest received on the original amount paid to them by the annuitant; second, the principal amount paid by the annuitant; and third, the money paid to the company by annuitants whose deaths occur before the sum they have contributed to the company has been repaid to them in full.

Not many people will buy a life annuity who do not expect to live long enough to recover their principal. Hence there is a natural selection of persons who expect to live longer than the average. As a result insurance

companies do not use the same mortality tables for determining the rate to be charged on life insurance policies, and the payment to be made by them on annuity contracts It is perfectly logical for such a distinction to be made In the text, however, the CSO Table is used for both annuities and life insurance policies, since the purpose is to teach and illustrate methodology The 1937 standard annuity table now in use for determining annuity rates is considered to be out of date, and will probably be superseded in a relatively few years

Life annuities and annuities certain

The mathematical aspects of life annuities are derived from the principles previously developed in discussing annuities certain, probabilities, and mathematical expectations Three types of problems were discussed in annuities certain finding the periodic payment, the amount, and the present worth of both ordinary annuities and annuities due Payments under a life annuity continue only during the life of a given person, the annuitant, therefore the number of payments is not certain, but is contingent upon the life of the annuitant Life annuities are sometimes classed as *contingent annuities* Since payments are made only while the annuitant lives, the present value of such future payments will depend on the probability of the annuitant s living It is perhaps necessary to repeat again that by the use of the mortality table the company can predict how many of a given group will die each year The question of who will die is still unanswered, but the question of how many will die can be determined

The value of a life annuity will thus depend on (1) the periodic sum to be received, (2) the rate of interest at which it is evaluated, and (3) the probability of the annuitant's living

In our consideration of life annuities it is assumed that the valuation, which is referred to as the *net single premium*, the *present value*, or the *net purchase price*, is the same whether viewed from the standpoint of the annuitant or the insurance company It is implicit in such an assumption that there are no expenses, losses, profits, or costs of operation for the insurance company, and that all premiums and interest income received by the seller of the life annuities are returned to the annuitant

Patently such an assumption is not warranted as a matter of practical application At the same time it must be recognized that in the development of such a type of operation this would be the logical point at which to begin That is, first it would be desirable to estimate exactly the cost of the life annuities To this would be added estimates of expenses of selling, operating, and administration Then the rates would be deter-

mined at a level sufficiently high to furnish some degree of safety and some assurance of profitable operation.

A second assumption implicit in the use of the CSO Tables presented in the appendix is that the company is able to earn at $2\frac{1}{2}\%$ converted annually on any funds left with it. This is a conservative rate presently adopted by the majority of companies.

A third assumption is that the mortality tables in use are accurate, that is, that annuitants will not live beyond the periods shown in the table. As pointed out earlier, because of the natural selection by people who expect to live longer than average, mortality tables may not be strictly accurate when applied to annuitants.

Ordinary whole life annuity

In discussing annuities certain the ordinary annuity was defined as one in which the first payment was made at the end of the first year. Similarly an ordinary whole life annuity, or a whole life annuity immediate, is a series of payments which begin at the end of the first year and continue so long as the annuitant lives. The symbol used for the present value of a series of future payments is $a_{\overline{n}|i}$. If each payment will be made only if the person involved is alive, the present value of each payment, such as the one at the end of k periods, will be multiplied by the probability of living $\frac{l_{x+k}}{l_x}$. Thus the present value is the sum of a series of values.

Actuaries are interested in saving time. Since the CSO Tables cover only 100 years, it is a fairly simple matter to compute a table showing the present value or the net single premium for a whole life annuity immediate of 1 per year issued to a person aged x. The work is further simplified by the fact the table assumes only one rate of interest, namely $2\frac{1}{2}\%$.

If it is assumed that l_x persons all aged x want to establish a fund from which each will receive \$1 annually so long as he shall live, the total contribution would then be l_xa_x if a_x is the cost to each annuitant. Here is an example of an equation of payment in which the amount in the fund is equal to the present value of all future payments to be made from the fund.

$$l_xa_x = vl_{x+1} + v^2l_{x+2} + v^3l_{x+3} + \cdots$$

to the end of the table.

To complete the calculation of tabular values, multiply both sides by v^x and solve for a_x.

$$v^x l_x a_x = v^x (v l_{x+1} + v^2 l_{x+2} + v^3 l_{x+3} + \quad)$$

or
$$a_x = \frac{v^{x+1} l_{x+1} + v^{x+2} l_{x+2} + v^{x+3} l_{x+3} +}{v^x l_x}$$

Commutation symbols

Such a formula is a cumbersome and difficult device with which to work Actuaries have found that much time can be saved by adopting a single notation to represent the more complicated symbols Thus, rather than writing the product $v^x l_x$, the symbol D_x is used to stand for the product, and the tabular values for all possible $v^x l_x$ values at $2\frac{1}{2}\%$ are calculated and included in the table The symbol D_x is referred to as a *commutation symbol*

When the symbol of D_x is substituted in the formula for $v^x l_x$, the value of a_x can be written

$$a_x = \frac{D_{x+1} + D_{x+2} + D_{x+3} + \quad \text{to the end of the table}}{D_x}$$

It is still necessary, even with the values of D_x already known, to add up the values of D_x from the age following the age of issue of a life annuity, that is, from D_{x+1} to the end of the table, in order to find the present value of a whole life annuity at age x These values too can be computed and added as another column to the table by beginning at the end of the table and adding each earlier year to the previous total

The symbol N_x has been adopted as the commutation symbol to represent these summations of all consecutive values of D_x

An illustration of the computation of the tabular values of D_x and N_x for the upper reaches of the table should be helpful in understanding them To compute the value of D_x when x is equal to 99, it is necessary to know the value of v^{99} and l_{99} The value of the first is found in the compound interest tables in which $(1 + 2\frac{1}{2}\%)^{-99}$ is found to be 0 08676355 The value of l_{99} from the CSO Table is 125 Their product is 10 84544377 ($125 \times 0\ 08676355$) This is the value for D_{99} In the table under the commutation symbol of D_x for 99 years the value has been rounded to 10 845444 The tabular value of D_{98} is equal to $v^{98} l_{98} = 0\ 08893264 \times 454 = 40\ 375419$ The tabular value of D_{97} is equal to $v^{97} l_{97} = 0\ 09115596 \times 1{,}005 = 91\ 611740$

If the value of N_x is to be the summation of the D_x values, then $N_{99} = D_{99} = 10\ 845444$, and $N_{98} = N_{99} + D_{98} = 10\ 845444 + 10\ 375419 = 51\ 2209$, and $N_{97} = N_{98} + D_{97} = 51\ 2209 + 91\ 6117 = 142\ 8326$

In order to compute the present value or net single premium of a whole life annuity it is necessary to use only the simple formula

$$a_x = \frac{N_{x+1}}{D_x}$$

The values for N_x and D_x are shown in the commutation columns of the CSO Table in the appendix. With these columns, the present worth at $2\frac{1}{2}\%$ of an ordinary whole life annuity for any age can be found by looking up the value of N_{x+1} and D_x and performing one division.

Illustration: A person aged 20 wants to purchase a whole life annuity of $2,500 per year. What will be the net single premium?

Here $x = 20$ and $x + 1 = 21$. The value for $N_{21} = 15{,}163{,}553$, and the value for $D_{20} = 580{,}662.42$.

$$a_{20} = \frac{N_{21}}{D_{20}} = \frac{15{,}163{,}553}{580{,}662.42} = 26.1142$$

The value of an annuity of $1 issued at age 20 would be 26.1142. The value of an annuity of $2,500 per year would be $\$2{,}500 \times 26.1142 =$ $65,285.50.

EXERCISE 15.4

1. Using the compound amount table and the CSO Table, verify the tabular value for D_x when $x = 25$.

2. Using the compound amount table and the CSO Table verify the tabular value for D_x when $x = 50$.

3. Compute the net single premium for a whole life annuity of $2,500 per year issued at age 50.

4. A man aged 24 is to receive $1,500 a year beginning 1 year hence for the rest of his life. What is the present worth of the annuity?

5. The widow of a reserve officer is to receive a life pension of $2,000 annually from the government so long as she does not remarry. If she marries on her thirtieth birthday, what is the present value on that date of the future income she has given up?

6. Ellen Sterling inherited a life interest in an estate of $258,000 at age 21. That is, she is to receive the income from the estate annually so long as she lives. If interest averages 4%, what is the value of her inheritance when she receives it?

7. A man seeks an income of $3,000 a year at age 65. What is the net single premium for a whole life annuity immediate for such an amount?

8. A student aged 21 took out a life insurance policy on which he agreed to pay $100 a year at the end of each year so long as he lived. What single payment at age 21 would have been equivalent to his obligation to the insurance company?

9 An instructor aged 32 agreed to make annual contributions to a pension plan of \$250 at the end of each year so long as he lived What single payment at age 32 would be equivalent to his payments to the pension fund?

10 A girl aged 18 won a contest which will pay her \$1,000 at the end of each year for life What is the value of her winnings now?

Other types of life annuities

The expression *whole life* when applied to annuities refers to the term and indicates that payments to the annuitant will continue for the remainder of his life The expression *ordinary*, indicates that payment is to be made at the end of one period and periodically thereafter If the annuity is classed as an *annuity due*—and many life annuities are so classed—the first payment is to be made at once and periodically thereafter The symbol $\ddot{a}_x$ is used to refer to a whole life annuity due Since payments begin one period earlier than in an ordinary life annuity, the present value is the same as for an ordinary life annuity plus one payment now Thus

$$\ddot{a}_x = a_x + 1$$

The first payment is made immediately for a life annuity due Hence there is no risk that the annuitant will not receive it The net single premium would be found by using the preceding formula

$$\ddot{a}_x = a_x + 1 = \frac{N_{x+1}}{D_x} + 1 = \frac{N_{x+1} + D_x}{D_x} = \frac{N_x}{D_x}$$

the present value of an annuity due of \$1 per year

Illustration Find the net single premium for a whole life annuity due of \$2,400 a year for a man at age 45

Here $x = 45$, so the net single premium is

$$\$2{,}400\ddot{a}_{45} = \$2{,}400\frac{N_{45}}{D_{45}} = \$2{,}400 \times \frac{5\ 161{,}996}{280{,}638\ 95} = 18\ 393 \times \$2{,}400$$
$$= \$44{,}143\ 20$$

The formulas $A = Ra_x$ and $\ddot{A} = R\ddot{a}_x$ can be solved for R if A or $\ddot{A}$ are known Thus

$$R = \frac{A}{a_x} = A \times \frac{D_x}{N_{x+1}}, \quad R = \frac{\ddot{A}}{\ddot{a}_x} = \ddot{A} \times \frac{D_x}{N_x}$$

Illustration A man aged 50 buys a whole life annuity due for \$12,000 How much will he receive at the beginning of each year for life?

Here $x = 50$ and $\ddot{A} = \$12{,}000$. Thus

$$R = \$12{,}000 \frac{D_{50}}{N_{50}} = \$12{,}000 \times \frac{235{,}925.04}{3{,}849{,}488} = \$735.44$$

Often it is wise to purchase a life annuity, or to evaluate such an annuity, on which the payments do not begin for some time in the future. If the first payment is to occur more than one year after the specified age x, it is called a *deferred life annuity.* The period of deferment is represented by the letter k, and the present value is denoted by $_{k|}\ddot{a}_x = \frac{N_{x+k}}{D_x}$.

The preceding formula shows the present value for a whole life annuity deferred whose first payment occurs k years hence. This would thus correspond to a k-year deferred whole life annuity due.

Illustrations:

a. When Professor Jackson was 50 years old he gave up a teaching position to become a United States senator. At that time he had contributed \$30,000 to a retirement fund. What will be the size of the payments which he will receive from a whole life annuity purchased with the \$30,000, the first annual payment to be made when he reaches age 65?

Here $x = 50$; $k = 15\,(65 - 50)$; and $_k\ddot{A}_x = \$30{,}000$. Thus

$$\$30{,}000 = R \times {_{15|}}\ddot{a}_{50} = R \times \frac{N_{65}}{D_{50}} \quad \text{and} \quad R = \$30{,}000 \times \frac{D_{50}}{N_{65}}$$

Therefore $R = \$30{,}000 \times \dfrac{235{,}925.04}{1{,}172{,}130} = \$6{,}038.37.$

b. Find the present value to a man aged 35 of an annuity of \$2,400 a year beginning at age 60?

Here $x = 35$; $x + k = 60$; and $R = \$2{,}400$. Therefore

$$_{25|}\ddot{A}_{35} = \$2{,}400 \times \frac{N_{60}}{D_{35}} = \$2{,}400 \times \frac{1{,}865{,}614}{381{,}995.63} = \$11{,}721.27.$$

Under the conditions of some life annuity contracts the payments continue for a stipulated number of years, or until the death of the annuitant, whichever is first. Thus a provision may be made that a widow with a minor child is to receive a payment of so many dollars per year for 15 years or until her death, whichever is first. Such an annuity is called a *temporary life annuity.* The present value of an n year temporary life annuity of 1 per year issued to a life aged x is denoted by the symbol $a_{x:\overline{n}|}$. The formula for determining its present value is

$$a_{x:\overline{n}|} = \frac{N_{x+1} - N_{x+n+1}}{D_x}$$

The present value for an n-year temporary whole life annuity due of 1 per year to a person aged x is

$$\ddot{a}_{x\,\overline{n|}} = \frac{N_x - N_{x+n}}{D_x}$$

Under certain circumstances it might be desirable to have an n-year temporary whole life annuity deferred k years The present value of such an annuity to a person aged x is

$$_{k|}a_{x\,\overline{n|}} = \frac{N_{x+k} - N_{x+k+n}}{D_x}$$

Illustrations

a Find the present value of a temporary life annuity of 10 annual payments of \$1,000 each to a man aged 20 if the first payment is due at the end of the year

Here $x = 20$, $n = 10$, and $R = \$1{,}000$ Thus

$$A_{20\,\overline{10|}} = \$1{,}000 \times \frac{N_{21} - N_{31}}{D_{20}} = \$1{,}000 \times \frac{15{,}163{,}553 - 10{,}153\ 180}{580{,}662\ 42}$$
$$= \$8{,}628\ 21$$

b Find the present value of a temporary life annuity of 10 annual payments of \$1,000 each to a man aged 20 if the first payment is due immediately

Here $x = 20$, $n = 10$, and $R = \$1{,}000$ Thus

$$\ddot{A}_{20\,\overline{10|}} = \$1{,}000 \times \frac{N_{20} - N_{30}}{D_{20}} = \$1{,}000 \times \frac{15{,}744{,}216 - 10{,}591{,}280}{580{,}662\ 42}$$
$$= \$8{,}869\ 09$$

c In order to assure that his son will be able to go to college, a father establishes a temporary 4-year life annuity to pay \$2,000 a year to his son, beginning at age 19 What is the present value when the son is aged 10?

Here $x = 10$, $n = 4$, $k = 9\,(19 - 10)$, and $R = \$2{,}000$ Thus

$$_{9|}\ddot{A}_{10\,\overline{4|}} = \$2{,}000 \times \frac{N_{19} - N_{23}}{D_{10}} = \$2{,}000 \times \frac{16{,}340{,}808 - 14{,}018{,}471}{759{,}171\ 73}$$
$$= \$6{,}039\ 04$$

General annuity formula

The life annuity formulas already presented include

Present value of an ordinary whole life annuity of 1. $a_x = \dfrac{N_{x+1}}{D_x}$

Present value of a whole life annuity due of 1. $\ddot{a}_x = \dfrac{N_x}{D_x}$

Present value of a deferred whole life annuity of 1, first payment to be made k years hence: $_{k|}\ddot{a}_x = \dfrac{N_{x+k}}{D_x}$

Present value of an n-year temporary life annuity immediate of 1: $a_{x:\overline{n}|} = \dfrac{N_{x+1} - N_{x+n+1}}{D_x}$

Present value of an n-year temporary life annuity due of 1: $\ddot{a}_{x:\overline{n}|} = \dfrac{N_x - N_{x+n}}{D_x}$

Present value of a deferred n-year temporary life annuity of 1, first payment to be made k years hence: $_{k|}\ddot{a}_{x:\overline{n}|} = \dfrac{N_{x+k} - N_{x+k+n}}{D_x}$

The similarity of these formulas is striking. In each case the denominator is D_x, in which x corresponds to the age of the annuitant at the time of valuation. It should also be observed that the value of the subscript to the first N symbol in the numerator corresponds to the age of the annuitant at the time he is to receive or the insurance company to make the first payment if he is alive.

When there is a second term in the numerator it shows the value of the subscript to N corresponding to the age of the annuitant 1 year *after* the last payment is made or received. In the case of whole life annuities the value of the second N term is obviously at the end of the mortality table and 0, and hence no deduction is made.

On the basis of these relationships a general formula for finding the present value of any life annuity may be presented as:

$$\text{Present value} = R \times \frac{N_y - N_z}{D_x}$$

Where x = age of annuitant on the date of the valuation, y = age of annuitant at the time the first payment is made by the insurance company, or received by the annuitant, z = age of annuitant one year after the date of last payment. $N_z = 0$ in the case of whole life annuities.

Illustration: A man aged 34 is to receive \$2,400 a year starting at age 60 for a period of 15 years. What is the present value of the life annuity?

Here $x = 34$; $y = 60$; $z = 75$ $(60 + 15)$. Hence

$$\text{Present value} = \$2{,}400 \times \frac{N_{60} - N_{75}}{D_{34}} = \frac{1{,}865{,}614 - 324{,}618.9}{393{,}256.29} = \$9{,}404.52$$

If alive he will receive \$2,400 when he is 60, 61, 62, and so on. His last payment will be received when he is 74, but he will receive nothing when he is 75 even if still alive.

EXERCISE 155

Find the present value of a life annuity of $1,000 a year if

1. $x = 35$, $y = 36$, $z = 100$
2. $x = 42$, $y = 42$, $z = 100$
3. $x = 28$, $y = 66$, $z = 100$
4. $x = 30$, $y = 31$, $z = 70$
5. $x = 27$, $y = 64$, $z = 90$

6. A man now 34 is to receive $1,200 a year for the rest of his life, starting 1 year hence What is the present value of this annuity?

7. A boy 14 years of age is to receive $800 a year for life, the first payment to be made on his twenty first birthday Find the present value

8. A man aged 42 agrees to pay $200 a year for the next 20 years if he is alive What single cash payment made today would be of equivalent value on the basis of the CSO Table?

9. A man aged 27 will receive $1,500 a year starting at age 54 for a period of 20 years What is the present value of this annuity?

10. When first married, a man took out a policy which provided upon his death for the payment of $2,500 per year to his wife for the remainder of her life He died at the age of 54 and his wife was still living If she were 3 years younger than he, what was the value of the annuity at the time of his death if the first payment were made immediately?

11. In settlement for damages in an accident claim, a girl of 20 is to receive $2,500 a year for life, the first payment to be made immediately What would have been an equivalent cash settlement?

12. By making an annual gift of $3,000 to his son aged 31, and annually thereafter, a father was able to reduce his taxes For 10 years the gift was placed in a savings account which paid $2\frac{1}{2}\%$ payable annually At the time of the tenth payment the money was to be used to purchase a whole life annuity payable at age 65 What would be the amount of the annual payments to be received by the son if he lived to be 65?

13. A man aged 60 has an option of selecting a pension of $2,500 a year for 10 years beginning at once, or a life annuity of $2,000 beginning 1 year hence If all payments are contingent on his survival, which has the greater present value?

14. A man aged 30 has $10,000 to invest in a life annuity What annual payments would he receive if he purchased (a) a whole life annuity immediate, (b) a whole life annuity starting at age 64?

15. A man aged 27 has $5,000 to invest in a life annuity What annual payments would he receive if he purchased (a) a whole life annuity starting at age 61, (b) a deferred 15-year temporary life annuity, first payment to be made at age 61?

Pure endowment

In the discussion of life annuities, the assumption is that more than one payment is to be made in the future to a man if he is still alive. A single payment to be made to a certain person at the end of a specified number of years if he is still alive is defined as a *pure endowment*. The present value of A dollars which a person will receive n years hence, if he survives, will depend upon the rate of interest and the probability that he will be alive at that time. It has already been shown that the probability a life aged x of surviving n years is $\frac{l_{x+n}}{l_x}$. In the chapter on compound interest it was shown that the present value of a future sum is $v^n = (1 + i)^{-n}$. Thus the present value of 1 payable in the future if a life aged x survives n years is $v^n \times \frac{l_{x+n}}{l_x}$. Multiplying both numerator and denominator by v^x, we have

$$\frac{v^{x+n}l_{x+n}}{v^x l_x}$$

The symbol used to denote the value of an n-year pure endowment for a life aged x is ${}_nE_x$. Thus stated in terms of commutation symbols,

$${}_nE_x = \frac{D_{x+n}}{D_x}$$

Illustration: Aurora Branesky, aged 35, is to receive \$20,000 when she reaches the age of 45. Find the present value of the pure endowment.

$$\text{Present value} = \$20{,}000 \times {}_{10}E_{35} = \$20{,}000 \times \frac{D_{45}}{D_{35}}$$

$$= \$20{,}000 \times \frac{280{,}638.95}{381{,}995.63} = \$14{,}693.31$$

Problems involving pure endowments, like problems involving all other aspects of actuarial science, may be worked using any mortality table or any interest rate. Variations in rates of mortality or interest will be reflected in the values found. When using the tables in the appendix, the assumptions implicit in every solution are that the rate of interest is $2\frac{1}{2}\%$ converted annually, and that the mortality rate as estimated by the CSO Table is followed. The values for all commutation symbols in the appendix is based on these two assumptions.

EXERCISE 15.6

1. Find the annual payments on a life annuity whose present value is \$6,000, to a man aged 35 if the first payment is to be received at age 55.

2. A 48-year-old man buys a life annuity for $36,000 Find his annual income from the annuity if the first payment is at age 60

3. Determine the annual income from a 20-year life annuity purchased for $25,000 by a man aged 30

4. A man aged 45 pays $5,000 for a life annuity that will start at age 63 to run for 20 years How much will he receive annually?

5 At the age of 45 a man pays $3,000 for a life annuity that is deferred 15 years What will he receive annually?

6. A retired engineer aged 67 is receiving $1,500 a year for life How much more per year would he receive annually if he changed it to a temporary annuity for the next 10 years?

7. Find the annual income to the annuitant aged 24 who paid $3 500 for a life annuity beginning at age 60

8. An estate of $30,000 is to be turned into cash and used to purchase a life annuity for an heir aged 48 What annual payments should the heir expect?

9. What is the annual income under the CSO Table if a person aged 54 purchases a deferred 20-year temporary life annuity, if the first payment is to be made at age 63?

10. A man aged 28 receives an inheritance of $2,400 every year, payable at the beginning of each year If an inheritance tax of 4% is to be paid on its present value, find the tax that must be paid

11. Find the present value of a pure endowment of $1,200 to be paid in 10 years to a man now aged 50, if he is then alive

12. Find the present value of a pure endowment of $10,000 to be paid to a man aged 40 when he reaches 65, if he is still alive

13. A man aged 25 inherits $1,000 With this he buys a 20-year pure endowment How much will he receive if living at age 45?

14. The state levied a tax of 2% on the present value of an inheritance A boy aged 18 is left $25,000 to be paid at age 25 if he is still living Find the amount of the tax

15. A group of 100 men aged 25 decide to begin an investment club by making $1,000 deposits Those who survive to age 60 are to divide the funds equally among them If the fund earns interest at $2\frac{1}{2}\%$, and mortality is exactly equal to the CSO Table, how much will each survivor receive?

Life insurance

Annuities have always been a part of the life insurance business in the United States, but in the last two decades the number of annuities in force has increased fivefold Even after this great increase in the number

of annuities, the number of persons covered by life insurance was almost 20 times as great as the number covered by annuity contracts.

A life insurance *policy* is a contract issued to an individual, called the *policyholder*, by an insurance company. In the contract the obligations of the person insured, called the *insured*, are outlined as well as the obligations and responsibilities of the company, called the *insurer* or the *carrier*. The company agrees that if death occurs while the policy is in force it will pay upon presentation of proof of death of the insured an amount called the *face* of the policy, or the *benefit*, to whosoever is designated in the policy as the *beneficiary*. The insured, on the other hand, agrees to pay a certain sum of money to the company. The payment or payments made to the company are called *premiums*. Some policies call for a single payment, others call for annual premiums, some quarterly, some monthly, and even some weekly.

When a life insurance policy is issued to an insured, the premium is calculated on the basis of his age at his nearest birthday. The policy is effective on what is known as the *issue date*, and *policy years* are measured from this effective date. In the computations of rates and premiums the assumption is made that payment of any death benefits will occur at the end of the policy year.

The actuary thinks of premium under two classifications. One, called the *gross premium*, includes the total periodic payment received by the company, whether it is to be used to meet overhead expenses, pay salesmen's commissions, pay taxes, or for any other purpose. Part of the gross premium is the *net premium*, which is sometimes referred to as the *true cost of insurance*. In the remainder of this chapter when the term premium is used it is to be interpreted as meaning the net premium.

It is not difficult to understand why our discussion is limited to the net premium. The assumptions on which net premiums are calculated can be definitely stated. They are:

1. That the computation of premiums is based on a single mortality table (here the CSO Table is used) and that the insured lives will follow the exact pattern depicted in the table.

2. That a single rate of interest—in the case of the CSO Table a rate of $2\frac{1}{2}\%$ payable annually—is used.

3. That all death benefits will be paid at the end of the policy year in which they occur.

4. That the amount of money collected by the company plus all interest income is distributed in death benefits to the beneficiaries.

In determining gross premium, on the other hand, the actuary must first determine the net premium. To find the gross premium he adds to

the net premium an amount sufficient to pay all costs of acquiring the insurance, collecting the premiums, supervising the investments, managing the company, meeting all expenses, including an estimated allowance for profit, and some additional fees to provide for an adequate margin of safety He may also choose to make allowances for higher interest rates and lower or higher mortality rates than the tables show Thus while gross premiums will vary considerably among companies, the net premiums based on the same mortality table and the same interest rate will be uniform Hence our study is limited to the factors which go to determine the net premium

Life insurance is the only medium by which existing financial arrangements can be carried out in spite of the death of the insured As long as he continues to live, his income can be used to meet his obligations, but when he dies his income may cease It is the purpose of life insurance to provide a method by which, when the insured dies, the beneficiary will receive, under the terms of the policy, an amount which should be sufficient to offset the economic loss occasioned by the death of the insured

Life insurance amounts basically to having a group of people contribute to a common fund which will pay to the beneficiary a certain sum if the insured should die during the stated period Thus the mortality rate determines the cost of insurance For instance, if the CSO Table is used as the basis to determine the cost of insuring the lives of 10,000 healthy persons aged 18 for \$1,000 each for one year, it would be necessary for each person insured to put \$2 24 into a common fund When invested for one year at $2\frac{1}{2}\%$ this would furnish a fund of \$23,000 If during the year 23 of the 10,000 persons died, as indicated by the CSO Table, there would be in the fund just enough to pay each beneficiary \$1,000 at the end of the policy year The net cost of the insurance would be \$2 24

Presupposing a large enough number of a given age group, it can be seen that the probability of death and hence the cost of insurance varies according to the age of the person insured The following table shows selected data from the CSO mortality table From this brief table it can be seen that as the probability of death increases, the cost of insurance rises slowly from age 10 to 20, doubles from 20 to 36, is four times as great at 56 as at 36, doubles again between 56 and 65, and doubles again in the next 7 years The number of deaths per 1,000 persons of a given age, which was only 2 43 at age 20, is 132 at age 80, 194 at age 85, 280 at age 90, and 371 at age 94 In order to establish a limit to the table it is assumed that the number of deaths per thousand aged 99 will be 1,000

The short table shows the mortality rate per 1,000 persons of selected ages as shown by the CSO Table, and assumed cost per $1,000 of single-year term insurance issued at the selected ages.

Age	*Mortality Rate per 1,000*	*1 Year Term Net Premium at Given Age*
10	1.97	$1.92
20	2.43	2.37
30	3.56	3.47
36	4.86	4.74
40	6.18	6.03
50	12.32	12.09
56	19.43	18.96
60	26.59	25.94
65	39.64	38.67
70	59.30	57.85
80	131.85	128.64
85	194.13	189.40
90	280.99	274.14

From the study of the mortality table, two facts should be noted. First, the true cost of insurance represented by the premium per thousand dollars of insurance increases steadily. Second, if premiums were collected annually to meet only the true cost of insurance, the amount collected would be just sufficient to pay the beneficiaries of those who die each year; thus a member of the group who does not die loses his entire premium in the sense that if he seeks insurance the next year he must pay another premium, which will be higher because of the increased mortality rate. Under such a plan he would have to pay an ever-increasing annual premium.

Whole life insurance policy issued for net single premium

An understanding of the computation of net single premiums can perhaps best be introduced by illustrating a type of policy rarely issued. This is called a whole life insurance policy with a net single premium. The characteristics of such a policy are that for the payment of a single net premium the insurance company agrees to pay the face amount of the policy to the beneficiary at the death of the insured regardless of when death occurs. On the basis of the assumptions which were stated earlier, the company would just break even on such an operation. Thus the value of the amount paid to the company in premiums must just be equal to the value of the amount to be disbursed on a selected date.

If each of l_x persons aged x bought a whole life insurance policy of $1 at the same time, the total amount that the company would collect would be l_xA_x, if A_x is used to represent the net single premium At the end of the first year the company would make payments to beneficiaries of d_x persons at the end of the next year d_{x+1}, at the end of the next year d_{x+2} to the end of the table Obviously the value of these future payments to be made by the company on the date the policy is issued will be the discounted values of d_x, so they will be equal to

$$A_x = \frac{vd_x + v^2d_{x+1} + v^3d_{x+2} + \quad \text{to the end of the table}}{l_x}$$

At this point a third commutation symbol, C_x, used by actuaries is introduced It represents the value of $v^{x+1}d_x$ By multiplying both the numerator and denominator of the right hand side by v^x, the equation becomes

$$A_x = \frac{v^{x+1}d_x + v^{x+2}d_{x+1} + v^{x+3}d_{x+2} + \quad \text{to the end of the table}}{v^xl_x}$$

Substituting the commutation symbol C_x for the $v^{x+1}d_x$ values here, the equation becomes

$$A_x = \frac{C_x + C_{x+1} + C_{x+2} + \quad \text{to the end of the table}}{D_x}$$

At this point a fourth commutation symbol, M_x, is introduced It represents the summation of the C_x values to the end of the table

$$M_x = C_x + C_{x+1} + C_{x+2} + \quad \text{to the end of the table}$$

The substitution of these two new symbols reduces the work still further, since again by making the computation once for a stated interest rate, here $2\frac{1}{2}\%$, tabular values can be easily computed for all values of C_x and M_x The equation then for the value of a single net premium whole life insurance policy at age x becomes

$$A_x = \frac{M_x}{D_x}$$

and the entire computation with the use of the table reduces itself to a single division

Illustration A man aged 25 buys a $1,000 whole life insurance policy What is the net single premium?

Here $x = 25$, $F = \$1{,}000$ Hence

$$\text{Net single premium} = \$1{,}000 \times A_{25} = \$1{,}000 \times \frac{M_{25}}{D_{25}}$$

$$= \$1{,}000 \times \frac{189{,}700\ 88}{506{,}594\ 02} = \$374\ 46$$

EXERCISE 15.7

Find the net single premium for the following:

1. A \$1,000 whole life policy for a boy aged 16.
2. A \$1,000 whole life policy for a man aged 79.
3. A whole life insurance policy for \$5,000 issued to a man aged 21.
4. A whole life insurance policy for \$500 issued to a girl aged 18.
5. A \$10,000 whole life policy for a man aged 47.

Term insurance

It often happens that insurance protection is needed for a definite period of time. Though as a usual thing life insurance is not taken for just 1 year, such policies may be issued. A policy issued for a single year, or for a definite period of years (5 and 10-year policies are now common) is known as a *term policy* and such insurance is called *term insurance.*

The net single premium charged for a single year of coverage can be shown by assuming each of l_x persons aged x bought a single year term policy of \$1 at the same time. Let the actuary's symbol c_x represent the amount paid. The net single premium for a one-year term issued for \$1 to a life aged x, is called the *natural premium,* is represented by the symbol c_x, and is shown in the more complete mortality tables.

$$l_x c_x = v d_x \quad \text{or} \quad l_x = \frac{v d_x}{l_x}$$

Multiplying both members of the right-hand side by v^x,

$$c_x = \frac{v^{x+1} d_x}{v^x d_x} = \frac{C_x}{D_x}$$

The cost of a one-year term at age 20 would be, per \$1,000:

$$\$1{,}000 \times {}_1A_{20} = \$1{,}000 \times \frac{C_{20}}{D_{20}} = \$1{,}000 \times \frac{1{,}376.5331}{580{,}662.42} = \$2.37$$

If such a policy were issued at age 65, the cost would be much higher, \$39.64, since the probability of dying is greater at age 65 than at 20. Term insurance furnishes protection only for the stated period. The premium is paid for protection only, and on a given date the policy expires.

Level premiums

Although the true cost of insurance increases each year, the same sum is usually collected annually when term policies are issued for a period of years. This is called a *level premium;* it means simply that the premium

does not change from year to year If a term policy were issued for 5 years at age 43, the cost of insurance, assuming 5 years of different natural premiums, would be

Age	*Net Single Premium for $1,000 for One Year at Age x*
43	$ 7 32
44	7 84
45	8 40
46	9 00
47	9 67
Total	$42 23

If the natural premium were collected each year, at the end of 5 years a total of $42 23 would have been collected, or an average of $8 45 a year Had a single premium been collected at the time the policy was issued, *the net single premium would have the present value of the* 5 annual premiums, or $39 42

The net single premium for an m-year term insurance policy issued to a person aged x is represented by the symbol $A_{x\,\overline{m|}}$ The formula for finding the premium is

$$A_{x\,\overline{m|}} = \frac{M_x - M_{x+m}}{D_x}$$

Thus for the net single premium for a 5 year term policy issued at age 43 with a face of $1,000, the value is

$$\$1{,}000 \times A_{43\,\overline{5|}} = \$1{,}000 \times \frac{M_{43} - M_{48}}{D_{43}}$$

$$= \$1{,}000 \times \frac{159{,}205\ 35 - 147{,}398\ 48}{299{,}485} = \$39\ 42$$

The question arises of how much should have been collected if an annual level premium were to be paid The level annual premium can be determined, since it constitutes an m-year temporary life annuity To find the net annual premium $P_{x\,\overline{m|}}$, equate the present value of the temporary life annuity due to that of the net single payment Thus

$$P_{x\,\overline{m|}} \times \ddot{a}_{x\,\overline{m|}} = A'_{x\,\overline{m|}}, \quad \text{so} \quad P_{x\,\overline{m|}} = \frac{A'_{x\,\overline{m|}}}{a_{x\,\overline{m|}}}$$

We know that $a_{x\,\overline{m|}} = \dfrac{N_x - N_{x+m}}{D_x}$, and that $A'_{x\,\overline{m|}} = \dfrac{M_x - M_{x+m}}{D_x}$

Hence

$$P'_{x\,\overline{m|}} = \frac{M_x - M_{x+m}}{D_x} \times \frac{D_x}{N_x - N_{x+m}} = \frac{M_x - M_{x+m}}{N_x - N_{x+m}}$$

is the formula for the net annual premium for an m payment m-year term policy.

Illustration: Find the net annual premium for a 5-year term policy for \$1,000 issued at age 43.

Here $x = 43$; $m = 5$; $F = \$1{,}000$. Therefore the net annual premium is

$$\$1{,}000 \times P'_{43:\overline{5|}} = \$1{,}000 \times \frac{M_{43} - M_{48}}{N_{43} - N_{48}}$$
$$= \$1{,}000 \times \frac{159{,}205.35 - 147{,}398.48}{5{,}751{,}467 - 4{,}347{,}548} = \$8.41$$

The level annual premium of \$8.41 is of equal value to both the single payment of \$39.42 and the annual payment total \$42.23.

The principle of collecting equal or level premiums is customary for all policies whether term or otherwise. It can be seen that during the first part of the period the company collects more than the true cost of insurance, but that in later years it may collect less.

EXERCISE 15.8

1. Find the net annual premium for a 5-year term insurance policy for \$1,000 purchased at age 22.
2. Find the net single premium for a 5-year term insurance policy for \$1,000 purchased at age 22.
3. What is the net annual premium on a 10-year term policy for \$10,000 taken by a man aged 43?
4. A man aged 45 purchased a single-year term policy for \$10,000. The gross premium paid was \$106.00. What was the net premium?
5. What is the net annual premium for a 20-year term policy of \$10,000 taken at age 31?

Ordinary life net annual premium

Although life insurance may be thought of primarily as a method of protection against the financial losses occasioned by death, it often serves a second basic purpose, that of providing a plan for systematic savings. All policies provide protection; some also make provisions for savings. All forms of term insurance give protection only; their sole purpose is to protect others in the event of the death of the insured. Systematic savings, in addition to protection, are provided for by what are commonly called ordinary (or whole) life policies, endowment policies, and retirement income policies. In these policies the savings aspect is emphasized perhaps even more than the protective aspect.

When policies are written for a whole life, that is, to furnish insurance for the policyholder until his death, the natural premium or true cost of insurance rises as he grows older just as it does under a term policy, but the disadvantage of a similar prohibitive rise in rates is alleviated in three ways First, through the practice of collecting a level premium, the burden is spread over the entire life of the insured In the earlier years of an ordinary life policy, the premium collected is higher than the true cost of insurance The other two ways to alleviate the prohibitive raise in rates will be referred to soon

We have previously seen that the single net premium for a whole life policy at age x is $A_x = \frac{M_x}{D_x}$ The annual premium, represented by the symbol P_x, would be the annual payments of a whole life annuity due whose present value would be equal to the net single premium Thus $P_x \times a_x = A_x$ so $P_x = \frac{A_x}{a_x}$ Since $A_x = \frac{M_x}{D_x}$ and $a_x = \frac{N_x}{D_x}$, then $P_x = \frac{M_x}{N_x}$, the formula for the net annual premium for the ordinary life policy

The net annual premium for a \$1,000 ordinary life policy issued at age 20 would be

$$\$1{,}000 \times \frac{M_{20}}{N_{20}} = \$1{,}000 \times \frac{196{,}657\ 17}{15{,}744{,}216} = \$12\ 49$$

If an annual premium of \$12 49 is collected in the early years of the policy, in fact, until the insured reaches the age of 51, the annual premium exceeds the true cost of insurance The difference between the natural premium and the amount collected is in the nature of a liability of the insurance company which provides funds for the company to invest The fact that such reserves are built up and retained by the company for the benefit of the policy holders makes life insurance an important vehicle for savings

The following table shows the excess of the level premiums over the natural premium for the whole life policy issued at age 20

Age	*Natural Premium*	*Level Premium*	*Excess*
20	\$2 37	\$12 49	\$10 12
21	2 45	12 49	10 04
22	2 52	12 49	9 97
23	2 61	12 49	9 88
24	2 70	12 49	9 79
25	2 81	12 49	9 68

Let us assume further the same policy was issued to 951,483 persons aged 20. The first year premiums collected would amount to:

951,483 × \$12.49 =	\$11,884,022.67
Interest for 1 year at $2\frac{1}{2}\%$ =	297,100.56
Total collected and earned	\$12,181,123.23
Number dying during year, 2,212; payment at \$1,000 each	2,212,000.00
Total in reserve	\$ 9,969,123.23

Reserve for each life aged 21 is thus \$10.51. The reserve is built up each year until at the end of 20 years it equals approximately \$240. The fact that the insurance company holds a reserve of about \$240 for the benefit of the policyholder means that when a beneficiary receives the \$1,000, the face amount of the policy, the insurance company is called on to pay only \$760 and to return the reserve of \$240. Consequently, only \$760 of insurance protection is actually furnished during the twenty-first year of the policy. The true cost of insurance for that year, that is, at age 41, is really not \$6.09, the natural premium based on the mortality table, but only $\frac{760}{1000}$ of that amount, or \$4.63. The balance of the \$12.49 premium collected goes to increase still further the reserve behind the policy.

As the reserves build up, there is a decrease in the net insurance, that is, the difference between the reserve and the face amount of the policies. Thus the second factor in an ordinary life policy which alleviates the burden of increasing costs is that, as the reserve builds up, the reduction in the net insurance necessary tends to offset the rise in the mortality rate.

Under an ordinary life policy, the existence of the reserve makes possible a third factor which largely counteracts the increase in the true cost of insurance. The funds which make up the reserve are invested, and the income from these funds augments the annual premiums collected from the policyholder. A reserve of \$240, mentioned in a preceding paragraph, invested at $2\frac{1}{2}\%$, would go far toward furnishing the amount necessary to meet the cost of net insurance for 1 year at age 41.

By the time a policyholder is 60 years of age, the reserve which is back of a \$1,000 policy issued at age 20 is approximately \$550. Thus the net insurance necessary is only \$450, and the net annual premium of \$12.49 is augmented by \$13.75, the interest earned on the reserve fund, assuming a return of $2\frac{1}{2}\%$. The premium alone more than meets the cost of the net insurance protection furnished at age 60.

As the reserve increases, it earns more income; and at the same time the amount of insurance needed decreases. As a result, the reserve is

built up rapidly, and eventually the reserve equals the face amount of the policy The policy is then said to have matured This means, in effect, that the policyholder by making periodic payments has saved an amount equal to the face of the policy The insured can withdraw the face amount of a matured policy at any time he desires The CSO Table assumes that death will occur at age 99 Therefore, according to this table, the reserve behind a policy should amount to the face of the policy by age 99, since under it death is presumed to occur before the insured reaches his 100th year

Terminal reserve

The reserve back of a policy at the end of any policy year, just before the next premium payment is due, is called the *terminal reserve* for the policy *for the year just ending The terminal reserve is the accumulated* value of past net premiums minus the accumulated value of the past insurance benefit To determine the terminal reserve for any type of policy, at the end of t years, first find the net annual premium on the policy issued at age x Then find the net single premium for a policy of the same amount issued to a life aged $x + t$ The net single premium shows the amount that the company would have to receive to issue the policy at the end of t years At this point the future premium payments form an annuity due The present value of these payments for a whole life policy is denoted by the formula

$$Pa_x = P\frac{N_{x+t}}{D_{x+t}}$$

The terminal reserve V_{x+t} at the end of t years should be represented by

$$V_{x+t} = F \times \frac{M_{x+t}}{D_{x+t}} - P \times \frac{N_{x+t}}{D_{x+t}}$$

Where V_{x+t} is the terminal reserve for the policy at the end of t years x is the age when the policy was issued, P is the net level premium, t is the number of years lapsed since the policy was issued

Illustration Find the terminal reserve for a whole life policy of $1,000 issued at age 20 (a) at the end of 20 years, (b) at the end of 40 years

a The net level annual premium is

$$P = \$1,000 \times \frac{M_{20}}{N_{20}} = \$1,000 \times \frac{196,657\ 17}{15,744,216} = \$12\ 49$$

The net single premium at age 40 is

$$\$1,000 \times \frac{M_{40}}{D_{40}} = \$1,000 \times \frac{165\ 359\ 89}{328,983\ 61} = \$502\ 64$$

Present value of the premiums still due is

$$\$12.49 \times \ddot{a}_{40} = \$12.49 \times \frac{N_{40}}{D_{40}} = \$12.49 \times \frac{6{,}708{,}573}{328{,}983.61} = \$254.69$$

Therefore

$$V_{40} = \$502.64 - 254.69 = \$238.69$$

b. From (a) the net level annual premium is \$12.49. The net single premium at age 60 is

$$\$1{,}000 \times \frac{M_{60}}{D_{60}} = \$1{,}000 \times \frac{108{,}543.46}{154{,}046.23} = \$704.62$$

Present value of the premiums still due is

$$\$12.49 \times \ddot{a}_{60} = \$12.49 \times \frac{N_{60}}{D_{60}} = \$12.49 \times \frac{1{,}865{,}614}{154{,}041.23} = \$151.26$$

Therefore

$$V_{60} = \$704.62 - 151.26 = \$553.36$$

Limited payment life

Two methods of paying for a whole life policy have been considered. The method of payment may be a compromise between these two extreme methods, of the net single premium on the one hand, and payments throughout life on the other. The insured may elect to pay for a limited period of n years, such as 10 or 20 years. Such policies are known as 10-payment life or 20-payment life. In them as in all insurance policies the premium is paid only as long as the insured lives.

The size of the net annual premiums for a limited payment life policy is found in the same way as in the ordinary policy. That is, the net annual premiums constitute an n-year temporary life annuity due. In this case

$${}_nP_x \times \ddot{a}_{x:\overline{n}|} = A_x, \quad \text{or} \quad {}_nP_x = \frac{A_x}{\ddot{a}_{x:\overline{n}|}}$$

But

$$A_x = \frac{M_x}{D_x} \text{ and } \ddot{a}_{x:\overline{n}|} = \frac{N_x - N_{x+n}}{D_x}$$

so

$${}_nP_x = \frac{M_x}{N_x - N_{x+n}}$$

Illustration: Find the net annual premium for a 10-payment whole life policy for \$5,000 issued at age 25. Find the terminal reserve at ages 30 and 40.

Here $x = 25$, $n = 10$, $t = 5$ or 15 Thus

Net annual premium $= \$5{,}000\,{}_{10}P_{25} = \$5{,}000 \times \dfrac{M_{25}}{N_{25} - N_{35}}$

$= \$5{,}000 \times \dfrac{189{,}700\ 875}{12{,}992{,}619\ 10 - 8{,}510\ 443\ 06} = \dfrac{189{,}700\ 875}{4{,}482{,}175\ 9} \times \$5{,}000$

$= \$211\ 62$

Terminal reserve at age 30 $= \$5\ 000 \times \dfrac{M_{30}}{D_{30}} - 211\ 62 \times \dfrac{N_{30} - N_{35}}{D_{30}}$

$= \$5{,}000 \times \dfrac{182{,}403\ 50}{440{,}800\ 58} - 211\ 62 \times \dfrac{10{,}591{,}280\ 39 - 8{,}510{,}443\ 06}{440{,}800\ 58}$

$= \$1{,}068\ 57$

Terminal reserve at age 40 $= \$5\ 000 \times \dfrac{M_{40}}{D_{40}} = \$5{,}000 \times \dfrac{165{,}359\ 89}{328{,}983\ 61}$

$= \$2{,}513\ 19$

since the policy was paid up at age 35

EXERCISE 15.9

1. Find the net level annual premium for an \$8,000 whole life policy for a man aged 27 Find the terminal reserve at ages 40 and 60
2. Find the net level annual premium for a \$2,000 whole life policy for a man aged 36 Find the terminal reserve at ages 50 and 70
3. Find the net level annual premium for a 20-payment \$5,000 whole life policy for a man aged 32 Find the terminal reserve at ages 45 and 60
4. Find the net level annual premium for a 30-payment \$10,000 whole life policy for a man aged 24 Find the terminal reserve at ages 40 and 65
5. *Find the net level annual premium for a 10-payment \$5,000 whole* life policy for a man aged 52 Find the terminal reserve at ages 60 and 70

Endowment insurance

An endowment policy is a term policy plus a pure endowment clause If the policyholder dies during the specified period, or before a specified age, his beneficiary will be paid the amount of the policy, while if the insured is alive at the end of the term he will receive the amount of the *endowment*

The net annual premium on an m year endowment policy on which the period of premium payments corresponds to the endowment period may be found as follows Since

$$P_{x\,\overline{m}|} \times a_{x\,\overline{m}|} = A^{1}_{x\,\overline{m}|} + {}_mE_x$$

then

$$P_{x\,\overline{m}|} = \frac{M_x - M_{x+m} + D_{x+m}}{N_x - N_{x+m}}$$

If the period of the premium is only n years ($n < m$) and the period of the endowment is m, the net annual premium can be found by using the formula

$$_nP_{x:\overline{m}|} = \frac{M_x - M_{x+m} + D_{x+m}}{N_x - N_{x+n}}$$

Illustrations:

a. Find the net annual premium for a 20-payment 30-year endowment policy for \$5,000 issued at age 28. Find the terminal reserve at age 40.

Here $x = 28$; $m = 30$; $n = 20$; $F = \$5,000$. Thus

$$\text{Net annual premium} = \$5,000 \times {}_{20}P_{28:\overline{30}|} = \$5,000 \times \frac{M_{28} - M_{58} + D_{58}}{N_{28} - N_{48}}$$

$$= \$5,000 \times \frac{185,385.34 - 116,185.34 + 169,777.17}{11,513,853.25 - 4,347,547.83}$$

$$= \$5,000 \times \frac{238,977.17}{7,166,305.42} = \$166.74$$

$$\text{Terminal reserve at age 40} = \$5,000 \times \frac{M_{40} - M_{58} + D_{58}}{D_{40}}$$

$$- 166.74 \times \frac{N_{40} - N_{48}}{D_{40}} = \$5,000 \times \frac{165,359.89 - 116,185.34 + 169,777.17}{328,983.61}$$

$$- 166.74 \times \frac{6,708,572.66 - 4,347,547.83}{328,983.61} = \$2,131.06$$

b. Find the net annual premium for a \$10,000 endowment at 70 policy, issued at age 28. Find terminal reserve at age 70.

Here $x = 28$; $x + m = 70$; $F = \$10,000$. Thus

$$\text{Net annual premium} = \$10,000 \times {}_{42}P_{28:\overline{42}|} = \$10,000 \times \frac{M_{28} - M_{70} + D_{70}}{N_{28} - N_{70}}$$

$$= \$10,000 \times \frac{185,385.34 - 64,517.79 + 80,706.62}{11,513,853.25 - 663,742.06} = \$185.78$$

$$\text{Terminal reserve at age 70} = \$10,000 \times \frac{M_{70} - M_{70} + D_{70}}{D_{70}} = \$10,000$$

That is, the terminal reserve at age 70 is the amount sent that day to the policyholder, since he is still alive.

EXERCISE 15.10

1. Find the net annual premium for a 30-year endowment policy for \$10,000 for a man aged 24. Find the terminal reserve at ages 45 and 54.

2. Find the net annual premium for a 25-year endowment policy for \$5,000 for a man aged 28. Find the terminal reserves at ages 42 and 53.

3. Find the net annual premium for a 20-payment 30-year endowment policy for $2,000 for a man aged 21 Find the terminal reserves at ages 30, 41, and 51

4. Find the net annual premium for a 15 payment 25-year endowment policy for $1,500 for a man aged 32 Find the terminal reserve at ages 40, 47, and 57

5. Find the net annual premium for an endowment at 65 policy for $10,000 for a man aged 30 Find the terminal reserve at ages 50 and 65

6. Find the net annual premium for an endowment at 75 policy for $20,000 for a man aged 38 Find the terminal reserve at ages 55 and 75

7. Find the net annual premium for an endowment at 80 policy for $7,500 for a man aged 45 Find the terminal reserve at ages 65 and 80

8. Find the net annual premium for a 30-payment endowment at 70 policy for $5,000 for a man aged 25 Find the terminal reserve at ages 45, 55, and 70

9. Find the net annual premium for a 20-payment endowment at 65 policy for $2,500 for a man aged 40 Find the terminal reserve at ages 55, 60, and 65

10. Find the net annual premium for a 25-payment endowment at 85 policy for $10,000 for a man aged 24 Find the terminal reserve at ages 30, 40, 50, 60, 70, 80, and 85

REVIEW PROBLEMS

Chapters 14 and 15

1. A 3% bond due in $7\frac{1}{2}$ years is bought to yield 4% to maturity. If interest is paid semiannually, find the price of the bond. Find the book value immediately after the fifth semiannual payment of interest has been recorded.

2. A 4% bond due in 8 years is bought to yield 5% to maturity. If interest is paid semiannually, find the price of the bond. Find the book value immediately after the fifth semiannual payment of interest has been recorded.

3. Ten years before maturity a $1,000, 5% bond with interest payable semiannually is bought to yield 4% payable semiannually. Find the price of the bond. Find the book value immediately after the ninth payment of interest has been recorded.

4. A $3\frac{1}{2}$% bond due in 4 years is bought to yield 5% to maturity. If interest is paid annually, construct a schedule showing the accumulation of the bond discount.

5. Find the value of a $1,000, 4% bond due in 4 years, interest payable semiannually, if bought to yield 3% converted semiannually. Construct an amortization schedule.

6. Six years before maturity a $1,000, 4% bond with interest payable semiannually is bought to yield 3% payable semiannually. Find the value of the bond, and without using an annuity table, construct an amortization schedule.

7. Find the value 10 years before maturity of a $1,000, 5% bond with interest payable annually priced to yield 4%. Find the book value of the bond just after the sixth interest payment has been received.

8. A $5\frac{1}{2}$% bond with interest paid semiannually due in 12 years is quoted at 102. Find the approximate yield.

9. A 4% bond with interest paid semiannually is purchased 12 years before maturity at 108. Find the approximate yield.

10. A $3\frac{1}{2}$% bond due in 15 years is quoted at 95. If interest is paid semiannually, find the approximate yield.

11. A $1,000, 3% bond is due February 15, 1973. If the bond is bought to yield $3\frac{1}{2}$% on May 15, 1958, find the "and interest" price.

12. A $10,000 mN $2\frac{1}{2}$% municipal bond due November 1, 1965, is purchased on April 15, 1959, to yield 4%. Find the quoted price.

13. Find the approximate yield on Servomechanisms, Inc., 5% bonds due December 1, 1966, bought 6 years before maturity at 108.

14. Find the approximate yield on 6% bonds of Alaska Telephone Corporation bought $7\frac{1}{2}$ years before maturity at $94\frac{1}{4}$

15. Find the approximate yield on Tennessee Gas Corporation 5% bonds bought 9 years before maturity at $107\frac{1}{2}$

16 A 3%, $1,000 bond is bought 15 years before maturity at $1,129 Find the approximate yield

17. A $1,000, 4% bond is bought 6 years before maturity at $94\frac{7}{8}$ Find the approximate yield

18 A $1,000, 5% bond with semiannual interest payments is bought 11 years before maturity at 92 *Find the approximate yield*

19 Find the approximate yield on a $1,000, 4% bond with semi annual coupons bought $5\frac{1}{2}$ years before maturity at $102\frac{1}{2}$

20. The bond table shows that a $1,000, 4% bond bought 4 years before maturity at $964 15 yields 5% to maturity Construct a schedule showing the accumulation of the bond discount

21. The bond table shows that a $1,000, $2\frac{1}{2}$% bond bought 2 years 6 months *before maturity to yield 2% costs $1,012 13* Construct a schedule to show the book value of the bond

22. Construct a schedule showing the accumulation of the discount on a $1,000, 3% bond bought $4\frac{1}{2}$ years before maturity to yield 4%

23. Find the cost of a 5% bond, $1,000 denomination, with interest payable semiannually, if bought to yield the purchaser 6% exactly $15\frac{1}{2}$ years before maturity

24. A city issued 3% bonds due in 20 years Interest is paid semiannually and the bonds are to be redeemed at par on September 1, 1975 Find the purchase price on April 1, 1960, to yield 2% converted semi annually

25. International Harvester has outstanding some preferred stock on which a dividend of $7 a share is paid regularly Assuming that the annual return of $7 will continue in perpetuity, what is the value of a share of this stock when investors expect an annual return of (a) $3\frac{1}{2}$%, (b) $4\frac{1}{2}$%, (c) 5%?

26. How much is needed to endow a chair at a university for $10,000 a year if money is worth 4% converted annually?

27. How much must be left in a perpetual trust at 3% converted annually to guarantee a yearly stipend of $200 to care for a cemetery plot?

28. What is the capitalized cost of a home that can be bought for $20,000, with an estimated expense of $500 a year for maintenance, at an interest rate of 5% converted semiannually?

29. What is the capitalized cost, if money is worth 5%, of a street light which costs $500 to install, $7.50 a year to maintain, and must be replaced every 30 years?

30. If money is worth 4%, find the capitalized cost of a water line which costs $1,560 and which must be replaced at the same cost every 30 years?

31. In the construction of a given section of a factory, sheet metal or aluminium can be used. The sheet metal will cost $1,200 and last 1 year. The aluminum will last 12 years. If money is worth 4%, how much could justifiably be spent for the aluminum?

32. A gas company found that one section of its gas main is in a leaky condition. In replacing the main, an untreated line will cost $8,100 and last 20 years; the pipe, if treated, will last 10 years longer. If money is worth 5%, how much can economically be spent in treating the pipe?

33. The cost accountant of a construction company is asked to determine which is more economical: 2 bulldozers at $12,000 each, costing $1,800 each to operate annually and lasting 6 years; or a single earth mover costing $24,000, having a life of 5 years, and an annual operating expense of $2,400. Money is worth $4\frac{1}{2}$%.

34. Find the annual investment expense of a temporary classroom building under the following conditions: original cost to build, $150,000; expected life of the structure, 15 years; annual upkeep, $3,000; and cost of razing it at the end of its useful life, $5,000. Money is worth 5% converted annually.

35. Underground conduit will cost $25,000 to install and $200 a year in annual upkeep. Find the capitalized cost if money is worth 3% converted annually.

36. A machine costing $500 has a life of 10 years and no scrap value. Find the capitalized cost, exclusive of maintenance, if money is worth 4% effective.

37. Control valves in a sprinkler system cost $15 each and have a life of 5 years. How much would one be justified in spending for a valve which has a life of 8 years if money is worth 5% converted annually?

38. An engineer has the choice of selecting one of two types of tanks to be used in a water system. Type A costs $75,000, has an estimated life of 20 years, a replacement cost of $85,000, and an annual maintenance cost of $1,000. Type B tank costs only $30,000, has an annual upkeep cost of $2,500, and replacement every 5 years of $15,000. Which type of tank is less expensive in the long run if money is worth 4% converted annually?

39. A machine costing \$10,000 depreciates 25% per year What is its value after 3 years?

40 If the rate of depreciation is 30% per year, how long will it take a \$5,000 machine to depreciate to \$840 35?

41. A piece of equipment costing \$15,000 has an estimated life of 8 years Construct a depreciation schedule using the sum of the digits method

42. A machine which cost \$840 has an estimated life of 5 years and a scrap value of \$120 Under the straight-line method, find the annual depreciation charge and the book value at the end of the third year

43. Your employer is considering the purchase of \$100,000 worth of equipment with an estimated useful life of 10 years You are asked to construct a depreciation schedule using the straight line depreciation method

44 Your employer is considering the purchase of \$100,000 worth of equipment with an estimated useful life of 10 years You are asked to construct a depreciation schedule, using a 20% constant percentage method

45. Your employer is considering the purchase of \$100,000 worth of equipment with an estimated useful life of 10 years You are asked to construct a depreciation schedule using the sum of the digits method

46. From the mortality tables determine the probability that a person aged 25 will live to be 75 What is the probability that he will not live to be 75?

47. From the mortality tables determine the probability that a boy aged 18 will live to be 25 and the probability that his father aged 45 will live to be 52

48. What is the probability that a young man of 20 will live to age 25? to age 45?

49. Compare the probability that a boy aged 18 will live to 60 with the probability that a man aged 45 will live to be 60

50. What is the probability that a man aged 35 will live to 65

51. A sailor aged 40 retired after 20 years of service and is to receive \$1,200 a year, beginning one year hence, for the rest of his life If money is worth $2\frac{1}{2}$%, what is the present worth of the life annuity?

52. For his eighteenth birthday a boy received a life insurance policy from his father Under the terms of the policy the holder agrees to pay an annual premium to the insurance company of \$50 a year so long as he lives What single payment may the father make when the boy is 18 which will satisfy his obligation to the insurance company?

53. Under the terms of a pension trust set up in the bank where he works, a man aged 35 agreed to make annual contributions to a pension plan of $500 a year so long as he lived. At age 35, what single sum would be equivalent to his payments to the fund?

54. A girl, aged 20, is to receive payments from a trust fund of $10,000 a year for life. The first payment is to be received 5 years hence. What is the present value of this annuity?

55. A $10,000 whole life policy is purchased at age 20. What is the annual premium?

56. A 20-payment $8,000 whole life policy is purchased at age 32. What is the size of each premium payment?

57. A $20,000, 30-year term policy is purchased at age 27. What is the annual premium payment?

58. A $10,000, 30-year endowment policy is purchased at age 19. What is the cash purchase price?

59. A man aged 34 purchases a $10,000 endowment at age 70 policy. What is the annual premium payment?

60. Find the annual premiums and the policy reserve at age 45 for a $5,000 whole life policy purchased at age 25.

61. Find the annual premiums and the policy reserve at age 48 for a $40,000, 30-year term policy purchased at age 35.

62. Find the annual premiums and the policy reserve at age 40 for a $10,000 endowment at 65 policy purchased by a man aged 24.

63. Find the annual premiums and the policy reserve at age 45 for a $10,000 endowment at 85 policy purchased by a man aged 27.

64. A man purchases a $1,000-a-year life annuity starting at age 66. He is now aged 26. Find the annual premiums he must pay and the policy reserve at age 50.

65. A $2,500-a-year life annuity starting at age 70 will cost how much per year starting at age 32? What is the reserve at age 65?

66. What is the cash value of a paid-up $12,000 whole life policy held by a man aged 42?

67. What is the cash value of a paid-up $2,000 a year life annuity starting at age 61 if the man is now aged 47?

68. What is the cash value of a $15,000 whole life policy held by a man aged 37 who started making payments at age 26?

69. What is the cash value of a $18,000, 40-year endowment policy held by a man aged 42 who started making payments at age 28?

70. A man, aged 22 when he took out his present policy, is paying on a $8,000 endowment at 65 policy. At age 45 he decides to convert it into a paid-up whole life policy. What is the face of the new policy?

71. A man, aged 24, takes out a $12,000 policy that is a term policy for the first 20 years, then becomes a $8,000 whole life policy If he pays for it during the term period, what are his payments?

72 A man aged 22 takes out a $20,000 policy that is a term policy for the first 25 years, then becomes a $10 000 endowment at 70 policy If he pays the entire premium during the first 25 years, what are his annual payments?

73. Find the annual income from a 15-year life annuity purchesed for $20,000 by a man aged 48

74. Find the annual income from a whole life annuity purchased for $35,000 by a man aged 34

75. Find the face value of an endowment at 80 policy a man can buy if he can make annual payments of $200 starting at age 32

Answers

Chapter 1

Exercise 1.3

1. 34; 48; 43; 34; 33; 42; 41; 42; 41; 30; 41; 40; 39; 53; 52; 57.
2. 47; 33; 45; 59; 43; 50; 33; 44; 51; 46; 52; 33; 41; 36; 57; 50.
3. 41; 34; 37; 34; 33; 27; 30; 46; 36; 30; 41; 43; 42; 46; 36; 43.
4. 46; 57; 39; 45; 41; 39; 39; 56; 53; 47; 43; 38; 49; 45; 38; 44.

Exercise 1.4

1. 96. **2.** 125. **3.** 114. **4.** 103. **5.** 115. **6.** 103.
7. 85. **8.** 120. **9.** 275. **10.** 250. **11.** 379. **12.** 369.
13. 311. **14.** 369. **15.** 334. **16.** 381.

Exercise 1.5

1. 81. **2.** 98. **3.** 193. **4.** 213. **5.** 348. **6.** 237.
7. 290. **8.** 195. **9.** 141. **10.** 285.

Exercise 1.6 (Sum of answers only)

1. 143. **2.** 132. **3.** 695. **4.** 1,084. **5.** 1,089.
6. 1,214.

Exercise 1.7

1. 8,109. **2.** 19,738. **3.** 25,000. **4.** 18,080. **5.** 154,092.
6. 204,881. **7.** 310,246. **8.** 154,327. **9.** 4,184,216.
10. 2,493,855. **11.** 3,113,011. **12.** 3,231,008. **13.** 55,141.
14. 75,341. **15.** 80,783. **16.** 63,761. **17.** 66,414.

Exercise 1.8

1. 16,062. **2.** 13,614. **3.** 13,588. **4.** 28,570.
5. 185,403. **6.** 194,583. **7.** 304,027. **8.** 285,723.
9. 297,405. **10.** 334,541. **11.** 324,263. **12.** 441,896.

Exercise 1 9

1. 1, 2, 1, 1, 2, 1, 2, 7, 1, 2, 9, 6, 10, 8 2. 2, 7, 0, 2, 1, 1, 7, 4 3. 10, 1, 6, 9, 1, 9, 1 4. 1, 3, 0, 3; 7, 2

Exercise 1 10

1. 252,270 2. 232,338 3. 293,400 4. 315,260
5. 2,217,400 6. 1,314,977 7. 3,588,135 8. 2,646,165

Exercise 1 12

1. 3,649 2. 23,447 3. 59,127 4. 2,178 5. 64,796
6 465,015 7. 22,757 8. 35,766 9. 6,177. 10. 6,139
11. 19,430 12. 23,634 13. 18,807 14. 87,775
15. 9,298 16 22,577 17. 7,382 18. 41,519 19. 22,036.
20. 18,188 21. 28,658 22. 93 961 23. 31,260
24. 5,101 25. 66,227 26. 58,594 27. 79,329
28. 16,909 29. 3,575 30. 9,408

Exercise 1 13

1. 277 2. 481 3. 31 4. 262 5. 406 6. 746
7. 219 8. 819 9. 295 10. 249 11. 189 12. 393
13. 1,493 14. 5,902 15. 3,889 16. 1,219 17. 1,999
18. 173,468 19. 223,528 20. 2,065,889

Exercise 1 14

1. 16,669 2. 9,844 3. 12,184 4. 9,127 5. 8,657
6. 6,010 7. 7,382 8. 2,081 9. 8,889 10. 7,945
11. 8,833 12. 7,905 13. 4,289 14. 32,824 15. 7,877

Exercise 1 15

1. 3,568 2. 46,852 3. 641,835 4. 105,691
5. 37,167 6. 220,389 7. 128,783 8. 124,772

Exercise 1 16

1. 74,338 2. 2,138 3. 70,124 4. 3,788 5. 54,597.
6. 55,737 7. 16,710 8 82,674 9. 6,683 10. 27,266

Exercise 1 17

1. Debit $1,050 24 2. Credit $1,680 90 3. Credit $4,599 18
4. Debit $2,525 15 5. Debit $1,687 11 6. Debit $3,799 83

Exercise 1.18

1. 215. 2. 2,326. 3. 37,615. 4. \$42,382. 5. \$3,368.11.

Exercise 1.19

1. 28,978. 2. 51,308. 3. 901,990. 4. 11,420.
5. 7,960. 6. 1,109,507; 894,107; 1,001,958; 1,001,656.
7. \$223.34. 8. \$425.35. 9. \$188.00. 10. \$1,303.97.
11. \$612.06. 12. \$2,809.59; \$1,334.71; \$1,391.58; \$2,752.72.

Chapter 2

Exercise 2.3

1. 30,861. 2. 7,906. 3. 17,748. 4. 61,855.
5. 344,080. 6. 246,720. 7. 415,226. 8. 68,432.
9. 367,268. 10. 482,258. 11. 470,862. 12. 1,007,552.
13. 4,310,229. 14. 2,512,548. 15. 2,000,960. 16. 5,142,078.
17. 1,874,080. 18. 3,578,211. 19. 3,783,608. 20. 8,166,438.

Exercise 2.4

1. 38,400. 2. 202,800. 3. 191,400. 4. 213,500.
5. 488,400. 6. 392,400. 7. 14,400,000. 8. 13,574,000.
9. 405,900. 10. 600,000. 11. 22,542. 12. 87,971.
13. 43,160. 14. 28,325. 15. 88,274. 16. 99,297.
17. 245,000. 18. 17,945. 19. 23,040. 20. 301,950.
21. 420. 22. 14,500. 23. 3,890,000. 24. 377,000.
25. 9,420. 26. 4,730. 27. 387,100. 28. 435,000.
29. 3,210,000. 30. 4,100,000. 31. 900. 32. 11,350.
33. 31,850. 34. 154,600. 35. 33,500. 36. 11,000.
37. 17,100. 38. 400,000. 39. 1,862,000. 40. 14,180,000.
41. 6,699. 42. 602. 43. 405. 44. 1,066. 45. 4,876.
46. 4,872. 47. 2,016. 48. 726. 49. 456. 50. 4,080.

Exercise 2.5

1. 10; 130; 120; 160; 8,510. 2. 0; 280; 950; 1,970; 1,250.
3. 0; 200; 500; 900; 1,000; 100; 200; 0; 400; 2,500. 4. 0; 0; 1,000; 1,000; 1,000; 0; 35,000; 6,000. 5. 0; 4,000,000; 16,000,000; 1,000,000; 1,000,000.

Exercise 2.6

1. 320,000. 2. 10,000,000. 3. 14,000,000. 4. 32,000,000.
5. 20,000,000. 6. 27,000,000. 7. 20,000,000. 8. 300,000,000,000.
9. 2,000,000,000. 10. 1,800,000,000.

Exercise 2 7

1. 169,068 2. 271,982 3. 723,456 4. 187,308
5. 392,368 6. 17,332,308 7. 471,546 8. 318,546
9. 62,032,264 10. 22,150,160 11. 125,879 12. 4,631,202
13. 12,227,292 14. 11,908,312 15. 42,791,895
16. 33,978,546 17. 10,840,592 18. 23,493,089
19. 7,804,048 20. 33,753,786 21. 31,449,763
22. 47,939,850 23. 86,412,252 24. 33,570,306
25. 46,082,790 26. 35,786,929 27. 38,165,511.
28. 24,455,316 29. 23,569,891 30. 14,593,735

Exercise 2 8

1. 9,3, 6,6 and 3,12 and 9,4 and 18,2, 5,9 and 15,3, 9,6 and 18,3 and 27,2, 9,8 and 18,4 and 36,2 and 24,3 and 6,12 2. 6,7 and 2 21 and 3,14, 26,2 and 13 4, 7,8 and 14,4 and 28,2, 8,8 and 4,16 and 2,32, 6,4 and 3,8 and 12,2 3. 9,9 and 3,27, 15,5 and 25,3, 4,8 and 16,2, 13,3, 17,3 4. 8,10 and 5 16 and 4,20 and 2,40, 8,12 and 3,32 and 4,24 and 6,16 and 2,48, 12,13 and 6,26 and 4,39 and 3,52 and 2,78, 91,2, 21,10 and 7,30 and 42,5 and 105,2 and 3,70 and 6 and 35 5 17,17, 152,2 and 76,4 and 38,8 and 19,16, 3,107, 49,2 and 7,14, 12,12 and 6,24 and 3,48 and 4,36 and 2,72 and 8,18 and 9,16

Exercise 2 9

1. 137 2. 55 3. 467 4. 602 5. 64 6. 931
7. 711 8. 93 9. 304 10. 305 11. 103 12. 41
13. 806 14 1,412 15. 810 16. 4 17. 16 18. 151
19. 307 20. 6 21. 81 22. 155 23. 711 24. 15
25. 91 26. 91 27. 8 28. 81 29. 27 30. 94
31. 34 32. 17 33. 28 34. 122 35. 111 36. 69
37. 19 38. 342 39. 48 40. 14 41. 18 42. 19
43. 18 44. 152 45. 24 46. 420 47. 190 48. 4,286
49. 2,567 50. 1,041 51. 5,892 52. No 53. No
54. 497 55. No

Exercise 2 10

1. 178 2. 106 3. 30 4. 78 5. 210 6. 49
7. 144 8. 71 9. 53 10. 43 11. 4 384 12. 4,157
13. 418 14. 427. 15. 842 16. 430 17. 194
18. 356 19. 324 20. 303

Exercise 2.11

1. $872\frac{3}{8}$; 872.375. **2.** $85\frac{23}{28}$; 85.821. **3.** $170\frac{29}{64}$; 170.453125. **4.** $61\frac{3}{4}$; 61.75. **5.** $542\frac{1}{2}$; 542.5. **6.** $4{,}157\frac{2}{5}$; 4,157.4. **7.** $418\frac{461}{926}$; 418.497. **8.** $303\frac{1}{2}$; 303.5. **9.** $255\frac{1}{8}$; 255.125. **10.** $24\frac{1969}{5807}$; 24.339.

Exercise 2.12

1. 709. **2.** 71. **3.** 43. **4.** 265. **5.** 603. **6.** 60. **7.** 483. **8.** 85. **9.** 48. **10.** 152. **11.** $42\frac{1}{42}$. **12.** $33\frac{13}{47}$. **13.** 40. **14.** 309. **15.** $35\frac{19}{44}$. **16.** $35\frac{5}{9}$. **17.** 53. **18.** $130\frac{87}{109}$. **19.** $336\frac{65}{263}$. **20.** $407\frac{290}{391}$.

Exercise 2.13

1. 69. **2.** 4,361. **3.** $2{,}751\frac{3}{4}$. **4.** $927\frac{2}{5}$. **5.** $1{,}300\frac{1}{5}$.

Exercise 2.14

1. 13. **2.** 43. **3.** 127. **4.** 384. **5.** 224. **6.** 8.2. **7.** 38.6. **8.** 7.28. **9.** 9.1157. **10.** 27.38.

Chapter 3

Exercise 3.1

1. $1\frac{3}{5}$. **2.** $2\frac{1}{7}$. **3.** $2\frac{3}{4}$. **4.** $3\frac{1}{5}$. **5.** $2\frac{1}{3}$. **6.** $6\frac{3}{4}$. **7.** $4\frac{3}{8}$. **8.** $4\frac{2}{3}$. **9.** 8. **10.** 4. **11.** $\frac{7}{3}$. **12.** $\frac{17}{5}$. **13.** $\frac{29}{8}$. **14.** $\frac{17}{7}$. **15.** $\frac{17}{3}$. **16.** $\frac{29}{12}$. **17.** $\frac{25}{16}$. **18.** $\frac{38}{9}$. **19.** $\frac{25}{4}$. **20.** $\frac{64}{15}$.

Exercise 3.2

1. $\frac{1}{2}$; $\frac{3}{4}$; $\frac{1}{2}$; $\frac{3}{4}$; $\frac{9}{11}$. **2.** $\frac{3}{4}$; $\frac{1}{3}$; $\frac{7}{12}$; $\frac{16}{21}$; $\frac{1}{10}$. **3.** $\frac{17}{18}$; $\frac{17}{72}$; $\frac{7}{26}$; $\frac{5}{13}$; $\frac{3}{8}$. **4.** $\frac{1}{5}$; $\frac{5}{12}$; $\frac{13}{18}$; $\frac{18}{29}$; $\frac{1}{2}$. **5.** $\frac{9}{16}$; $\frac{5}{6}$; $\frac{4}{5}$; $\frac{3}{4}$; $\frac{13}{18}$. **6.** $\frac{4}{32}$; $\frac{3}{12}$. **7.** $\frac{3}{15}$; $\frac{4}{40}$. **8.** $\frac{4}{14}$; $\frac{28}{42}$. **9.** $\frac{15}{40}$; $\frac{25}{45}$. **10.** $\frac{30}{72}$; $\frac{36}{64}$. **11.** $\frac{6}{8}$; $\frac{9}{12}$; $\frac{15}{20}$; $\frac{21}{28}$; $\frac{48}{64}$; $\frac{75}{100}$. **12.** $\frac{4}{6}$; $\frac{6}{9}$; $\frac{10}{15}$; $\frac{12}{18}$; $\frac{16}{24}$; $\frac{18}{27}$; $\frac{20}{30}$; $\frac{24}{36}$. **13.** $\frac{10}{12}$; $\frac{15}{18}$; $\frac{20}{24}$; $\frac{30}{36}$; $\frac{35}{42}$; $\frac{45}{54}$; $\frac{55}{66}$; $\frac{65}{78}$. **14.** $\frac{6}{10}$; $\frac{12}{20}$; $\frac{18}{30}$; $\frac{24}{40}$; $\frac{45}{75}$; $\frac{48}{80}$; $\frac{54}{90}$; $\frac{60}{100}$. **15.** $\frac{7}{14}$; $\frac{16}{32}$; $\frac{39}{78}$; $\frac{72}{144}$; $\frac{118}{236}$; $\frac{291}{582}$.

Exercise 3.3

1. 36. **2.** 180. **3.** 90. **4.** 90. **5.** 72. **6.** 84. **7.** 1,200. **8.** 36. **9.** 360. **10.** 120.

Exercise 3 4

1. $\frac{7}{12}$, $\frac{5}{8}$, $\frac{2}{3}$, $\frac{7}{10}$, $\frac{3}{4}$ 2. $\frac{3}{10}$, $\frac{1}{3}$, $\frac{3}{8}$, $\frac{5}{12}$, $\frac{7}{16}$ 3. $\frac{9}{16}$, $\frac{7}{12}$, $\frac{3}{5}$, $\frac{5}{8}$, $\frac{2}{3}$ 4. $\frac{1}{12}$, $\frac{5}{36}$, $\frac{3}{20}$, $\frac{3}{16}$, $\frac{1}{5}$ 5. $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, $\frac{8}{9}$ 6. $\frac{11}{12}$, $\frac{7}{8}$, $\frac{13}{16}$, $\frac{17}{24}$ 7. $\frac{7}{12}$, $\frac{9}{16}$, $\frac{11}{20}$, $\frac{13}{24}$ 8. $\frac{5}{6}$, $\frac{7}{8}$, $\frac{3}{4}$, $\frac{2}{3}$ 9. $\frac{4}{9}$, $\frac{5}{12}$, $\frac{2}{5}$, $\frac{3}{8}$, $\frac{1}{3}$ 10. $\frac{4}{25}$, $\frac{5}{32}$, $\frac{3}{20}$, $\frac{2}{15}$, $\frac{1}{8}$

Exercise 3 5

1. $1\frac{1}{12}$ 2. $\frac{23}{24}$ 3 $\frac{37}{45}$ 4. $\frac{23}{36}$ 5. $\frac{5}{12}$ 6. $1\frac{7}{24}$
7. $1\frac{23}{40}$ 8. $1\frac{3}{8}$ 9. $1\frac{1}{4}$ 10. $1\frac{31}{48}$ 11. $1\frac{5}{24}$ 12. $\frac{27}{32}$
13. $1\frac{2}{15}$ 14. $2\frac{9}{100}$ 15. $2\frac{1}{32}$

Exercise 3 6

1. $8\frac{5}{16}$ 2. $8\frac{5}{12}$ 3. $14\frac{1}{6}$ 4. $22\frac{47}{48}$ 5. $11\frac{7}{24}$ 6. $6\frac{1}{48}$
7. $4\frac{5}{48}$ 8. $6\frac{3}{4}$ 9. $7\frac{11}{18}$ 10. $7\frac{17}{24}$ 11. $5\frac{7}{16}$ 12. $5\frac{5}{24}$
13. $12\frac{2}{13}$ 14. $16\frac{15}{28}$ 15. $10\frac{11}{24}$ 16. $7\frac{5}{24}$ 17. $6\frac{43}{48}$
18. $8\frac{1}{2}$ 19. $54\frac{11}{12}$ 20. $14\frac{2}{9}$

Exercise 3 7

1. 5/24 2. 27/80 3. 5/18 4. 11/24 5. 11/90
6. 1/4 7. 9/16 8. 3/20 9. 7/24 10. 13/40
11. 1/6 12. 3/20 13. 1/10 14. 11/16 15. 1/12
16. 13/30 17. 1/14 18. 1/12 19. 2/9 20. 5/12

Exercise 3 8

1. $1\frac{3}{8}$ 2. $3\frac{11}{48}$ 3 $4\frac{5}{24}$ 4. $2\frac{5}{24}$ 5. $1\frac{13}{15}$ 6. $10\frac{13}{14}$
7. $88\frac{7}{12}$ 8. $2\frac{35}{48}$ 9. $12\frac{5}{12}$ 10. $4\frac{5}{16}$ 11. $14\frac{5}{8}$
12 $15\frac{5}{8}$ 13. $2\frac{7}{16}$ 14. $\frac{11}{20}$ 15. $\frac{19}{24}$

Exercise 3 9

1. $3\frac{3}{4}$ 2. $2\frac{2}{3}$ 3. $6\frac{2}{3}$ 4. $3\frac{1}{3}$ 5. $1\frac{1}{2}$ 6. $4\frac{1}{2}$
7. 14 8. $3\frac{3}{4}$ 9. 9 10. 6/7 11. 1/8 12. 3/20
13. 1/4 14. 2/45 15. 35/72

Exercise 3 10

1. 1/8 2. 5/6 3. 10 4. 1/16 5. 1 6. 2/27
7. 2/9 8. 5/32 9. 5/18 10. 1/16

Exercise 3 11

1. $7\frac{3}{5}$ 2. 170 3. 171 4. $7\frac{4}{11}$ 5. 9 6. 115
7. 102 8. $881\frac{1}{12}$ 9. $45\frac{7}{9}$. 10. $177\frac{5}{6}$ 11. 18
12. $658\frac{1}{3}$ 13. 310 14. $24\frac{281}{776}$ 15. 4,200

Exercise 3.12

1. 3/4. **2.** 5/6. **3.** $1\frac{3}{4}$. **4.** 16/21. **5.** $1\frac{13}{14}$. **6.** 1/2. **7.** $2\frac{4}{13}$. **8.** $1\frac{1}{24}$. **9.** 8. **10.** 9. **11.** 16. **12.** 18. **13.** 1/20. **14.** 3/128. **15.** 1/40. **16.** 4/9. **17.** $1\frac{1}{2}$. **18.** $3\frac{1}{3}$. **19.** $1\frac{1}{2}$. **20.** 8/9. **21.** $1\frac{2}{3}$. **22.** $1\frac{2}{5}$. **23.** $3\frac{2}{9}$. **24.** 4/5. **25.** 24. **26.** $13\frac{1}{8}$. **27.** $39\frac{1}{5}$. **28.** $9\frac{3}{4}$. **29.** 24. **30.** $13\frac{1}{2}$.

Exercise 3.13

1. 45/52. **2.** 9/14. **3.** 21/22. **4.** 9/10. **5.** 3/4. **6.** 5/6. **7.** 2. **8.** 5/6. **9.** 10. **10.** $1\frac{1}{6}$. **11.** 3/80. **12.** $6\frac{2}{33}$. **13.** 15/32. **14.** 2. **15.** $3\frac{109}{225}$.

Chapter 4

Exercise 4.1

1. 123.033. **2.** 50.494. **3.** 134.2308. **4.** 67.839. **5.** 9.5058. **6.** 70.2030. **7.** 67.1749. **8.** 40.488. **9.** 140.293. **10.** 18.72201. **11.** $128.56. **12.** $164.88. **13.** $279.31. **14.** $330.95. **15.** $173.35; $187.84; $221.78; $320.73; $903.70. **16.** 12.998. **17.** 134.732. **18.** 7.2226. **19.** 4.917. **20.** 0.054. **21.** 32.52. **22.** 28.49. **23.** 4.952. **24.** 44.148. **25.** 9.267. **26.** 18.351. **27.** $15.41. **28.** $12.04. **29.** $7.78. **30.** $18.69. **31.** $288.65; $234.73; $53.92.

Exercise 4.2

1. 2. **2.** 3. **3.** 5. **4.** 5. **5.** 2. **6.** 3. **7.** 7. **8.** 1. **9.** 4. **10.** 1. **11.** 416.2. **12.** 0.006420. **13.** 0.003721. **14.** 2,452. **15.** 0.08384. **16.** 16,190,000. **17.** 3,785,000. **18.** 82,320. **19.** 0.0008283. **20.** 3.008. **21.** 2,099. **22.** 160. **23.** 72.5. **24.** 28.41. **25.** 19.8. **26.** 3,130. **27.** 126,000. **28.** 67.38. **29.** 130.9. **30.** 1,790. **31.** 0.000191. **32.** 0.0000185. **33.** 0.00267. **34.** 0.00012. **35.** 0.0001943. **36.** 0.4434. **37.** 19.9. **38.** 42.9. **39.** 0.0353. **40.** 9.633.

Exercise 4.3

1. 17,100. **2.** 4,450. **3.** 0.283. **4.** 0.000208. **5.** 17.5. **6.** 146.1. **7.** 150. **8.** 243.0. **9.** 44.74. **10.** 1250.

Exercise 4 4

1. 4 9 2. 1 79 3. 645 4. 17 9 5. 2,080
6. 0 1659 7. 37 7 8. 194 9. 0 0672 10. 264
11. 46 12. 0 0000239 13. 18 0 14. 6 81 15. 51 4
16. 18 7 17. 12 00 18. 26 32 19. 1 482 20. 7 01
21. 3 782 22. 1 545 23. 49 96 24. 65 30 25. 0 01500

Exercise 4 5

1. 12/25 2. 7/200 3. 3/400 4. 3/5,000 5. 7/8
6 1/160 7. 17/400 8 2/125 9. 9/16 10. 1/8
11. 2/45 12 151/600 13. 9/140 14. 23/200,000
15. 7/1,000 16 0 5833 17. 0 225 18. 0 1166
19. 0 004 20. 0 875 21. 0 5625 22. 0 266
23. 0 32 24. 0 5 25. 0 4 26. 0 025 27. 0 05
28 0 33 29. 0 16 30. 0 1

Exercise 4 6

1. 1/2 2. 1/3 3 1/6 4. 1/8 5. 1/9 6. 1/12
7. 1/15 8. 1/16 9. 1/24 10. 1/30 11. 1/40
12. 1/60 13. 1/150 14. 5/8 15. 16/25 16. 2/3
17. 3/4 18. 4/5 19. 33/40 20. 7/8

Exercise 4 7

1. 4,400 2. 8,000 3. 81,333$\frac{1}{3}$ 4. 54,700 5. 547
6. 16,200 7. 46,250 8. 60,900 9. 500,000 10. 5,283$\frac{1}{3}$
11. 90 12. 55,000

Exercise 4 8

1. \$9 2. \$9 3. \$9 4. \$8 5. \$8 6. \$31,50
7. \$49 8. \$15 9. \$24 10. \$10 11. \$54 12. \$60
13. \$35 14. \$48 15. \$40 16. \$31 75 17. \$8
18. \$16 20 19. \$16 67 20. \$17 50

Exercise 4 9

1. 3 2. 14 4 3. 7 245 4. 7 668 5. 14
6. 58 44 7. 346 133 8. 34,571 2 9. 19 68
10. 249 9 11. 345 6, 230 4, 172 8, 115 2, 86 4 12. 23 04, 19 2, 11 52, 10 8, 9 6 13. 256, 12 8, 32,000, 6,400, 96
14. 80, 400, 2,000, 900, 360 15. 22 88, 14 56, 12 32, 114 4, 72 8

Chapter 5

Exercise 5.1

1. 0.01 and 1%. **2.** 0.06 and 6%. **3.** 0.6 and 60%. **4.** 0.5 and 50%. **5.** 0.375 and $37\frac{1}{2}$%. **6.** 0.75 and 75%. **7.** 0.4166··· and $41\frac{2}{3}$%. **8.** 0.666··· and $66\frac{2}{3}$%. **9.** 0.875 and $87\frac{1}{2}$%. **10.** 0.095 and $9\frac{1}{2}$%. **11.** 0.0833··· and $8\frac{1}{3}$%. **12.** 0.125 and $12\frac{1}{2}$%. **13.** 0.166··· and $16\frac{2}{3}$%. **14.** $0.46\frac{2}{3}$ and $46\frac{2}{3}$%. **15.** 0.01833··· and $1\frac{5}{6}$%. **16.** $0.58\frac{1}{3}$ and $58\frac{1}{3}$%. **17.** 0.28125 and $28\frac{1}{8}$%. **18.** 0.625 and $62\frac{1}{2}$%. **19.** 0.109375 and $10\frac{15}{16}$%. **20.** 0.96875 and $96\frac{7}{8}$%. **21.** 0.25 and 25%. **22.** 0.4 and 40%. **23.** 0.0625 and $6\frac{1}{4}$%. **24.** 0.3125 and $31\frac{1}{4}$%. **25.** 0.6875 and $68\frac{3}{4}$%. **26.** 0.266··· and $26\frac{2}{3}$%. **27.** $0.333\frac{1}{3}$ and $33\frac{1}{3}$%. **28.** $0.433\frac{1}{3}$ and $43\frac{1}{3}$%. **29.** 0.20833··· and $20\frac{5}{6}$%. **30.** 0.1875 and $18\frac{3}{4}$%. **31.** 0.175 and $17\frac{1}{2}$%. **32.** 0.15625 and $15\frac{5}{8}$%. **33.** 2.5 and 250%. **34.** 0.005 and $\frac{1}{2}$%. **35.** 0.0325 and $3\frac{1}{4}$%. **36.** 0.001 and $\frac{1}{10}$%.

Exercise 5.2

1. 10%. **2.** $6\frac{2}{3}$%. **3.** $5\frac{5}{9}$%. **4.** $\frac{2}{3}$%. **5.** $18\frac{3}{4}$%. **6.** 70%. **7.** $4\frac{1}{11}$%. **8.** $2\frac{2}{27}$%. **9.** 16%. **10.** $11\frac{1}{9}$%. **11.** $4\frac{1}{6}$%. **12.** $58\frac{1}{3}$%. **13.** $31\frac{1}{4}$%. **14.** $1\frac{1}{4}$%. **15.** $2\frac{11}{12}$%. **16.** 9/16%. **17.** $\frac{103}{288}$% or 0.3576%. **18.** $4\frac{42}{97}$%. **19.** $1\frac{118}{125}$% or 1.94%. **20.** $21\frac{194}{961}$% or 21.2%. **21.** $66\frac{2}{3}$%. **22.** $27\frac{7}{9}$%. **23.** $33\frac{1}{3}$%; 25%. **24.** 75%. **25.** 15.57%; 7.97%.

Exercise 5.3

1. 0.45. **2.** 8.5. **3.** 116.4. **4.** 52.65. **5.** 21,250. **6.** 45. **7.** 76.5625. **8.** 3. **9.** \$16.20. **10.** \$18.06. **11.** \$48.81. **12.** \$3.15. **13.** 3.5014. **14.** 0.555. **15.** \$29.87. **16.** 2,340. **17.** 99. **18.** 8,475. **19.** 1,050. **20.** \$10,333.33. **21.** \$2,304; \$4,224; \$5,120; $4\frac{1}{2}$%; \$576. **22.** \$49,011.22. **23.** \$1,996,109.74. **24.** \$20.63. **25.** \$28.84.

Exercise 5.4

1. 1/10. **2.** 1/20. **3.** 3/100. **4.** 1/40. **5.** 7/200. **6.** 3/40. **7.** 1/12. **8.** 4/75. **9.** 1/6. **10.** 9/40. **11.** 9/25. **12.** 1/80. **13.** 3/500. **14.** 7/1,600. **15.** 1/240. **16.** 1/480. **17.** 1/800. **18.** 9/2,000. **19.** 11/400. **20.** 7/160.

Exercise 5 5

1. $18\frac{3}{4}$% 2. $43\frac{3}{4}$% 3. $91\frac{2}{3}$% 4. $29\frac{1}{6}$% 5. $45\frac{5}{6}$%
6. $81\frac{1}{4}$% 7. 35% 8. $36\frac{2}{3}$% 9. $17\frac{1}{2}$% 10. $15\frac{5}{8}$%
11. 17/80 12. 9/200 13. 3/40 14. 13/300 15. 37/200
16. 17/40 17. 9/20 18. 1/320 19. 11/2,400 20. 11/75

Exercise 5 6

1. 50 2. 75 3. 1,800 4. 39 5. 1,518 52 6. 16
7. $50 8 1 9. $24,444\frac{4}{9}$ 10. 480 11. $3,072
12 $20,114 29 13 $100 80 14. $19,440 15. $4 800
16 $22 628 56 17. $345 600 18. $60 32 19. $346 26
20. $128 57 21. $4,500 22. $75 23. $24,074 29
24. $1,818 18 25. $52,857 14 26. $1,875, $2 500, $3,125, $3,750, $5,000, $6,250 27. $80,000 28 2079 69%
29 $2 000,000 30. 1% decrease

Exercise 5 7

1. $253 33 2 $20 58 3. $303 12, $468 75, $750
4. $208 12 $716 88 5. $1 91 6. 28%, $70 7. $37\frac{1}{2}$%
8. $110 40, $369 60 9 The second offer $223 50 v s $218 40
10. The second offer $26 77 v s $26 67

Exercise 5 8

1. a 32%, b 43%, c 30%, d $21\frac{1}{4}$%, e 39 2%, f 53 08%, g 43 3%, h 49 07%, i 51 55%, 2. a $1323 00, b $176 89, c $567 00, d $874 95, e $307 20, f $925 00, g $194 40, h $1107 62, i $2,550 00, j $1451 27 3. $45 90 4. $365 47, $884 53 5. $59 34
6. 90 ¢ 7. $447 12 8. $173 33, $151 67 9 $72 29
10. $153 58 11. $19 74 12. $67 63

Exercise 5 9

	Cash Discount	Amount Paid		Cash Discount	Amount Paid
1.	$27 76	$897 58	6.	$17 48	$565 20
2	$10 50	$514 30	7.	$4 85	$237 77
3.	$12 85	$1,271 71	8.	$64 69	$2,091 79
4.	$13 68	$670 44	9.	$6 56	$321 19
5.	$3 93	$192 32	10.	$7.39	$731 43

Exercise 5.10

1. 25.78. **2.** 36.57. **3.** $16\frac{2}{3}$. **4.** 4. **5.** $6\frac{2}{3}$. **6.** 15.
7. 1/4. **8.** 1.5625. **9.** 200. **10.** 620. **11.** \$800.
12. \$1,680. **13.** \$3,200. **14.** \$1,200. **15.** \$1,800.
16. \$2,400. **17.** \$3,600. **18.** \$90. **19.** \$108. **20.** \$4,800.
21. \$1.50 an hour. **22.** $6\frac{1}{4}\%$. **23.** \$412.50. **24.** 60%; $37\frac{1}{2}\%$.
25. \$274.43. **26.** 22,959. **27.** Second machine; $2\frac{1}{2}\%$.
28. \$728; \$546; \$182. **29.** 34,343. **30.** 1.44%.
31. 125,000; 150,000; 225,000. **32.** 10,370,305. **33.** 11.76%.
34. 54,600,000; 128,100,000; 27,300,000. **35.** \$186.26; \$364.15; \$555.93. **36.** 20.91%. **37.** 15%. **38.** \$145,000.
39. \$600. **40.** \$66. **41.** \$80. **42.** \$3.25. **43.** \$800.
44. \$54.35. **45.** \$1.18.

Exercise 5.11

1.

Assets	*Per Cent of Total 1 year ago*	*Per Cent of Total Today*
Cash	23	14
Accounts receivable	7	13
Inventory	12	21
Total current assets	42	48
Fixed assets	58	52
Total assets	100	100
Liabilities		
Current liabilities	15	16
Long-term liabilities	22	32
Capital stock	53	42
Surplus	10	10
Total liabilities	100	100

2.

Assets	
Cash	76
Accounts receivable	229
Inventory	216
Total current assets	143
Fixed assets	115
Total assets	127
Liabilities	
Current liabilities	133
Long-term liabilities	185
Common stock	100
Surplus	133
Total liabilities	127

3. A, + 40%, B, + 8⅓%, C, − 8½%, D, + 1 2%, E, + 22%
4. $2,737 50, gain 5 27% 5. $5,600

6. Assets	This Year	Last Year	Net Change Decrease	Net Change Increase	Per Cent of Change
Cash	$ 4,244	$ 3,847		$ 397	+ 10 3
U S Gov't securities	7,927	4,320		3,607	+ 83 5
Accounts receivable	6,040	8,660	$2,620		− 30 3
Inventory	28,549	25,925		2,624	+ 10 1
Property, plant, etc	16,559	16,456		103	+ 0 6
Total assets	$63,319	$59,208		$4,111	+ 6 9
Liabilities					
Current liabilities	$17,670	$20,101	$2,431		− 12 1
Long term debt	*10,000*	*2,000*		*$8,000*	*+ 400 0*
Preferred stock	0	8,000	8,000		− 100 0
Common stock	8 000	7,500		500	+ 6 7
Earnings retained	27,649	21,607		6,042	+ 28 0
Total liabilities	$63,319	$59,208		$4,111	+ 6 9

7.	This Year Amount and Per Cent of Net Sales		Last Year Amount and Per Cent of Net Sales	
Net sales	$165,710	100	$135,000	100
Selling, general, and admin expense	30,650	18 5	24,643	18 3
Interest	1,017	0 6	494	0 4
Federal income tax	10,474	6 3	13,815	10 2
Other taxes	1,050	0 6	3,785	2 8
Net profit	7,931	4 8	8,620	6 4

8. Domestic 36 2%, Agricultural 7 2%, Commercial 21 2%, Industrial 25 2%, Public authorities 6 5%, Sales for resale 1 7%, Railway 0 8%, Other 1 1% 9. Bonds 48 8%, Pref stocks 23 8%, Common stocks 27 4% 10. Outlet A, + 100%, Outlet B, − 10%, Outlet C, − 12 5%, Outlet D, + 5 96%, Outlet E, − 10%, Total sales 2nd year $1,672,648, Total sales 1st year $1,628,286, Net change, increase $44 362, % of change of total, + 2 7% 11. $0 75
12. $1,270 20 13. $9,000, $84,000 14. $1,529 50 15. 0 4%
16. $3,025 17. $4,524 68 18. 1 942%, $0 02
19. $121,366 20. 62 1%, 1 66%, 0 83%, 1 16%

Chapter 6

Exercise 6.1

1. 18. **2.** 18. **3.** 29. **4.** 11. **5.** 2. **6.** 10. **7.** 13.
8. 32. **9.** 32. **10.** 2. **11.** 10. **12.** 22. **13.** $5\frac{1}{3}$.
14. 15. **15.** 10. **16.** $3\frac{2}{3}$. **17.** 2. **18.** 1. **19.** 3.
20. 1.

Exercise 6.2

1. $19x$. **2.** $4y$. **3.** $4a$. **4.** $3cd$. **5.** $13ab$. **6.** $3ni$.
7. $33ab$. **8.** $30cd$. **9.** $14f$. **10.** $6b$. **11.** $8a + 8c$.
12. $7ab + 4cd$. **13.** $2xy + 5ab$. **14.** $2pq + 5p$. **15.** $4r + 2s$.
16. $2w + 4$. **17.** $ab + 3$. **18.** $9x + 3y + 3$. **19.** $3e$.
20. $6a + 2b + 2c$.

Exercise 6.3

1. $9x + 10y + 12z$. **2.** $10a + 4b + 2c$. **3.** $7ab + 6bc + 6cd$.
4. $11a + 4b + 5$. **5.** $7xy + ab + 5$. **6.** $6x + 3y + 5$.
7. $16a + 6b + 4c$. **8.** $10w + 4v + 4q$. **9.** $5x + 4y + 5z$.
10. $2ab + 6cd + 3ef$.

Exercise 6.4

1. -4. **2.** -15. **3.** -3. **4.** $+9$. **5.** -4.
6. -7. **7.** -3. **8.** -13. **9.** $+46$. **10.** -67.
11. $19x$. **12.** $-26y$. **13.** $1\frac{1}{9}d$. **14.** $-\frac{2}{11}z$. **15.** a.
16. $0.73w$. **17.** $-12.7c$. **18.** $1.24y$. **19.** $-1.375a$.
20. $3.25cd$.

Exercise 6.5

1. -2. **2.** $+7$. **3.** -11. **4.** $+1$. **5.** $+5$.
6. $+15$. **7.** -8. **8.** -15. **9.** -26. **10.** $+39$.
11. $+2/9$. **12.** $+1/32$. **13.** $-8/11$. **14.** $+59/72$.
15. $-2a$. **16.** $-7.8x$. **17.** $-40f$. **18.** $-23a$.
19. $-14f$. **20.** $+6xz$.

Exercise 6.6

1. $2x + 3y$. **2.** $-7x - 5y$. **3.** $-x + 8y$. **4.** $3x$.
5. $-5x$. **6.** $-2x - 3y$. **7.** $7x + 5y$. **8.** $x - 8y$. **9.** $-3x$.
10. $5x$.

Exercise 6 7

1. $+4$ 2. $+7$ 3. -13 4. $+6$ 5. -13
6. -5 7. 0 8. $+14$ 9. $+15$ 10. -139
11. -5.83 12. $+0.09$ 13. $+10.83$ 14. $1\frac{2}{5}$ 15. $8\frac{1}{2}$
16. $9wz$ 17. $4w$ 18. 0 19. $-11cd$ 20. $-40f+21g$

Exercise 6 8

1. -12 2. $+35$ 3. -21 4. $+24$ 5. -40
6. $+42$ 7. -24 8. $+30$ 9. $+54$ 10. -48
11. -28 12. $+40$ 13. $+30$ 14. -120
15. $+168$ 16. -48 17. $+160$ 18. $+105$
19. -360 20. -81 21. -288 22. $+1008$
23. $+320$ 24. -324 25. -240 26. $+96$
27. -120 28. $+28$ 29. $+288$ 30. $+360$

Exercise 6 9

1. $+4$ 2. -7 3. -12 4. $+9$ 5. -7
6. $+5$ 7. $+8$ 8. -5 9. $+7$ 10. -6 11. $+7$
12. $+9$ 13. -5 14. $+7$ 15. $+9$ 16. -11
17. -9 18. $+9$ 19. -3 20. $+25$ 21. $+4$
22. $+3$ 23. -2 24. -9 25. -6 26. -4
27. -6 28. $-2\frac{2}{3}$ 29. -3 30. -4

Exercise 6 10

1. x^6 2. a^9 3. y^{14} 4. a^3b^4 5. x^4y^4 6. x^2.
7. y^2 8. $\frac{1}{x^3}$ 9. 1 10. $\frac{1}{w^4}$ 11. $8\sqrt{3}$ 12. $3\sqrt{17}$
13. $5\sqrt[3]{15}$ 14. $9\sqrt{14}$ 15. $3\sqrt[4]{9}$ 16. $7\sqrt{6}+3\sqrt{7}$
17. $9+9\sqrt{2}$ 18. $3\sqrt[3]{4}+3\sqrt[3]{12}$ 19. $\sqrt{5}+3\sqrt[3]{5}$
20. $3\sqrt[3]{2}+6\sqrt[3]{3}$

Exercise 6 11

1. $56x^3$ 2. $15a^5$ 3. $10a^4$ 4. $12y^4$ 5. $6a^3b^3$
6. $40x^3y^4$ 7. $35xw^5$ 8. $45a^6b$ 9. $54a^5b^2$ 10. $45x^3y^2$
11. $28x^2+21x-56$ 12. $12x^3+60x^2-36$ 13. $4x^3+12x^2$
$+8x$ 14. $21a^3-15a+9b$ 15. $10x^3-6x^2+4x$
16. $84x^5-56x^4+28x^3-14x^2$ 17. $15a^2b^3+24a^3b^2$
18. $12x^3y^3-8x^4y^2-12x^2y^2$ 19. $20a^3b^3-32a^4b^3+8a$
20. $-12w^3z^5+9w^4z^3+6w^2z^4$

Exercise 6.12

1. $x^2 - 6x + 9$. 2. $9x^2 + 12xy + 4y^2$. 3. $4x^2 - 20xy + 25y^2$. 4. $16a^2 - 8ab + b^2$. 5. $9a^2 - 12ab + 4b^2$. 6. $25x^2 - 30xy + 9y^2$. 7. $a^2 - 25$. 8. $4x^2 - 9y^2$. 9. $4a^2 - 9b^2$. 10. $9a^2 - 4c^2$. 11. $4x^2 - 25y^2$. 12. $9x^2 - 16$. 13. $x^2 + 7x + 12$. 14. $x^2 - 2x - 15$. 15. $x^2 - 4x - 21$. 16. $x^2 - 6x + 8$. 17. $x^2 + 3x - 4$. 18. $x^2 + 2x - 15$. 19. $6x^2 + x - 12$. 20. $8x^2 - 10x + 3$. 21. $6a^2 - 11ab - 10b^2$. 22. $15x^2 - 14xy - 8y^2$. 23. $6x^2 + 5xy - 6y^2$. 24. $2x^2 + 3xy - 5y^2$. 25. $2x^2 + 5xy + 2y^2$. 26. $6x^2 + xy - y^2$. 27. $6x^2 - 19xy + 10y^2$. 28. $6a^2 + 17ab + 12b^2$. 29. $10a^2 + 9ab - 9b^2$. 30. $21w^2 - 13wz - 20z^2$.

Exercise 6.13

1. $4x + 2$. 2. $a + 9$. 3. $-c + 2d$. 4. $2w - 13$. 5. $-3w - 2s + 13$. 6. $2x - 6$. 7. $a + 2b - 2c$. 8. $-2x - 3y + 7$. 9. $x - 14$. 10. $7a - 3$. 11. $-x - 4y + 11$. 12. $5a - 8b - c$. 13. $5x + 5$. 14. $-5w - x + 8$. 15. $-x - 4y - 9$. 16. $2x + 3$. 17. $x + 4$. 18. $5a + 2b$. 19. $3x - 10y$. 20. $-15x - 8$. 21. $3(x + 7)$. 22. $2(x - 4)$. 23. $2(2x - 5)$. 24. $-2(x - 2)$. 25. $-5(x + 3)$. 26. $4(2x + y - 5)$. 27. $2(3a + 4b - 2c)$. 28. $3(a + 2b) - 5(c + 2d)$. 29. $-4(a - 2b) + 3(x - 3y)$. 30. $a(x + y) + b(w - 2v)$.

Exercise 6.14

1. $ax(a - x)$. 2. $4x(2x - 3)$. 3. $3x^2(x + 2y)$. 4. $3yz(3y - z)$. 5. $7x^2(2 - y)$. 6. $7(1 - 2x^2)$. 7. $4x(x^2 + 2xy^2 - 3y^3)$. 8. $5x^2(1 + 2y - 4y^3)$. 9. $4ab(3abc^2 - 2c - a)$. 10. $w(3w^2 - 5wz - 4z^3 + 1)$.

Exercise 6.15

1. $(x + 5)(x - 5)$. 2. $(2x + y)(2x - y)$. 3. $(3a + 4b)(3a - 4b)$. 4. $(4x + 5y)(4x - 5y)$. 5. $(6a + 5b)(6a - 5b)$. 6. $(3x + 2a)(3x - 2a)$. 7. $(3x^2 + 4y)(3x^2 - 4y)$. 8. $(2ab + 3c)(2ab - 3c)$. 9. $(4xyz + wv)(4xyz - wv)$. 10. $(ab + 2cd)(ab - 2cd)$.

Exercise 6.16

1. $(x + 2)^2$. 2. $(x - 4)^2$. 3. $(x - 5)^2$. 4. $(x + 1)^2$. 5. $(x - 6)^2$. 6. $(2x - 3)^2$. 7. $(3x + 1)^2$. 8. $(4x - 3)^2$. 9. $(2x + 5)^2$. 10. $(6x + 1)^2$.

Exercise 6 7

1. $+4$ 2. $+7$ 3. -13 4. $+6$ 5. -13
6. -5 7. 0 8. $+14$ 9. $+15$ 10. -139
11. -5.83 12. $+0.09$ 13. $+10.83$ 14. $1\frac{2}{5}$ 15. $8\frac{1}{2}$
16. $9wz$ 17. $4w$ 18. 0 19. $-11cd$ 20. $-40f+21g$

Exercise 6 8

1. -12 2. $+35$ 3. -21 4. $+24$ 5. -40
6. $+42$ 7. -24 8. $+30$ 9. $+54$ 10. -48
11. -28 12. $+40$ 13. $+30$ 14. -120
15. $+168$ 16. -48 17. $+160$ 18. $+105$
19. -360 20. -81 21. -288 22. $+1008$
23. $+320$ 24. -324 25. -240 26. $+96$
27. -120 28. $+28$ 29. $+288$ 30. $+360$

Exercise 6 9

1. $+4$ 2. -7 3. -12 4. $+9$ 5. -7
6. $+5$ 7. $+8$ 8. -5 9. $+7$ 10. -6 11. $+7$
12. $+9$ 13 -5 14. $+7$ 15. $+9$ 16. -11
17. -9 18. $+9$ 19. -3 20. $+25$ 21. $+4$
22. $+3$ 23. -2 24. -9 25. -6 26. -4
27. -6 28. $-2\frac{2}{3}$ 29. -3 30. -4

Exercise 6 10

1. x^6 2. a^9 3. y^{14} 4. a^3b^4 5. x^4y^4 6. x^2
7. y^2 8. $\frac{1}{x^3}$ 9. 1 10. $\frac{1}{w^4}$ 11. $8\sqrt{3}$ 12. $3\sqrt{17}$
13 $5\sqrt[3]{15}$ 14. $9\sqrt{14}$ 15. $3\sqrt[4]{9}$ 16. $7\sqrt{6}+3\sqrt{7}$
17. $9+9\sqrt{2}$ 18. $3\sqrt[3]{4}+3\sqrt[3]{12}$ 19. $\sqrt{5}+3\sqrt[3]{5}$
20. $3\sqrt[3]{2}+6\sqrt[3]{3}$

Exercise 6 11

1. $56x^3$ 2. $15a^5$ 3. $10a^4$ 4 $12y^4$ 5. $6a^3b^3$
6. $40x^3y^4$ 7. $35xw^5$ 8. $45a^6b$ 9. $54a^5b^2$ 10. $45x^3y^2$
11. $28x^2+21x-56$ 12. $12x^3+60x^2-36$ 13. $4x^3+12x^2$
$+8x$ 14. $21a^3-15a+9b$ 15. $10x^3-6x^2+4x$
16. $84x^5-56x^4+28x^3-14x^2$ 17. $15a^2b^3+24a^3b^2-12ab$
18. $12x^3y^3-8x^4y^2-12x^2y^2$ 19. $20a^3b^3-32a^4b^3+8a^3b^4$.
20. $-12w^3z^5+9w^4z^3+6w^2z^4$

Exercise 6.12

1. $x^2 - 6x + 9$. **2.** $9x^2 + 12xy + 4y^2$. **3.** $4x^2 - 20xy + 25y^2$. **4.** $16a^2 - 8ab + b^2$. **5.** $9a^2 - 12ab + 4b^2$. **6.** $25x^2 - 30xy + 9y^2$. **7.** $a^2 - 25$. **8.** $4x^2 - 9y^2$. **9.** $4a^2 - 9b^2$. **10.** $9a^2 - 4c^2$. **11.** $4x^2 - 25y^2$. **12.** $9x^2 - 16$. **13.** $x^2 + 7x + 12$. **14.** $x^2 - 2x - 15$. **15.** $x^2 - 4x - 21$. **16.** $x^2 - 6x + 8$. **17.** $x^2 + 3x - 4$. **18.** $x^2 + 2x - 15$. **19.** $6x^2 + x - 12$. **20.** $8x^2 - 10x + 3$. **21.** $6a^2 - 11ab - 10b^2$. **22.** $15x^2 - 14xy - 8y^2$. **23.** $6x^2 + 5xy - 6y^2$. **24.** $2x^2 + 3xy - 5y^2$. **25.** $2x^2 + 5xy + 2y^2$. **26.** $6x^2 + xy - y^2$. **27.** $6x^2 - 19xy + 10y^2$. **28.** $6a^2 + 17ab + 12b^2$. **29.** $10a^2 + 9ab - 9b^2$. **30.** $21w^2 - 13wz - 20z^2$.

Exercise 6.13

1. $4x + 2$. **2.** $a + 9$. **3.** $-c + 2d$. **4.** $2w - 13$. **5.** $-3w - 2s + 13$. **6.** $2x - 6$. **7.** $a + 2b - 2c$. **8.** $-2x - 3y + 7$. **9.** $x - 14$. **10.** $7a - 3$. **11.** $-x - 4y + 11$. **12.** $5a - 8b - c$. **13.** $5x + 5$. **14.** $-5w - x + 8$. **15.** $-x - 4y - 9$. **16.** $2x + 3$. **17.** $x + 4$. **18.** $5a + 2b$. **19.** $3x - 10y$. **20.** $-15x - 8$. **21.** $3(x + 7)$. **22.** $2(x - 4)$. **23.** $2(2x - 5)$. **24.** $-2(x - 2)$. **25.** $-5(x + 3)$. **26.** $4(2x + y - 5)$. **27.** $2(3a + 4b - 2c)$. **28.** $3(a + 2b) - 5(c + 2d)$. **29.** $-4(a - 2b) + 3(x - 3y)$. **30.** $a(x + y) + b(w - 2v)$.

Exercise 6.14

1. $ax(a - x)$. **2.** $4x(2x - 3)$. **3.** $3x^2(x + 2y)$. **4.** $3yz(3y - z)$. **5.** $7x^2(2 - y)$. **6.** $7(1 - 2x^2)$. **7.** $4x(x^2 + 2xy^2 - 3y^3)$. **8.** $5x^2(1 + 2y - 4y^3)$. **9.** $4ab(3abc^2 - 2c - a)$. **10.** $w(3w^2 - 5wz - 4z^3 + 1)$.

Exercise 6.15

1. $(x + 5)(x - 5)$. **2.** $(2x + y)(2x - y)$. **3.** $(3a + 4b)(3a - 4b)$. **4.** $(4x + 5y)(4x - 5y)$. **5.** $(6a + 5b)(6a - 5b)$. **6.** $(3x + 2a)(3x - 2a)$. **7.** $(3x^2 + 4y)(3x^2 - 4y)$. **8.** $(2ab + 3c)(2ab - 3c)$. **9.** $(4xyz + wv)(4xyz - wv)$. **10.** $(ab + 2cd)(ab - 2cd)$.

Exercise 6.16

1. $(x + 2)^2$. **2.** $(x - 4)^2$. **3.** $(x - 5)^2$. **4.** $(x + 1)^2$. **5.** $(x - 6)^2$. **6.** $(2x - 3)^2$. **7.** $(3x + 1)^2$. **8.** $(4x - 3)^2$. **9.** $(2x + 5)^2$. **10.** $(6x + 1)^2$.

Exercise 6 17

1. $(x+3)(x+5)$ 2. $(x-6)(x-2)$ 3. $(x+3)(x+1)$
4. $(x-5)(x-2)$ 5. $(x+3)(x-2)$ 6 $(x-5)(x+4)$
7. $(x-8)(x+3)$ 8 $(x+7)(x-4)$ 9. $(x+5)(x+8)$
10. $(x-5)(x-7)$

Exercise 6 18

1. $(2x-3)(x-2)$ 2. $(2x+3)(x-2)$ 3. $(2x+3)(x+2)$
4. $(2x-3)(x+2)$ 5. $(3x-2)(x-5)$ 6. $(3x-2)(x+5)$
7. $(3x+5)(x-2)$ 8 $(6x-1)(x-4)$ 9. $(6x+1)(x-2)$
10. $(6x+1)(x+2)$

Exercise 6 19

1. $3(x+2y)$ 2. $5x(x-3)$ 3. $P(1+i)$
4. $2a(a+3-2a^2)$ 5. $(a+2b)(a-2b)$ 6. $(4x+3y)(4x-3y)$
7. $(27+24)(27-24) = 51\ 3 = 153$ 8. $(x+2)^2$ 9. $(x-y)^2$
10. $(30+2)^2 = 900+120+4 = 1{,}024$, $(30+5)^2 = 900+300+25 = 1{,}225$ 11. $(2a-5x)^2$ 12. $(x-3)(x-4)$
13. $(x+4)(x-3)$ 14. $(x+5)(x-2)$ 15. $(x-5)(x-2)$
16. $(2x+y)(x-2y)$ 17. $(2x+3y)(2x+y)$
18. $(4x+3)(3x-1)$ 19. $(3x+2y)(2x+y)$
20. $(3x+1)(2x-3)$

Chapter 7

Exercise 7 1

1. Identical 2. Conditional 3. Conditional 4. Identical
5. Conditional 6. Identical 7. Identical 8. Conditional
9. *Identical* 10. *Identical*

Exercise 7 2

1. 4 2. 5 3. 15 4. 1 5. 11 6. 25 7. 13
8. 9 9. -3 10. -5 11. 4 12. -5 13. 20
14. 6 15. -12 16. -16 17. 9 18. $4\frac{1}{2}$ 19. $-4\frac{1}{2}$
20. $-4\frac{4}{5}$

Exercise 7 3

1. *4* 2. 6 3. -3 4. -2 5. -6 6. 6
7. 1 8. 9 9. -12 10. -15 11. 32 12. 5
13. 3 14. 9 15. 3 16. $\frac{1}{4}$ 17. 1 18. 2
19. $-\frac{3}{8}$ 20. 5 21. $-9\frac{1}{2}$ 22. -1 23. 5 24. $\frac{1}{3}$
25. 2 26. -16 27. -5 28. 5 29. *1* 30. $-1\frac{1}{2}$
31. 24 32. 6 33. $\frac{1}{2}$ 34. -6 35. 2 36. 3
37. 2 38. -4 39. $3\frac{1}{3}$ 40. 13

Exercise 7.4.

1. 6. 2. 24. 3. $12\frac{1}{2}$. 4. 6. 5. -10. 6. 6.
7. 6. 8. 4. 9. 10. 10. $13\frac{1}{5}$. 11. $1\frac{1}{2}$. 12. $4\frac{2}{3}$.
13. 5. 14. 3. 15. $1\frac{1}{3}$.

Exercise 7.5

1. 7. 2. 2. 3. -2. 4. 4. 5. 0. 6. $1\frac{1}{2}$. 7. 1.
8. 13. 9. $-7\frac{1}{2}$. 10. $\frac{1}{2}$. 11. -20. 12. $1\frac{1}{2}$. 13. 11.
14. 5. 15. 3. 16. 10. 17. $5\frac{1}{4}$. 18. $2\frac{1}{10}$. 19. 3.
20. 3. 21. 1. 22. 1. 23. 9. 24. -8. 25. -12.
26. $-2\frac{1}{3}$. 27. 1. 28. 3. 29. 4. 30. 8.

Exercise 7.6

1. $S-P$. 2. $S-I$. 3. $\dfrac{C}{Pq}$. 4. $\dfrac{C}{Pr}$. 5. $\dfrac{C}{qr}$.
6. $S-a-c$. 7. $S-a-b$. 8. $\dfrac{S}{(1+i)}$. 9. $\dfrac{S-P}{P}$.
10. $W-ab$. 11. $\dfrac{W-c}{b}$. 12. $\dfrac{2A}{b}$. 13. $\dfrac{k}{2\pi H}$.
14. $\dfrac{b}{b-3}$. 15. $\dfrac{3a}{a-1}$. 16. $Wn-n$. 17. $\dfrac{m}{W-1}$.
18. $\dfrac{Vb}{V-1}$. 19. $\dfrac{Va-a}{V}$. 20. $\dfrac{dD}{d-D}$. 21. $\dfrac{CD}{C-D}$.
22. $CR+Cnr$. 23. $\dfrac{E-CR}{Cn}$. 24. $\dfrac{E-Cnr}{C}$. 25. $\dfrac{wb}{a}$.
26. $\dfrac{wb}{c}$. 27. $\dfrac{ac}{b}$. 28. $\dfrac{AD}{360}$. 29. $\dfrac{360a}{D}$. 30. $\dfrac{360a}{A}$.

Exercise 7.7

1. 15; $x+5$. 2. $3x+1$; $2x-2$. 3. y^2. 4. y^2+10.
5. $x+1$; even. 6. $x+2$; $x+2-5$ or $x-3$. 7. $100-x$.
8. $0.1x$. 9. $x-300$. 10. $2\% \cdot 300 + 1\frac{1}{2}\%(x-300)$ or
$\$6 + 0.015x - \$4.50$. 11. $0.02x$. 12. $0.01(x - \$10{,}000)$;
$\$200 + 0.01x - \$100$. 13. $0.4y$; 0. 14. $\dfrac{0.4y}{y+100}$.
15. $16{,}000 - x$. 16. $\frac{1}{5}$; $\frac{1}{8}$; $\frac{13}{40}$. 17. $\dfrac{1}{x}$; $\dfrac{1}{y}$; $\dfrac{x+y}{xy}$. 18. $\dfrac{x}{5}$.
19. $\dfrac{x}{5}$; $\dfrac{4x}{5}$. 20. $\dfrac{x}{2}$; $\dfrac{x}{8}$; $\dfrac{3x}{8}$.

Exercise 7 8

1. \$25,000 bonds, \$25,000 mortgages 2. \$10,000 bonds, *\$15,000 stock* 3. *80 at 75¢, 20 at \$1* 4. *20 pounds*
5. 75 pounds 6. $66\frac{2}{3}$ quarts 30-cent and $33\frac{1}{3}$ quarts 45-cent oil
7. 4 quarts 8. $17\frac{1}{2}$ ounces 9. 14 of 70¢, 21 of 90¢
10. 36 ounces 11. $\frac{1}{15}$ quart 12. 10,000 at \$1, 70,000 at \$2
13. 508 at \$1, 127 at \$1 50 14. 15 women, 5 men
15. $2\frac{1}{4}$ gallons water, $1\frac{3}{4}$ gallons 80% pure alcohol 16. \$64,000 at *5%, \$36,000 at 4%* 17. *\$5,142 86 at 4%, \$6,857 14 at 3%*
18. $7\frac{1}{2}$ ounces of 40% developer, $2\frac{1}{2}$ ounces of 20% developer

Exercise 7 9

1. 24 dimes, 14 quarters 2. 19 nickels, 13 dimes
3. 40 dimes, 42 quarters 4. \$2,000 per lot, \$10,000 per house
5. 16 new cars, 16 used cars 6. 200 dimes, 400 pennies
7. 54 5 cent sales 8. 92 at 15¢, 56 at 25¢ 9. 100 oranges
10. 24 square yards at \$7 50, 18 square yards at \$9

Exercise 7 10

1. \$150, \$1,350 2. \$229 17 3. \$400 4. 56 25%, 36%
5 \$533 33 6. \$960 7. \$800 8. \$900 9. \$1,920
10. \$458 40 11. \$2,800 12. 120 games 13. \$121,366
14. 537 15. \$4,872 74

Exercise 7 11

1. 500 pounds 2. 160 pounds 3. $7\frac{1}{2}$ days 4. 50 men
5. 45 days 6. 15 markers 7. 8 and 6 inches 8. 24 by 24 and 32 by 18 inches 9. 32 miles per hour 10. $11\frac{1}{4}$ knots

Exercise 7 12

1. 15 2. 8 3. 4 4. 10 5. 15 6. $2\frac{2}{3}$
7. $15\frac{1}{5}$ 8. 11 52 9. 3 75 10. 650 11. 36 12. 80
13. $112\frac{1}{2}$ 14. $18\frac{3}{4}$ 15. $17\frac{1}{2}$ 16. 375 17. 55 18. 80
19. $266\frac{2}{3}$ 20. 7 5

Exercise 7 13

1. \$1 80 2 \$29 00 3. \$93 33 4. \$96
5. 1,260 square feet 6. \$991 67 7. 333 8. 15
9. $4\frac{4}{5}$ feet 10. $23\frac{1}{3}$ 11. 120 pounds 12. \$15, \$20
13. \$1,400, \$2,400 14. \$31,680 15. 525

Exercise 7.14

1. \$2,785; \$5,570; \$8,355; \$11,140. **2.** \$15,750; \$21,000; \$31,500. **3.** \$26,556.10; \$39,834.15; \$39,834.15. **4.** A \$97.30; B \$162.16; C \$291.89; D \$389.19; E \$259.46. **5.** A \$40,000; B \$6,400; C \$33,600; D \$ 96,000; E \$64,000. **6.** A \$1,200; B \$2,400; C \$3,600; D \$3,600; E \$6,000; F \$7,200. **7.** Wilson \$11,500; White \$14,375. **8.** A \$10,000; B \$15,000; C \$12,500. **9.** A \$4,063.42; B \$3,336.58. **10.** A \$6,546.09; B \$4,945.91.

Chapter 8

Exercise 8.1

1. $x = 7, y = 2$. **2.** $x = 2, y = -4$. **3.** $x = 3, y = 2$. **4.** $x = 4, y = -2$. **5.** $x = 4, y = 2$. **6.** $x = 5, y = 4$. **7.** $x = 0, y = 2$. **8.** $x = 1, y = 1$. **9.** $x = 2\frac{3}{8}, y = \frac{3}{4}$. **10.** $x = 8\frac{2}{23}, y = 4\frac{11}{23}$.

Exercise 8.2

1. $x = 3, y = 1$. **2.** $x = 4, y = -1$. **3.** $x = 1, y = -2$. **4.** $x = -3, y = -2$. **5.** $x = -2, y = 3$. **6.** $x = -\frac{1}{2}, y = -\frac{1}{2}$. **7.** $x = 2\frac{1}{2}, y = 1\frac{1}{2}$. **8.** $x = 1, y = -4$. **9.** $x = 3, y = \frac{1}{2}$. **10.** $x = \frac{3}{5}, y = -2\frac{1}{5}$.

Exercise 8.3

1. 24 and 20. **2.** 15 and 10. **3.** 40 and 24. **4.** 4 and 2. **5.** $\frac{1}{2}$ and -1. **6.** 3 and 2. **7.** 4 and 1. **8.** 85 and 58. **9.** 8 and 5. **10.** $\frac{1}{5}$ and $\frac{1}{7}$.

Exercise 8.4

1. 216 and 36 miles per hour, respectively. **2.** 24 up and 36 down. **3.** 18 and 5 miles per hour, respectively. **4.** 12 miles; 40 miles per hour. **5.** Plane 320 miles per hour; wind 40 miles per hour. **6.** 240 miles. **7.** Plane 220 miles per hour; wind 60 miles per hour. **8.** 20 and 12 miles per hour. **9.** 20 miles.

Exercise 8.5

1. 500 cubic centimeters water, 300 cubic centimeters solution. **2.** 40 at \$12,000, 8 at \$15,000. **3.** 560 at \$1, 240 at \$1.25. **4.** 1.6 gallons pure alcohol, 3.4 gallons water. **5.** $5\frac{13}{46}$ gallons first solution, $\frac{33}{46}$ gallons second solution. **6.** 20 at \$6,000, 34 at \$7,200. **7.** \$2,400 (Ford), \$3,000 (Mercury). **8.** 60 at \$8, 140 at \$6. **9.** 18 tons paper, 6 tons magazines.

Exercise 8 6

1. $4\frac{1}{2}$% and 9% 2. \$300,000 at 10%, \$500,000 at 9%
3. \$15,000 at 4%, \$10,000 at 5% 4. \$6,000 at 3%, \$8,000 at $3\frac{3}{4}$%
5. \$22,400 at $3\frac{3}{4}$%, \$7,600 at 4% 6. \$8,000 at $3\frac{1}{4}$%, \$12,000 at 4%
7. \$7,500 at 5% profit, \$2,100 at 12% loss 8 \$8 000 at 8% profit, \$4,000 at 20% loss 9. \$51,428 57 10. \$18,000 and \$12,000

Exercise 8 7

1. Taxes \$115,783 78, Bonus \$20,432 43 2. Federal tax \$1,900, State tax \$400 3. Federal tax \$3,200, State tax \$400
4. Taxes \$80,151 83, Bonus \$10,884 82

Exercise 8 8

1. $x = 1, y = 2, z = 2$ 2. $x = 1, y = -1, z = 2$
3. $x = 4\frac{1}{3}, y = 4, z = -2\frac{2}{3}$ 4. $x = 4, y = 3, z = \frac{1}{2}$
5. $a = 0, b = -1, c = -2$ 6. $a = 1, b = -2, c = -4$
7. $x = -2\frac{1}{9}, y = 1\frac{2}{3}, z = -\frac{4}{9}$ 8. $w = 3, x = 1, y = -1, z = 2$
9. $w = -15\frac{1}{5}, x = -1\frac{2}{5}, y = 1\frac{1}{5}, z = 5\frac{3}{5}$
10. $w = -\frac{4}{5}, x = 4, y = -\frac{3}{5}, z = 1\frac{3}{5}$

Exercise 8 9

1. \$0 01 loss per 100 screwdrivers, \$1 00 profit per 100 punches, \$2 00 profit per 100 chisels 2. Average daily sales volumes A \$480, B \$520, C \$380 3. Average weekly sales Y \$21,400, Z \$19,600, X \$18,500 4. A \$24,000, B \$11,368 42, C \$102,315 99 5. Federal tax \$1952 66, State tax in A \$200 56, State tax in B \$36 21
6. \$3,090 24 Commission, \$556 24 State tax, \$18,541 41 Federal tax

Exercise 8 10

1. $x = \pm 2$ 2. $x = \pm 3$ 3. $x = \pm 2\sqrt{3}$ 4. $x = \pm 2$
5. $x = \pm 9$ 6. $x = \pm 5$ 7. $x = \pm 4$ 8. $x = \pm \sqrt{7}$.
9. $x = \pm 3$ 10. $x = \pm 3$

Exercise 8 11

1. $0, 7$ 2. $0, -1\frac{1}{4}$ 3. $0, 2\frac{1}{3}$ 4. $3, 3$ 5. $5, 5$
6. $2, 2$ 7. $-4, 3$ 8. $-5, 2$ 9. $-5, -2$
10. $-5, 4$ 11. $6, -4$ 12. $-4, -7$ 13. $2, -\frac{1}{2}$
14. $-\frac{1}{2}, -1\frac{1}{2}$ 15. $\frac{3}{4}, -\frac{1}{3}$ 16. $-\frac{1}{3}, 1\frac{1}{2}$ 17. $-\frac{2}{3}, -1\frac{1}{2}$
18. $4, -2$ 19. $-\frac{1}{2}, 2$ 20. $\frac{1}{2}, -3$

Exercise 8.12

1. $1; \frac{3}{4}$. **2.** $4; -3\frac{3}{5}$. **3.** $-1 \pm \sqrt{\frac{3}{2}}$. **4.** $3; \frac{5}{9}$.
5. $-\frac{1}{3}; -1\frac{1}{3}$. **6.** $-1; 1\frac{3}{4}$. **7.** $3; \frac{1}{2}$. **8.** $\pm\sqrt{3}$.
9. $-2 \pm \sqrt{5}$. **10.** $x = -6, x = \frac{1}{4}$.

Exercise 8.13

1. 15 and 3. **2.** 4 and 16; -5 and 25. **3.** 9 and 16.
4. 90 by 40 feet. **5.** 30 by 10 feet. **6.** 10 and 15 days.
7. 30 miles per hour. **8.** 50 at $17.50. **9.** 40. **10.** 8 children; $7,800.

Chapter 9

Exercise 9.1

1. a^7. **2.** a^8. **3.** a^2. **4.** a^{-2}. **5.** a^3. **6.** a^4. **7.** a^5.
8. a^4. **9.** a^8. **10.** a^9. **11.** a^6. **12.** a^{-12}. **13.** $(1 + i)^6$.
14. $(1 + i)^{14}$. **15.** $(1 + i)^{24}$. **16.** $(1 + i)^{24}$. **17.** $(a + b)^7$.
18. $4^3 = 64$.

Exercise 9.2

1. 2.9487×10^4. **2.** 2.9487×10^2. **3.** 2.9487×10^{-1}.
4. 4.5987×10^4. **5.** 4.5987×10^0. **6.** 4.5987×10^{-3}.
7. 4.5987×10^{-1}. **8.** 4.5987×10^2. **9.** 6.72×10^4.
10. 6.72×10^{-3}. **11.** 6.72×10^{-5}. **12.** 6.72×10^2.

Exercise 9.3

1. 1. **2.** 2. **3.** 0. **4.** 3. **5.** 2. **6.** 2. **7.** 4.
8. 0. **9.** 2. **10.** 1. **11.** -1. **12.** -2. **13.** 0.
14. 5. **15.** 0. **16.** -1. **17.** -6. **18.** -4. **19.** 1.
20. 0.

Exercise 9.4

1. 2.401401. **2.** 3.410102. **3.** 3.424065. **4.** 0.477121.
5. 1.478422. **6.** 1.431364. **7.** 2.432167. **8.** 2.440122.
9. 9.461948 − 10. **10.** 8.473662 − 10. **11.** 3.415140.
12. 0.425860. **13.** 2.463146. **14.** 1.452859. **15.** 3.476976.
16. 2.426023. **17.** 2.417306. **18.** 8.446226 − 10.
19. 6.485153 − 10. **20.** 4.477700.

Exercise 9 5

1. 0 698336 **2.** 1 425257 **3.** 0 790820 **4.** 2 087888
5. 3 942911 **6.** 0 678864 **7.** 0 446008 **8** 2 912339
9. 4 821428 **10.** 8 768135 − 10 **11.** 7 621426 − 10
12. 0 621426 **13.** 3 338038 **14.** 1 637249 **15.** 0 940133
16. 4 921182 **17.** 0 903275 **18.** 4 416824
19. 6 903106 − 10 **20.** 5 823904 − 10

Exercise 9 6

1. 126 8 **2.** 2 050 **3.** 31 84 **4.** 0 3672 **5.** 4 196
6. 0 05190 **7.** 6,165 **8.** 73 13 **9.** 0 008326 **10.** 9 266
11. 12 444 **12.** 0 21623 **13.** 325 34 **14.** 0 037542
15. 4 2924 **16.** 5,326 6 **17.** 0 62729 **18.** 0 0073744
19. 84 348 **20.** 93,367

Exercise 9 7

1. 133 84 **2.** 459 68 **3.** 7,882 7 **4.** 2,420 5
5. 333 56 **6.** 9 2474 **7.** 0 030069 **8.** 0 046398
9. 15 416 **10.** 1 9847 **11.** 517 94 **12.** 1,644 7
13. 10,735 **14.** 0 017766 **15.** 0 045547 **16.** 1117 4
17. 4,920 0 **18.** 98 715 **19.** 12 437 **20.** 0 016255

Exercise 9 8

1. 6 0297 **2.** 6 6591 **3.** 6 3003 **4.** 0 30079
5. 0 070879 **6.** 0 020086 **7.** 11 843 **8.** 0 073621
9. 90 747 **10.** 0 042878 **11.** 0 43931 **12.** 2 0462
13. 11 690 **14.** 0 49754 **15.** 2 8466 **16.** 41 283
17. 123 84 **18.** 0 0069032 **19.** 3,122 8 **20.** 0 64217

Exercise 9 9

1. 60 540 **2.** 102 78 **3.** 22 834 **4.** 59 970 **5.** 35,269
6. 0 037074 **7.** 0 00023511 **8.** 0 27546 **9.** 0 013092
10. 0 00054093

Exercise 9 10

1. 7 5967 **2.** 4 2544 **3.** 48 857 **4.** 2 0767
5. 1 8698 **6.** 0 81743 **7.** 0 75947 **8.** 0 37662
9. 0 65882 **10.** 0 68724

Exercise 9 11

1. 28 **2.** 7 **3.** $8\frac{1}{2}$% **4.** 7% **5.** 16 **6.** 6%

Exercise 9.13

1. 8. **2.** 96. **3.** 54. **4.** 1,000. **5.** 2,500. **6.** 2,040. **7.** 351. **8.** 623. **9.** 627. **10.** 736. **11.** 24,900. **12.** 18.68. **13.** 17.77. **14.** 8.86. **15.** 0.499. **16.** 0.00498. **17.** 17.92. **18.** 377,000. **19.** 0.000754. **20.** 0.000902.

Exercise 9.14

1. 6. **2.** 3. **3.** 4. **4.** 5.5. **5.** 7. **6.** 0.7. **7.** 0.75. **8.** 3.58. **9.** 2.82. **10.** 2.33. **11.** 2.55. **12.** 1.77. **13.** 2.42. **14.** 6.67. **15.** 16.1. **16.** 0.717. **17.** 0.730. **18.** 0.341. **19.** 0.274. **20.** 0.0648. **21.** 0.1567. **22.** 0.0162. **23.** 106. **24.** 134.1. **25.** 0.553. **26.** 0.392. **27.** 6.22. **28.** 431. **29.** 0.000513. **30.** 62.9.

Exercise 9.15

1. 3. **2.** 21.9. **3.** 1.875. **4.** 16.8. **5.** 13.7. **6.** 17.1. **7.** 66.5. **8.** 41.8. **9.** 400. **10.** 243. **11.** 50.3. **12.** 67.3. **13.** 33.5. **14.** 14.48. **15.** 12.11. **16.** 10.77. **17.** 20.3. **18.** 1.304. **19.** 350. **20.** 711. **21.** 1.08. **22.** 0.137. **23.** 0.331. **24.** 0.274. **25.** 717. **26.** 866. **27.** 4.00. **28.** 25.4. **29.** 72.0. **30.** 126.4.

Exercise 9.16

1. 225; 484; 1,225; 1,850; 72.2; 10.6; 49; 81. **2.** 6.25; 33.7; 27.6; 27.9; 70.0; 88.5. **3.** 43,300; 113,800; 300,000; 146,000; 840,000; 2,110,000. **4.** 7.75; 14.76; 34.0; 76.3; 84.4. **5.** 0.00276; 0.00764; 0.0558; 0.143; 0.790. **6.** 12; 13; 17; 23; 32; 40. **7.** 1.967; 2.59; 3.04; 4.65; 5.90; 7.85. **8.** 69.1; 163.6; 495; 27.7; 81.4. **9.** 0.790; 0.879; 0.726; 0.897; 0.975; 0.256. **10.** 0.1118; 0.233; 0.00727; 0.00654; 0.00270.

Chapter 10

Exercise 10.1

1. 217 days, 215 days. **2.** 123 days, 120 days. **3.** 82 days, 81 days. **4.** 226 days, 224 days. **5.** 187 days, 185 days. **6.** 63 days, 62 days. **7.** 253 days, 248 days. **8.** 135 days, 133 days. **9.** 165 days, 162 days. **10.** 172 days, 169 days. **11.** 138 days, 136 days. **12.** 86 days, 84 days. **13.** 60 days, 60 days. **14.** 480 days, 474 days. **15.** 100 days, 99 days.

Exercise 10 2

1. \$28 88 **2.** \$7 49 **3.** \$19 12 **4.** \$12 75 **5.** \$29 42 **6.** \$3 49 **7.** \$339 73 **8.** \$581 25 **9.** \$16 41 **10.** \$861 74 **11.** \$2,503,184 93 **12.** \$1,811 **13.** \$379 38 **14.** Ordinary \$14 16, Bankers \$14 54, Exact \$14 34 **15.** \$78 61 **16.** \$65 88 **17.** \$8,219 18 **18.** \$875 50 **19.** \$921 **20.** 5 56%

Exercise 10 3

1. \$2 23 **2.** \$8 28 **3.** \$2 40 **4.** \$3 97 **5.** \$3 82 **6.** \$7 80 **7.** \$0 71 **8.** \$23 20 **9.** \$28 60 **10.** \$72 00 **11.** \$250 00 **12.** \$66 35 **13.** \$26 67 **14.** \$3 63 **15.** \$17 56 **16.** \$31 50 **17.** \$0 69 **18.** \$729 **19.** \$3,559 42 **20.** \$317 33

Exercise 10 4

1. \$4,017 26 **2.** \$2,505 53 **3.** \$859 15 **4.** \$429 41 **5.** \$1,190 51 **6.** \$1,775 15 **7.** \$1,990 32 **8.** \$2,391 03 **9.** \$460 28 **10.** \$326 23 **11.** \$974 03 **12.** \$3,561 10 **13.** \$103 75 **14.** \$1,823 32 **15.** \$807 10

Exercise 10 5

1. \$2,505 48 **2.** \$1,794 15 **3.** \$4,017 20 **4.** \$859 11 **5.** \$1,190 43 **6.** \$3,158 40 **7.** \$2,391 **8.** \$512 72 **9.** \$360 21 **10.** \$477 11 **11.** \$589 50 **12.** \$396 **13.** \$1,197 38 **14.** \$514 50 **15.** \$2,509 05 **16.** \$2,508 98 **17.** \$250 57 **18.** \$765 31 **19.** \$614 33 **20.** \$6,537 74

Exercise 10 6

1. \$99,502 49 **2.** Latter ($6\frac{1}{4}$% vs 6 185%) **3.** \$2,383 42 **4.** \$833 88 **5.** 7 124 + % **6.** 7 692 + % **7.** 8 1632 + % **8.** 6% **9.** No, it is 36 734 + % **10.** 74 226 + %

Exercise 10 7

1. \$1,095 72 **2.** \$1,391 13 **3.** \$328 07. **4.** \$260 90 **5.** \$282 11. **6.** \$3,105 82 **7.** By Merchant's Rule \$884 00, By U S Rule \$885 24 **8.** By Merchant's Rule \$564 05, By U.S Rule \$564 31 **9.** By Merchant's Rule \$6,981 33, By U S Rule \$7,394 18 **10.** \$112 31.

Exercise 10.8

1. 20.52%. 2. Residuary 19.28%, Constant-ratio 26.67%. 3. 30%. 4. 13.03%. 5. 20%. 6. The finance company offer is better ($26\frac{2}{3}$% vs. 30%). 7. 36.92%. 8. Merchant's Rule 20.51%, Constant-ratio 19.20%. 9. For \$50 27.43%; for \$100 32.00%; for \$175 21.10%. 10. 15.48%. 11. 44.4%; 44.4%.

Exercise 10.9

1. \$413.79; \$432.60. 2. \$828.95; \$840.00; \$845.60. 3. \$396.71. 4. \$1,029.85. 5. October 21. 6. July 19. 7. August 27. 8. January 23. 9. March 29. 10. July 17. 11. \$248.72. 12. \$5,011.48. 13. \$5,066.85. 14. \$13,502.87. 15. \$10,060.98.

Chapter 11

Exercise 11.1

1. \$910.44. 2. \$1,211.81. 3. \$320.57. 4. \$3,696.38. 5. \$421.44. 6. \$266.74. 7. \$850.93. 8. \$2,063.72; \$2,108.62.

Exercise 11.2

1. \$150; 100%; 50%. 2. \$15; 60%; 37.5%. 3. \$3.50; 41.18%; 29.17%. 4. 25¢; 41.67%; 29.4%. 5. \$1.40; 40%; 28.5%. 6. \$6.50; 35.14%; 26%. 7. \$450; 37.5%; 27.27%. 8. \$1.30; 59%; 37.1%. 9. 55¢; 44%; 30.56%. 10. \$4.75; 38%; 27.5%. 11. 50%; $33\frac{1}{3}$%. 12. 42.8%; 30%. 13. 28.5%. 14. 42.86%. 15. 25%; 20%.

Exercise 11.3

1. \$140; \$260. 2. \$93.75; \$156.25. 3. \$25; \$50. 4. \$15.93; \$29.57. 5. \$16.25; \$16.25. 6. \$8.38; \$19.57. 7. \$6.80; \$11.95. 8. \$2.20; \$6.59. 9. \$1.93; \$2.36. 10. \$260.96; \$554.54. 11. \$8.12. 12. \$2.84. 13. \$1.68. 14. \$6.60. 15. \$1.56.

Exercise 11.4

1. \$66.67. 2. \$47.79. 3. \$11.81. 4. \$1.09. 5. \$30. 6. \$2.80. 7. \$93.75. 8. \$357.14. 9. \$17.24. 10. \$66.81. 11. \$8.40. 12. \$1.10. 13. $\$0.83\frac{1}{3}$. 14. 32 cents. 15. \$121.60.

Exercise 11 5

1. $67 50 2. $87 50 3. $34 42 4. $3 33 5. $85
6. $3 7. $1 33 8. $7 91 9. $32 90 10. $267 19
11. $8 12. $1 80 13. 19 cents 14. 45 cents
15. 53 cents

Exercise 11 6

1. $26 67 2. $66 67 3. $17 54 4. $12 22 5. $24 58
6. $20 23 7. $1,000 8. $21 88 9. $81 08 10. $820
11. $3 12. $4 58 13. $3 97 14. $133 33 15. $344

Exercise 11 7

1. 25% 2. $33\frac{1}{3}$% 3. 42 86% 4. 53 85% 5. $66\frac{2}{3}$%
6. 81 82% 7. 100% 8. 79 86% 9. 200% 10. 400%
11. 73 91% 12. 300% 13. 128 57% 14. 92 31%
15. 59 06%

Exercise 11 8

1. $33\frac{1}{3}$% 2. $37\frac{1}{2}$% 3. 41 18% 4. 42 86%
5. 45 45% 6. 47 37% 7. 50% 8. 27 27% 9. 80%
10. 76 2% 11. 64 3% 12. 54 5% 13. 52 94%
14. $86\frac{2}{3}$% 15. 88 9%

Exercise 11 9

1. 31 8% 2. 35 8% 3. 53 3% 4. 25% 5. 44%
6. 41 18% 7. 41 43% 8. 40 91% 9. 35 91%
10. 150%

Exercise 11 10

1. 1 to 1 2. 1 to 1 3. 3 to 7, 9 at $32 50, 21 at $37 50
4. 9 to 5, 36 at $37 50, 20 at $44 50 5. 3 to 2 6. 3 to 5,
30 at $67 50, 50 at $79 50 7. 5 to 3, 25 at $18 50, 15 at $22 50
8. 1 for $4 50, 2 for $5 45 9. 2 at $6 95, 3 at $8 25

Exercise 11 11

1. 25 at $2 85, 75 at $2 45 2. 3 at $35, 2 at $45 3. 3 at $1,
7 at $1 30 4. 1 at $3 21, 2 at $3 75 5. 39 at $2 50, 49 at $3 00

Exercise 11 12

1. 47 6% 2. 38 7% 3. 40 2% 4. 53 5% 5. 69 7%
6. 47 7% 7. 34 2% 8. 39 5%

Exercise 11.13

1. 7.14%. **2.** 17.5%. **3.** 11.5%. **4.** 27.3%. **5.** 6.25%. **6.** 4.76%. **7.** 10.7%. **8.** 15.38%.

Exercise 11.14

1. 32.14%. **2.** 6.67%. **3.** 12%. **4.** 2.8%. **5.** 3.04%.

Chapter 12

Exercise 12.1

1. 4%; 2. **2.** $1\frac{7}{8}$%; 10. **3.** 7/8%; 24. **4.** 1/4%; 50. **5.** 1/6%; 40. **6.** 3/4%; 3. **7.** $1\frac{1}{4}$%; 17. **8.** $1\frac{1}{4}$%; 6. **9.** 1/4%; 208. **10.** 1/2%; 48. **11.** 3%; 3. **12.** $1\frac{1}{4}$%; 40. **13.** 3/8%; 300. **14.** 6%; 15. **15.** $\frac{11}{24}$%; 240.

Exercise 12.2

1. (1 + 2%); (1.02); 41; 2.2522004569. **2.** (1 + $2\frac{1}{2}$%); (1.025); 10; 1.2800845442. **3.** (1 + $\frac{1}{4}$%); (1.0025); 51; 1.1358041362. **4.** (1 + 1%); (1.01); 23; 1.2571630183. **5.** (1 + $\frac{1}{4}$%); (1.0025); 51; 1.1358041362. **6.** 2.3966; 2.396558. **7.** $644.01. **8.** $1,516.40. **9.** $10,938.07. **10.** $8,320.55; $8,125.00. **11.** $3,453.04. **12.** $860.29. **13.** $24,005.10. **14.** $12,500 is better ($12,800.84). **15.** $1,902.36.

Exercise 12.3

1. 11.9 years; $11\frac{11}{12}$ years. **2.** 2.54%. **3.** Gains $111.04 by selling now. **4.** 11.187%. **5.** 1.25%. **6.** Between $18\frac{1}{2}$ and 19 years. **7.** $100,000. **8.** 14.95%. **9.** Income property ($7.04 vs. $6 for each dollar invested). **10.** 54.798% compounded monthly.

Exercise 12.4

1. $S = \$2{,}400(1 + \frac{1}{6}\%)^{120}\,(1 + \frac{1}{6}\%)^{12}$; $\log S = \log 2{,}400 + 132 \cdot \log 1.00167$. **2.** $S = \$2{,}400(1 + \frac{1}{2}\%)^{80}\,(1 + \frac{1}{2}\%)^{80}$; $\log S = \log 2{,}400 + 160 \cdot \log 1.005$. **3.** $S = \$2{,}400(1 + \frac{1}{6}\%)^{100}\,(1 + \frac{1}{6}\%)^{80}$; $\log S = \log 2{,}400 + 180 \cdot \log 1.00167$. **4.** $S = \$4{,}000(1 + 1\%)^{120}\,(1 + 1\%)^{4}$; $\log S = \log 4{,}000 + 124 \cdot \log 1.01$. **5.** $S = \$1{,}000(1 + 5\%)^{40}$; $\log S = \log 1{,}000 + 40 \cdot \log 1.05$. **6.** $S = \$45{,}000(1 + 1\frac{1}{2}\%)^{100}\,(1 + 1\frac{1}{2}\%)^{100}$; $\log S = \log 45{,}000 + 200 \cdot \log 1.015$. **7.** $S = \$100(1 + \frac{1}{2}\%)^{120}$

$(1 + \frac{1}{2}\%)^{60}$, $\log S = \log 100 + 180 \quad \log 1\ 005$ **8.** $S = \$600(1 + \frac{1}{3}\%)^{120} (1 + \frac{1}{3}\%)^{24}$, $\log S = \log 600 + 144 \quad \log 1\ 00333$ **9.** $S = \$8{,}750(1 + 1\frac{1}{2}\%)^{80} (1 + 1\frac{1}{2}\%)^{70}$, $\log S = \log 8{,}750 + 150 \cdot \log 1\ 015$
10. $S = \$10{,}000(1 + 1\%)^{80} (1 + 1\%)^{72}$, $\log S = \log 10{,}000 + 152 \quad \log 1\ 01$

Exercise 12 5

1. $2,121 60 **2.** $1,142 39 **3.** $8,573 35 **4.** $11,070 94
5. $4,460 48 **6.** $1,436 51 **7.** $1,530 40 **8.** $444 31.
9. $4,829 18 **10.** $16,093 33

Exercise 12 6

1. 4 04% **2.** *Bond* 2 515625% vs 2 018436% **3.** *Stock* 3 0339% vs 2 97% **4.** 5 0945% **5.** 3 0416%, 3 0339%, 3 0225% **6.** 7 186% **7.** 5 0625% **8.** 8 30%
9. 8 243% **10.** 4 074%

Exercise 12 7

1. $4,437 24 **2.** $250 54 **3** $3,604 74 **4.** $377 22, $370 37, $400, $380 77 **5.** $4,797 08 **6.** $108,872 95
7. $4,800 79 **8.** $8,315 70 **9.** $3,894 80 **10.** $37,243 06

Exercise 12 8

1. $821 93, $924 56, $1,216 65 **2.** $1,049 01, $1,179 99, $1,552 79 **3.** $1,870 94, $2,104 55, $2,769 44 **4.** $743 90
5. $1,074 51 **6.** Yes, only $4,811 69 **7.** $2,243 75
8. $3,768 98 **9.** $19,473 01. **10.** $1,264 80

Chapter 13

Exercise 13 1

1. $515 23 **2.** $2,401 22, $401 22 **5.** $2,554 46
6 $7,113 87 **7.** $5,379 71 **8.** $1,273 95 **9.** $2,791 82, $3,277 59, $4,600 77 **10.** $6,141 43 **11.** $13,814 08
12. $11,561 84, $11,731 39 **13.** $18,009 16 **14.** $3,667 75
15. $7,250 50 **16.** $9,621 47 **17.** $7,359 86 **18.** $1,817 20
19. $21,899 44 **20.** $49,716 57

Exercise 13.2

1. \$2,105.91. **2.** Former—\$20,750.44. **3.** \$112,740.66. **4.** \$18,014.69. **5.** \$1,716.86. **6.** \$10,453.92. **7.** \$871.42. **8.** \$2,248.37. **9.** \$26,854.77. **10.** \$5,508.13. **11.** \$2,027.72. **12.** Latter—\$9,877.02; \$122.98.

Exercise 13.3

1. 22; \$99.90. **2.** 59; \$17.66. **3.** 8; \$120.92. **4.** 13; \$73.68. **5.** 37; \$1. **6.** 28; \$49.71.

Exercise 13.4

1. \$10,975.38. **2.** \$12,973.77. **3.** \$2,947.24. **4.** \$41,874.23. **5.** \$3,884.95.

Exercise 13.5

1. \$692.69; \$10,194.92. **2.** \$1,123.14. **3.** \$336.08. **4.** \$1,250.32.

5. \$467.57;

Period	*Outstanding Principal at the Beginning of the Period*	*Interest at 2%*	*Payment*	*Principal Repaid*
1	\$4,200.00	\$84.00	\$467.57	\$383.57
2	3,816.43	76.33	467.57	391.24
3	3,425.19	68.50	467.57	399.07
4	3,026.12	60.52	467.57	407.05
5	2,619.07	52.38	467.57	415.19
6	2,203.88	44.08	467.57	423.49
7	1,780.39	35.61	467.57	431.96
8	1,348.43	26.97	467.57	440.60
9	907.83	18.16	467.57	449.41
10	458.42	9.17	467.57	458.40

6.

Period	*Outstanding Principal at the Beginning of the Period*	*Interest at 1%*	*Payment*	*Principal Repaid*
1	\$2,250.00	\$22.50	\$350.00	\$327.50
2	1,922.50	19.22	350.00	330.78
3	1,591.72	15.92	350.00	334.08
4	1,257.64	12.58	350.00	337.42
5	920.22	9.20	350.00	340.80
6	579.42	5.79	585.21	000.00

7. \$501.36.

8.

Period	Outstanding Principal at the Beginning of the Period	Interest at 3%	Payment	Principal Repaid
1	$450 00	$13 50	$64 11	$50 61
2	399 39	11 98	64 11	52 13
3	347 26	10 42	64 11	53 69
4	293 57	8 81	64 11	55 30
5	238 27	7 15	64 11	56 96
6	181 31	5 44	64 11	58 67
7	122 64	3 68	64 11	60 43
8	62 21	1 87	64 08	62 21

9. $28 32 10. $426 02

Exercise 13 6

1. $2,035 75, $5,035 75 2. $90 83 3. $71 18
4. $953 33 5. $20,221 97 6. $6,167 84 7. $57,088 64
8. $25,851 75 9. $42,184 83 10. $95,422 88

Exercise 13 7

1.

End of Period	Periodic Payment	Interest at $3\frac{1}{2}$%	Periodic Increase in Fund	Amount of Sinking Fund
1	$939 05	——	$ 939 05	$ 939 05
2	939 05	$ 32 87	971 92	1,910 97
3	939 05	66 88	1,005 93	2,916 90
4	939 05	102 09	1,041 14	3,958 04
5	939 05	138 53	1,077 58	5,035 62
6	939 05	176 25	1,115 30	6,150 92
7	939 05	215 28	1,154 33	7,305 25
8	939 07	255 68	1,194 75	8,500 00

2.

End of Period	Periodic Payment	Interest at 6%	Periodic Increase in Fund	Amount of Sinking Fund
1	$3,547 93	——	$3,547 93	$ 3,547 93
2	3,547 93	$212 88	3,760 81	7,308 74
3	3,547 93	438 52	3,986 45	11,295 19
4	3,547.93	677 71	4,225 64	15,520 83
5	3,547 92	931 25	4,479 17	20,000 00

3. \$5.33. 4. \$731.17; \$2,293.69. 5. \$3,492.41; \$5,492.41; \$62,621.71. 6. \$42,620.68; \$168,436.36.

Exercise 13.8

1. \$7,435.00. 2. \$19,750.10. 3. \$50,132.82. 4. \$10,382.07. 5. \$6,802.70. 6. \$104,537.25. 7. \$5,198.42. 8. \$6,993.80. 9. \$575.37. 10. \$3,569.47. 11. \$41,771.48. 12. \$16,968.83. 13. \$4,010.89. 14. \$29,335.44. 15. \$282,553.36. 16. \$690,837.51.

Chapter 14

Exercise 14.1

1. \$135.77 discount. 2. \$44.00 premium. 3. \$127.44 premium. 4. \$30.80 discount. 5. \$53.11 premium. 6. \$13.85 discount. 7. \$12.74 premium. 8. \$12.87 premium. 9. \$11.90 discount. 10. \$28.49 discount. 11. \$960.15. 12. \$1,065.28. 13. \$1,081.11. 14. \$926.40. 15. \$1,000; \$918.24; \$1,090.23.

16. \$971.99;

At End of Period	*Interest Received*	*Accumulation Payment*	*2% Interest on Accumulation Fund*	*Increase in Fund*	*Size of Fund*	*Book Value of Bond*
0	——	——	——	——	——	\$ 971.99
1	\$15.00	\$ 4.44	——	\$ 4.44	\$ 4.44	976.43
2	15.00	4.44	\$0.09	4.53	8.97	980.96
3	15.00	4.44	0.18	4.62	13.59	985.58
4	15.00	4.44	0.27	4.71	18.30	990.29
5	15.00	4.44	0.36	4.81	23.11	995.10
6	15.00	4.44	0.46	4.90	28.01	1,000.00
Total		\$26.64	\$1.37	\$28.01		

17. \$955.29.

18. $983 50,

At End of Period	$1\frac{1}{2}$% of Book Value	Interest Received	Amount Added to Book Value	Book Value
0	——	——	——	$ 983 50
1	$ 14 76	$12 50	$ 2 26	985 76
2	14 79	12 50	2 29	988 05
3	14 82	12 50	2 32	990 37
4	14 86	12 50	2 36	992 73
5	14 89	12 50	2 39	995 12
6	14 93	12 50	2 43	997 55
7	14 96	12 50	2 45	1,000 00
Total	$104 01	$87 50	$16 50	

19. $932 63, $951 46   20. $964 65

21. $1,015 58,

At End of Year	Amortization Payment	3% of Fund	Increase in Fund	Size of Fund	Book Value of Bond
0	——	——	——	——	$1,015 58
1	$ 2 03	——	$ 2 03	$ 2 03	1,013 55
2	2 03	$0 06	2 09	4 12	1,011 46
3	2 03	0 12	2 15	6 27	1,009 31
4	2 03	0 19	2 22	8 49	1,007 09
5	2 03	0 25	2 28	10 77	1,004 81
6	2 03	0 32	2 35	13 12	1,002 46
7	2 07	0 39	2 46	15 58	1,000 00
Total	$14 25	$1 33	$15 58		

22. $1,039.18;

End of Period	*$1\frac{1}{2}\%$ of Book Value*	*Interest Received*	*Amount Subtracted from Book Value*	*Book Value*
0	——	——	——	$1,039.18
1	$ 15.59	$ 17.50	$ 1.91	1,037.27
2	15.56	17.50	1.94	1,035.33
3	15.53	17.50	1.97	1,033.36
4	15.50	17.50	2.00	1,031.36
5	15.47	17.50	2.03	1,029.33
6	15.44	17.50	2.06	1,027.27
7	15.41	17.50	2.09	1,025.18
8	15.38	17.50	2.12	1,023.06
9	15.34	17.50	2.16	1,020.90
10	15.31	17.50	2.19	1,018.71
11	15.28	17.50	2.22	1,016.49
12	15.25	17.50	2.25	1,014.24
13	15.21	17.50	2.29	1,011.95
14	15.18	17.50	2.32	1,009.63
15	15.14	17.50	2.36	1,007.27
16	15.11	17.50	2.39	1,004.88
17	15.07	17.50	2.43	1,002.45
18	15.04	17.50	2.45	1,000.00
Total	$275.81	$315.00	$39.18	

23. $1,035.33; $1,014.24. **24.** $10,248.19; $10,141.22. **25.** $1,100.15; $1,078.36.

Exercise 14.2

1. $921.23; $911.81. **2.** $1,069.99; $1,062.44. **3.** $927.22; $925.37. **4.** $920.64; $913.21. **5.** $1,083.87; $1,069.31. **6.** $975.16; $966.49. **7.** $977.54; $977.54. **8.** $1,034.00. **9.** $912.38; $902.38. **10.** $802.51; $802.51. **11.** $968.23. **12.** 1.95%. **13.** Quoted $10,874.97, Flat $10,812.47; No. **14.** $87,460.41. **15.** $11,273.31; $11,284.34.

Exercise 14.3

1. 2.86%. **2.** 3.51%; 3.26%. **3.** 3.85%. **4.** 4.43. **5.** 4.22. **6.** 3.28%. **7.** 2.91%. **8.** 4.21%. **9.** 3.75%. **10.** 4.25%. **11.** $4\frac{1}{2}\%$. **12.** $5\frac{3}{4}\%$. **13.** 2.32%. **14.** 3.06%. **15.** 4.13%.

Exercise 14 4

1. $200, $150, $120 2 $24,285 71 3. $50,000
4 $30,000 5. $109 09 6. $105,120 7. $72,000
8. $280,000 9. $50,000 10. $96,000 11. $4,619 50
12. $4,641 66 13. $10,000 14. $7,000 15. $5,495,632 52
16. $115 48 17. Latter by $6,850 30 18. Latter by $16,385 83
19. $30,951 26 20. $17,575 66 21. $55,962 56
22 $10,949 40 23. $405,889 79 24. Yes by $3,235 32
25. $92,195 93

Exercise 14 5

1. $2,312 50,

End of Year	*Annual Depreciation*	*Reserve for Depreciation*	*Book Value*
0	—	—	$20,000 00
1	$ 2,312 50	$ 2,312 50	17,687 50
2	2,312 50	4,625 00	15,375 00
3	2,312 50	6,937 50	13 062 50
4	2,312 50	9,250 00	10,750 00
5	2,312 50	11,562 50	8,437 50
6	2,312 50	13,975 00	6,025 00
7	2,312 50	16,287 50	3,712 50
8	2,312 50	18,500 00	1,500 00
Total	$18,500 00		

2. $275,

End of Year	*Annual Depreciation*	*Reserve for Depreciation*	*Book Value*
0	—	—	$1,850 00
1	$ 275 00	$ 275 00	1,575 00
2	275 00	550 00	1,300 00
3	275 00	825 00	1,025 00
4	275 00	1,100 00	750 00
5	275 00	1,375 00	475 00
6	275 00	1,650 00	200 00
Total	$1,650 00		

3. $94 29, $142 16 4. $200, $840. 5. 8 years
6. 10 years 7. $8,311 77, $96,233 18 8. $3,431 99, $84,675 79

9. $177.40;

End of Year	Annual Depreciation	Interest Earned	Amount added to Fund	Amount in Fund	Book Value
0	——	——	——	——	$1,850.00
1	$177.40	——	$177.40	$ 177.40	1,672.60
2	177.40	$10.64	188.04	365.44	1,484.56
3	177.40	21.93	199.33	564.77	1,285.23
4	177.40	33.88	211.28	776.05	1,073.95
5	177.40	46.55	223.95	1,000.00	$ 850.00

10. $232.01;

End of Year	Annual Depreciation	Interest Earned	Amount added to Fund	Amount in Fund	Book Value
0	——	——	——	——	$1,200.00
1	$232.01	——	$232.01	$ 232.01	967.99
2	232.01	$11.60	243.61	475.62	724.38
3	232.01	23.78	255.79	731.41	468.59
4	232.02	36.57	268.59	1,000.00	200.00

11. $r = 34.56\%$;

End of Year	Annual Depreciation	Total Depreciation Taken	Book Value
0	——	——	$12,500.00
1	$4,320.12	$4,320.12	8,179.88
2	2,827.04	7,147.16	5,352.84
3	1,849.97	8,997.13	3,502.87
4	1,210.63	10,207.76	2,292.24
5	792.24	11,000.00	1,500.00

12. $r = 33.126\%$;

End of Year	Annual Depreciation	Total Depreciation Taken	Book Value
0	——	——	$750.00
1	$248.44	$248.44	501.56
2	166.15	414.59	335.41
3	111.11	525.70	224.30
4	74.30	600.00	150.00

13.

	Straight-Line		Sinking Fund		Constant Percentage	
End of Year	Cumulative Deprecia-tion	Book Value	Cumulative Deprecia-tion	Book Value	Annual Deprecia-tion	Book Value
0	——	$20,000 00	——	$20,000 00	——	$20,000 00
1	$3 000 00	17,000 00	$2,714 62	17,285 38	$4,842 80	15,157 20
2	6,000 00	14 000 00	5,564 97	14,435 03	3,670 16	11,487 04
3	9,000 00	11,000 00	8,557 84	11,442 16	2,781 47	8,705 57
4	12,000 00	8,000 00	11,700 35	8,299 65	2,108 00	6,597 57
5	15,000 00	5,000 00	15,000 00	5,000 00	1,597 57	5,000 00

14.

	Straight-Line		Sinking Fund		Constant Percentage	
End of Year	Cumulative Deprecia-tion	Book Value	Cumulative Deprecia-tion	Book Value	Annual Deprecia-tion	Book Value
0	——	$16,000 00	——	$16,000 00	——	$16,000 00
1	$2,400 00	13,600 00	$2,215 52	13,784 48	$3,874 24	12,125 76
2	4,800 00	11,200 00	4,519 66	11,480 34	2,936 13	9,189 63
3	7,200 00	8,800 00	6,915 97	9,084 03	2,225 18	6,964 45
4	9,600 00	6,400 00	9,408 14	6,591 86	1,686 38	5,278 07
5	12,000 00	4,000 00	12,000 00	4,000 00	1,278 07	4,000 00

15. $342 36 **16.** $60 **17.** Annual charge $18 18, $16 36, $14 54, $12 73, $10 91, $9 09, $7 27, $5 46, $3 64, $1 82 Cumulative reserve $18 18, $34 54, $49 08, $61 81, $72 72, $81 81, $89 08, $94 54, $98 18, $100 00

18.

	Straight-Line		Sum of Digits	
	Annual Charge	Cumulative Reserve	Annual Charge	Cumulative Reserve
1	$200	$ 200	$333 33	$ 333 33
2	200	400	266 67	600 00
3	200	600	200 00	800 00
4	200	800	133 33	933 33
5	200	1,000	66 67	1,000 00

19.

	Annual Charge	*Cumulative Reserve*
1	$2,222.22	$ 2,222.22
2	1,944.44	4,166.66
3	1,666.67	5,833.33
4	1,388.89	7,222.22
5	1,111.11	8,333.33
6	833.33	9,166.66
7	555.56	9,722.22
8	277.78	10,000.00

20. By sum of digits $42,587; By straight-line $25,000; Difference of $17,587.

Chapter 15

Exercise 15.1

1. 0.66710. **2.** 0.84790. **3.** 0.3624; 0.04272; 0.02728; 0.5676. **4.** 0.00001696; 0.989437. **5.** 0.1619. **6.** 0.009575. **7.** 93. **8.** 0.6766; 0.0802; 0.0949. **9.** 0.3366. **10.** 0.46747.

Exercise 15.2

1 through **15**: Probability that a life aged: **1.** 40 will live to be 41. **2.** 20 will live to be 21. **3.** 21 will live to be 31. **4.** 55 will live to be 60. **5.** 65 will live to be 70. **6.** 18 will die in 5 years (before age 23). **7.** 21 will die in 10 years (before age 31). **8.** 20 will die in one year. **9.** 45 will die in one year. **10.** 30 will die between the ages of 35 and 36. **11.** 18 will die between the ages of 24 and 25. **12.** 46 will die between the ages of 50 and 51. **13.** 35 will die between the ages of 40 and 50. **14.** 60 will die between the ages of 65 and 75. **15.** 40 will die between the ages of 60 and 70. **16.** p_x. **17.** q_x. **18.** ${}_{5|5}q_{35}$. **19.** ${}_{6|}q_{24}$. **20.** ${}_{9|}q_{40}$.

Exercise 15.3

1. 0.998. **2.** 0.01516; 0.04329. **3.** 0.98664. **4.** 0.9879. **5.** 0.01472. **6.** 0.472; 0.3897. **7.** 0.00197; 0.00243; 0.00356; 0.00618; 0.01232; 0.02659; 0.05930; 0.13185; 0.28099. **8.** 0; 0.4532; 0.2323. **9.** 0.8905; 0.6088; 0.02413. **10.** 0.125; 0.06494.

Exercise 15 4

3. $38,291 42 4. $37,428 23 5. $46,068 36 6. $273,941 30
7. $30,290 69 8. $2,583 23 9. $5,587 22 10. $26,660 70

Exercise 15 5

1. $21,278 90 2. $19,604 40 3. $2,265 16 4. $21,528 42
5. $2,665 52 6. $25,969 15 7. $17,775 76 8. $2,906 95
9. $8,063 79 10. $39,738 50 11. $67,784 12. $9,158 08
13. $19,505 05, $22,221 48 14. $416 07, $3,401 36
15. $1,400 56, $1,667 67

Exercise 15 6

1. $832 00 2. $4,890 46 3. $1 685 04 4. $1,030 93
5. $451 13 6. $485 11 7. $976 86 8. $1,749 27
9. $148 38 10. $2,371 20 11. $783 54 12. $3,528 70
13. $1,815 15 14. $413 26 15. $3,375 65

Exercise 15 7

1. $313 00 2. $868 99 3. $1,727 77 4. $162 68
5. $5,714 61

Exercise 15 8

1. $2 71 2. $12 83 3. $100 62 4. $83 99 5. $63 72

Exercise 15 9

1. $124 60, $1,480 21, $4,128 94 2. $42 50, $510 62, $1,249 30
3. $141 50, $1,861 59, $3,523 08 4. $181 80, $2,957 76, $7,537 36
5. $374 13, $2,793 65, $3,997 05

Exercise 15 10

1. $249 35, $6,097 67, $10 000 2. $157 35, $2,385 89, $5,000
3. $64 53, $610 49, $1,577 00 4. $69 57, $574 25, $1,188 36, $1,500 5. $223 93, $4,690 86, $10,000 6. $500 69, $7,613 78, $20,000 7. $235 24, $3,741 19, $7,500 8. $100 55, $2,183 22, $3,685 15, $5,000 9. $100 12, $1,543 39, $2,225 50, $2,500
10. $214 20, $1,000 34, $3,351 73, $6,059 10, $7,105 51, $8,107 40, $9,118 17, $10,000

N	0	1	2	3	4	5	6	7	8	9	D
100	00 0000	00 0434	00 0868	00 1301	00 1734	00 2166	00 2598	00 3029	00 3461	00 3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	01 0300	01 0724	01 1147	01 1570	01 1993	01 2415	424
103	01 2837	01 3259	01 3680	01 4100	4521	4940	5360	5779	6197	6616	420
104	7033	7451	7868	8284	8700	9116	9532	9947	02 0361	02 0775	416
105	02 1189	02 1603	02 2016	02 2428	02 2841	02 3252	02 3664	02 4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	03 0195	03 0600	03 1004	03 1408	03 1812	03 2216	03 2619	03 3021	404
108	03 3424	03 3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	04 0207	04 0602	04 0996	397
110	04 1393	04 1787	04 2182	04 2576	04 2969	04 3362	04 3755	4148	4540	4932	393
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	390
112	9218	9606	9993	05 0380	05 0766	05 1153	05 1538	05 1924	05 2309	05 2694	386
113	05 3078	05 3463	05 3846	4230	4613	4996	5378	5760	6142	6524	383
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	06 0320	379
115	06 0698	06 1075	06 1452	06 1829	06 2206	06 2582	06 2958	06 3333	06 3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	373
117	8186	8557	8928	9298	9668	07 0038	07 0407	07 0776	07 1145	07 1514	370
118	07 1882	07 2250	07 2617	07 2985	07 3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	9182	9543	9904	08 0266	08 0626	08 0987	08 1347	08 1707	08 2067	08 2426	360
121	08 2785	08 3144	08 3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	9905	09 0258	09 0611	09 0963	09 1315	09 1667	09 2018	09 2370	09 2721	09 3071	352
124	09 3422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	10 0026	346
126	10 0371	10 0715	10 1059	10 1403	10 1747	10 2091	10 2434	10 2777	10 3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	341
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	11 0253	338
129	11 0590	11 0926	11 1263	11 1599	11 1934	11 2270	11 2605	11 2940	11 3275	3609	335
130	3943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	12 0245	330
132	12 0574	12 0903	12 1231	12 1560	12 1888	12 2216	12 2544	12 2871	12 3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	13 0012	323
135	13 0334	13 0655	13 0977	13 1298	13 1619	13 1939	13 2260	13 2580	13 2900	3219	321
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	316
138	9879	14 0194	14 0508	14 0822	14 1136	14 1450	14 1763	14 2076	14 2389	14 2702	314
139	14 3015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	6128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
141	9219	9527	9835	15 0142	15 0449	15 0756	15 1063	15 1370	15 1676	15 1982	307
142	15 2288	15 2594	15 2900	3205	3510	3815	4120	4424	4728	5032	305
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
144	8362	8664	8965	9266	9567	9868	16 0168	16 0469	16 0769	16 1068	301
145	16 1368	16 1667	16 1967	16 2266	16 2564	16 2863	3161	3460	3758	4055	299
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
148	17 0262	17 0555	17 0848	17 1141	17 1434	17 1726	17 2019	17 2311	17 2603	17 2895	293
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
150	6091	6381	6670	6959	7248	7536	7825	8113	8401	8689	289

N	0	1	2	3	4	5	6	7	8	9	D
50	17 6091	17 6381	17 6670	17 6959	17 7248	17 7536	17 7825	17 8113	17 8401	17 8689	289
51	8977	9264	9552	9839	18 0126	18 0413	18 0699	18 0986	18 1272	18 1558	287
52	18 1844	18 2129	18 2415	18 2700	2985	3270	3555	3839	4123	4407	285
53	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
54	7521	7803	8084	8366	8647	8928	9209	9490	9771	19 0051	281
55	19 0332	19 0612	19 0892	19 1171	19 1451	19 1730	19 2010	19 2289	19 2567	2846	279
56	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
57	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
58	8657	8932	9206	9481	9755	20 0029	20 0303	20 0577	20 0850	20 1124	274
59	20 1397	20 1670	20 1943	20 2216	20 2488	2761	3033	3305	3577	3848	272
60	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556	271
61	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
62	9515	9783	21 0051	21 0319	21 0586	21 0853	21 1121	21 1388	21 1654	21 1921	267
63	21 2188	21 2454	2720	2986	3252	3518	3783	4049	4314	4579	266
64	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
65	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
66	22 0108	22 0370	22 0631	22 0892	22 1153	22 1414	22 1675	22 1936	22 2196	22 2456	261
67	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
68	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
69	7887	8144	8400	8657	8913	9170	9426	9682	9938	23 0193	256
70	23 0449	23 0704	23 0960	23 1215	23 1470	23 1724	23 1979	23 2234	23 2488	2742	255
71	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
72	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
73	8046	8297	8548	8799	9049	9299	9550	9800	24 0050	24 0300	250
74	24 0549	24 0799	24 1048	24 1297	24 1546	24 1795	24 2044	24 2293	2541	2790	249
75	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
76	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
77	7973	8219	8464	8709	8954	9198	9443	9687	9932	25 0176	245
78	25 0420	25 0664	25 0908	25 1151	25 1395	25 1638	25 1881	25 2125	25 2368	2610	243
79	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
80	5273	5514	5755	5996	6237	6477	6718	6958	7198	7439	241
81	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
82	26 0071	26 0310	26 0548	26 0787	26 1025	26 1263	26 1501	26 1739	26 1976	26 2214	238
83	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
84	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
85	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
86	9513	9746	9980	27 0213	27 0446	27 0679	27 0912	27 1144	27 1377	27 1609	233
87	27 1842	27 2074	27 2306	2538	2770	3001	3233	3464	3696	3927	232
88	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
89	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
90	8754	8982	9211	9439	9667	9895	28 0123	28 0351	28 0578	28 0806	228
91	28 1033	28 1261	28 1488	28 1715	28 1942	28 2169	2396	2622	2849	3075	227
92	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
93	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
94	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
95	29 0035	29 0257	29 0480	29 0702	29 0925	29 1147	29 1369	29 1591	29 1813	29 2034	222
96	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
97	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
98	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
99	8853	9071	9289	9507	9725	9943	30 0161	30 0378	30 0595	30 0813	218
00	30 1030	30 1247	30 1464	30 1681	30 1898	30 2114	2331	2547	2764	2980	217

Proportional Parts

n\d	295	290	285	280
1	30	29	29	28
2	59	58	57	56
3	89	87	86	84
4	118	116	114	112
5	148	145	143	140
6	177	174	171	168
7	207	203	200	196
8	236	232	228	224
9	266	261	257	252

n\d	275	270	265	260
1	28	27	27	26
2	55	54	53	52
3	83	81	80	78
4	110	108	106	104
5	138	135	133	130
6	165	162	159	156
7	193	189	186	182
8	220	216	212	208
9	248	243	239	234

n\d	255	250	248	246
1	26	25	25	25
2	51	50	50	49
3	77	75	74	74
4	102	100	99	98
5	128	125	124	123
6	153	150	149	148
7	179	175	174	172
8	204	200	198	197
9	230	225	223	221

n\d	244	242	240	238
1	24	24	24	24
2	49	48	48	48
3	73	73	72	71
4	98	97	96	95
5	122	121	120	119
6	146	145	144	143
7	171	169	168	167
8	195	194	192	190
9	220	218	216	214

n\d	236	234	232	230
1	24	23	23	23
2	47	47	46	46
3	71	70	70	69
4	94	94	93	92
5	118	117	116	115
6	142	140	139	138
7	165	164	162	161
8	189	187	186	184
9	212	211	209	207

n\d	228	226	224	222
1	23	23	22	22
2	46	45	45	44
3	68	68	67	67
4	91	90	90	89
5	114	113	112	111
6	137	136	134	133
7	160	158	157	155
8	182	181	179	178
9	205	203	202	200

n\d	220	218	216	214
1	22	22	22	21
2	44	44	43	43
3	66	65	65	64
4	88	87	86	86
5	110	109	108	107
6	132	131	130	128
7	154	153	151	150
8	176	174	173	171
9	198	196	194	193

Six-Place Logarithms of Numbers 200-250

N	0	1	2	3	4	5	6	7	8	9	D
200	30 1030	30 1247	30 1464	30 1681	30 1898	30 2114	30 2331	30 2547	30 2764	30 2980	217
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
204	9630	9843	31 0056	31 0268	31 0481	31 0693	31 0906	31 1118	31 1330	31 1542	212
205	31 1754	31 1966	2177	2389	2600	2812	3023	3234	3445	3656	211
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
209	32 0146	32 0354	32 0562	32 0769	32 0977	32 1184	32 1391	32 1598	32 1805	32 2012	207
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
213	8380	8583	8787	8991	9194	9398	9601	9805	33 0008	33 0211	203
214	33 0414	33 0617	33 0819	33 1022	33 1225	33 1427	33 1630	33 1832	2034	2236	202
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
218	8456	8656	8855	9054	9253	9451	9650	9849	34 0047	34 0246	199
219	34 0444	34 0642	34 0841	34 1039	34 1237	34 1435	34 1632	34 1830	2028	2225	198
220	2423	2620	2817	3014	3212	3409	3606	3802	3999	4196	197
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	35 0054	194
224	35 0248	35 0442	35 0636	35 0829	35 1023	35 1216	35 1410	35 1603	35 1796	1989	193
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
229	9835	36 0025	36 0215	36 0404	36 0593	36 0783	36 0972	36 1161	36 1350	36 1539	189
230	36 1728	1917	2105	2294	2482	2671	2859	3048	3236	3424	188
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
234	9216	9401	9587	9772	9958	37 0143	37 0328	37 0513	37 0698	37 0883	185
235	37 1068	37 1253	37 1437	37 1622	37 1806	1991	2175	2360	2544	2728	184
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	38 0030	181
240	38 0211	38 0392	38 0573	38 0754	38 0934	38 1115	38 1296	38 1476	38 1656	1837	181
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	178
245	9166	9343	9520	9698	9875	39 0051	39 0228	39 0405	39 0582	39 0759	177
246	39 0935	39 1112	39 1288	39 1464	39 1641	1817	1993	2169	2345	2521	176
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
250	7940	8114	8287	8461	8634	8808	8981	9154	9328	9501	173

Six-Place Logarithms of Numbers 250-300

0	1	2	3	4	5	6	7	8	9	D
39 7940	39 8114	39 8287	39 8461	39 8634	39 8808	39 8981	39 9154	39 9328	39 9501	173
9674	9847	40 0020	40 0192	40 0365	40 0538	40 0711	40 0883	40 1056	40 1228	173
40 1401	40 1573	1745	1917	2089	2261	2433	2605	2777	2949	172
3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
9933	41 0102	41 0271	41 0440	41 0609	41 0777	41 0946	41 1114	41 1283	41 1451	169
41 1620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
4973	5140	5307	5474	5641	5808	5974	6141	6308	6474	167
6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
9956	42 0121	42 0286	42 0451	42 0616	42 0781	42 0945	42 1110	42 1275	42 1439	165
42 1604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
9752	9914	43 0075	43 0236	43 0398	43 0559	43 0720	43 0881	43 1042	43 1203	161
43 1364	43 1525	1685	1846	2007	2167	2328	2488	2649	2809	161
2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	159
7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
9333	9491	9648	9806	9964	44 0122	44 0279	44 0437	44 0594	44 0752	158
44 0909	44 1066	44 1224	44 1381	44 1538	1695	1852	2009	2166	2323	157
2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
7158	7313	7468	7623	7778	7933	8088	8242	8397	8552	155
8706	8861	9015	9170	9324	9478	9633	9787	9941	45 0095	154
45 0249	45 0403	45 0557	45 0711	45 0865	45 1018	45 1172	45 1326	45 1479	1633	154
1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
9392	9543	9694	9845	9995	46 0146	46 0296	46 0447	46 0597	46 0748	151
46 0898	46 1048	46 1198	46 1348	46 1499	1649	1799	1948	2098	2248	150
2398	2548	2697	2847	2997	3146	3296	3445	3594	3744	150
3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
9822	9969	47 0116	47 0263	47 0410	47 0557	47 0704	47 0851	47 0998	47 1145	147
47 1292	47 1438	1585	1732	1878	2025	2171	2318	2464	2610	146
2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145
7121	7266	7411	7555	7700	7844	7989	8133	8278	8422	145

Proportional Parts

n\d	170	169	168	167
1	17	17	17	17
2	34	34	34	33
3	51	51	50	50
4	68	68	67	67
5	85	85	84	84
6	102	101	101	100
7	119	118	118	117
8	136	135	134	134
9	153	152	151	150

n\d	166	165	164	163
1	17	17	16	16
2	33	33	33	33
3	50	50	49	49
4	66	66	66	65
5	83	83	82	82
6	100	99	98	98
7	116	116	115	114
8	133	132	131	130
9	149	149	148	147

n\d	162	161	160	159
1	16	16	16	16
2	32	32	32	32
3	49	48	48	48
4	65	64	64	64
5	81	81	80	80
6	97	97	96	95
7	113	113	112	111
8	130	129	128	127
9	146	145	144	143

n\d	158	157	156	155
1	16	16	16	16
2	32	31	31	31
3	47	47	47	47
4	63	63	62	62
5	79	79	78	78
6	95	94	94	93
7	111	110	109	109
8	126	126	125	124
9	142	141	140	140

n\d	154	153	152	151
1	15	15	15	15
2	31	31	30	30
3	46	46	46	45
4	62	61	61	60
5	77	77	76	76
6	92	92	91	91
7	108	107	106	106
8	123	122	122	121
9	139	138	137	136

n\d	150	149	148	147
1	15	15	15	15
2	30	30	30	29
3	45	45	44	44
4	60	60	59	59
5	75	75	74	74
6	90	89	89	88
7	105	104	104	103
8	120	119	118	118
9	135	134	133	132

n\d	146	145	144
1	15	15	14
2	29	29	29
3	44	44	43
4	58	58	58
5	73	73	72
6	88	87	86
7	102	102	101
8	117	116	115
9	131	131	130

N	0	1	2	3	4	5	6	7	8	9	D
300	47 7121	47 7266	47 7411	47 7555	47 7700	47 7844	47 7989	47 8133	47 8278	47 8422	145
301	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144
302	48 0007	48 0151	48 0294	48 0438	48 0582	48 0725	48 0869	48 1012	48 1156	48 1299	144
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
308	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
309	9958	49 0099	49 0239	49 0380	49 0520	49 0661	49 0801	49 0941	49 1081	49 1222	140
310	49 1362	1502	1642	1782	1922	2062	2201	2341	2481	2621	140
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	139
312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	138
316	9687	9824	9962	50 0099	50 0236	50 0374	50 0511	50 0648	50 0785	50 0922	137
317	50 1059	50 1196	50 1333	1470	1607	1744	1880	2017	2154	2291	137
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
320	5150	5286	5421	5557	5693	5828	5964	6099	6234	6370	136
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
322	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	135
323	9203	9337	9471	9606	9740	9874	51 0009	51 0143	51 0277	51 0411	134
324	51 0545	51 0679	51 0813	51 0947	51 1081	51 1215	1349	1482	1616	1750	134
325	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	133
326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4415	133
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	132
330	8514	8646	8777	8909	9040	9171	9303	9434	9566	9697	131
331	9828	9959	52 0090	52 0221	52 0353	52 0484	52 0615	52 0745	52 0876	52 1007	131
332	52 1138	52 1269	1400	1530	1661	1792	1922	2053	2183	2314	131
333	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	130
334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	130
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210	129
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	129
337	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788	129
338	8917	9045	9174	9302	9430	9559	9687	9815	9943	53 0072	128
339	53 0200	53 0328	53 0456	53 0584	53 0712	53 0840	53 0968	53 1096	53 1223	1351	128
340	1479	1607	1734	1862	1990	2117	2245	2372	2500	2627	128
341	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899	127
342	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167	127
343	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432	126
344	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693	126
345	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951	126
346	9076	9202	9327	9452	9578	9703	9829	9954	54 0079	54 0204	125
347	54 0329	54 0455	54 0580	54 0705	54 0830	54 0955	54 1080	54 1205	1330	1454	125
348	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701	125
349	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944	124
350	4068	4192	4316	4440	4564	4688	4812	4936	5060	5183	124

0	1	2	3	4	5	6	7	8	9	D
54 4068	54 4192	54 4316	54 4440	54 4564	54 4688	54 4812	54 4936	54 5060	54 5183	124
5307	5431	5555	5678	5802	5925	6049	6172	6296	6419	124
6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	123
7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	123
9003	9126	9249	9371	9494	9616	9739	9861	9984	55 0106	123
55 0228	55 0351	55 0473	55 0595	55 0717	55 0840	55 0962	55 1084	55 1206	1328	122
1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	121
5094	5215	5336	5457	5578	5699	5820	5940	6061	6182	121
6303	6423	6544	6664	6785	6905	7026	7146	7267	7387	120
7507	7627	7748	7868	7988	8108	8228	8349	8469	8589	120
8709	8829	8948	9068	9188	9308	9428	9548	9667	9787	120
9907	56 0026	56 0146	56 0265	56 0385	56 0504	56 0624	56 0743	56 0863	56 0982	119
56 1101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
2293	2412	2531	2650	2769	2887	3006	3125	3244	3362	119
3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
8202	8319	8436	8554	8671	8788	8905	9023	9140	9257	117
9374	9491	9608	9725	9842	9959	57 0076	57 0193	57 0309	57 0426	117
57 0543	57 0660	57 0776	57 0893	57 1010	57 1126	1243	1359	1476	1592	117
1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	116
4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	116
5188	5303	5419	5534	5650	5765	5880	5996	6111	6226	115
6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	115
7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	115
8639	8754	8868	8983	9097	9212	9326	9441	9555	9669	114
9784	9898	58 0012	58 0126	58 0241	58 0355	58 0469	58 0583	58 0697	58 0811	114
58 0925	58 1039	1153	1267	1381	1495	1608	1722	1836	1950	114
2063	2177	2291	2404	2518	2631	2745	2858	2972	3085	114
3199	3312	3426	3539	3652	3765	3879	3992	4105	4218	113
4331	4444	4557	4670	4783	4896	5009	5122	5235	5348	113
5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	113
6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	112
7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	112
8832	8944	9056	9167	9279	9391	9503	9615	9726	9838	112
9950	59 0061	59 0173	59 0284	59 0396	59 0507	59 0619	59 0730	59 0842	59 0953	112
59 1065	1176	1287	1399	1510	1621	1732	1843	1955	2066	111
2177	2288	2399	2510	2621	2732	2843	2954	3064	3175	111
3286	3397	3508	3618	3729	3840	3950	4061	4171	4282	111
4393	4503	4614	4724	4834	4945	5055	5165	5276	5386	110
5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	110
6597	6707	6817	6927	7037	7146	7256	7366	7476	7586	110
7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	110
8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	109
9883	9992	60 0101	60 0210	60 0319	60 0428	60 0537	60 0646	60 0755	60 0864	109
60 0973	60 1082	1191	1299	1408	1517	1625	1734	1843	1951	109
2060	2169	2277	2386	2494	2603	2711	2819	2928	3036	108

Proportional Parts

n\d	124	123	122
1	12	12	12
2	25	25	24
3	37	37	37
4	50	49	49
5	62	62	61
6	74	74	73
7	87	86	85
8	99	98	98
9	112	111	110

n\d	121	120	119
1	12	12	12
2	24	24	24
3	36	36	36
4	48	48	48
5	61	60	60
6	73	72	71
7	85	84	83
8	97	96	95
9	109	108	107

n\d	118	117	116
1	12	12	12
2	24	23	23
3	35	35	35
4	47	47	46
5	59	59	58
6	71	70	70
7	83	82	81
8	94	94	93
9	106	105	104

n\d	115	114	113
1	12	11	11
2	23	23	23
3	35	34	34
4	46	46	45
5	58	57	57
6	69	68	68
7	81	80	79
8	92	91	90
9	104	103	102

n\d	112	111	110
1	11	11	11
2	22	22	22
3	34	33	33
4	45	44	44
5	56	56	55
6	67	67	66
7	78	78	77
8	90	89	88
9	101	100	99

n\d	109	108
1	11	11
2	22	22
3	33	32
4	44	43
5	55	54
6	65	65
7	76	76
8	87	86
9	98	97

Six-Place Logarithms of Numbers 400-450

N	0	1	2	3	4	5	6	7	8	9	D
400	60 2060	60 2169	60 2277	60 2386	60 2494	60 2603	60 2711	60 2819	60 2928	60 3036	108
401	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118	108
402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	108
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274	108
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348	107
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	107
406	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	107
407	9594	9701	9808	9914	61 0021	61 0128	61 0234	61 0341	61 0447	61 0554	107
408	61 0660	61 0767	61 0873	61 0979	1086	1192	1298	1405	1511	1617	106
409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	106
410	2784	2890	2996	3102	3207	3313	3419	3525	3630	3736	106
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792	106
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	105
413	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895	105
414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	105
415	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	105
416	9093	9198	9302	9406	9511	9615	9719	9824	9928	62 0032	104
417	62 0136	62 0240	62 0344	62 0448	62 0552	62 0656	62 0760	62 0864	62 0968	1072	104
418	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	104
419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	104
420	3249	3353	3456	3559	3663	3766	3869	3973	4076	4179	103
421	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210	103
422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	103
423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	103
424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	102
425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	102
426	9410	9512	9613	9715	9817	9919	63 0021	63 0123	63 0224	63 0326	102
427	63 0428	63 0530	63 0631	63 0733	63 0835	63 0936	1038	1139	1241	1342	102
428	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	101
429	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	101
430	3468	3569	3670	3771	3872	3973	4074	4175	4276	4376	101
431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383	101
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388	100
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390	100
434	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389	100
435	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387	100
436	9486	9586	9686	9785	9885	9984	64 0084	64 0183	64 0283	64 0382	99
437	64 0481	64 0581	64 0680	64 0779	64 0879	64 0978	1077	1177	1276	1375	99
438	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366	99
439	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354	99
440	3453	3551	3650	3749	3847	3946	4044	4143	4242	4340	98
441	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324	98
442	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306	98
443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	98
445	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	97
446	9335	9432	9530	9627	9724	9821	9919	65 0016	65 0113	65 0210	97
447	65 0308	65 0405	65 0502	65 0599	65 0696	65 0793	65 0890	0987	1084	1181	97
448	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	97
449	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	97
450	3213	3309	3405	3502	3598	3695	3791	3888	3984	4080	96

0	1	2	3	4	5	6	7	8	9	D
65 3213	65 3309	65 3405	65 3502	65 3598	65 3695	65 3791	65 3888	65 3984	65 4080	96
4177	4273	4369	4465	4562	4658	4754	4850	4946	5042	96
5138	5235	5331	5427	5523	5619	5715	5810	5906	6002	96
6098	6194	6290	6386	6482	6577	6673	6769	6864	6960	96
7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	96
8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	95
8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	95
9916	66 0011	66 0106	66 0201	66 0296	66 0391	66 0486	66 0581	66 0676	66 0771	95
66 0865	0960	1055	1150	1245	1339	1434	1529	1623	1718	95
1813	1907	2002	2096	2191	2286	2380	2475	2569	2663	95
2758	2852	2947	3041	3135	3230	3324	3418	3512	3607	94
3701	3795	3889	3983	4078	4172	4266	4360	4454	4548	94
4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	94
5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	94
6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	94
7453	7546	7640	7733	7826	7920	8013	8106	8199	8293	93
8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	93
9317	9410	9503	9596	9689	9782	9875	9967	67 0060	67 0153	93
67 0246	67 0339	67 0431	67 0524	67 0617	67 0710	67 0802	67 0895	0988	1080	93
1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	93
2098	2190	2283	2375	2467	2560	2652	2744	2836	2929	92
3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	92
3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	92
4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	92
5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	92
6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	91
8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	91
9428	9519	9610	9700	9791	9882	9973	68 0063	68 0154	68 0245	91
68 0336	68 0426	68 0517	68 0607	68 0698	68 0789	68 0879	0970	1060	1151	91
1241	1332	1422	1513	1603	1693	1784	1874	1964	2055	90
2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	90
3047	3137	3227	3317	3407	3497	3587	3677	3767	3857	90
3947	4037	4127	4217	4307	4396	4486	4576	4666	4756	90
4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	90
5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	89
6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	89
7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	89
9309	9398	9486	9575	9664	9753	9841	9930	69 0019	69 0107	89
69 0196	69 0285	69 0373	69 0462	69 0550	69 0639	69 0728	69 0816	0905	0993	89
1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	88
1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	88
3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	88
4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	88
5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	87
6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	87
7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	87
8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	87
8970	9057	9144	9231	9317	9404	9491	9578	9664	9751	87

Proportional Parts

n\d	97	96
1	9.7	9.6
2	19.4	19.2
3	29.1	28.8
4	38.8	38.4
5	48.5	48.0
6	58.2	57.6
7	67.9	67.2
8	77.6	76.8
9	87.3	86.4

n\d	95	94
1	9.5	9.4
2	19.0	18.8
3	28.5	28.2
4	38.0	37.6
5	47.5	47.0
6	57.0	56.4
7	66.5	65.8
8	76.0	75.2
9	85.5	84.6

n\d	93	92
1	9.3	9.2
2	18.6	18.4
3	27.9	27.6
4	37.2	36.8
5	46.5	46.0
6	55.8	55.2
7	65.1	64.4
8	74.4	73.6
9	83.7	82.8

n\d	91	90
1	9.1	9.0
2	18.2	18.0
3	27.3	27.0
4	36.4	36.0
5	45.5	45.0
6	54.6	54.0
7	63.7	63.0
8	72.8	72.0
9	81.9	81.0

n\d	89	88
1	8.9	8.8
2	17.8	17.6
3	26.7	26.4
4	35.6	35.2
5	44.5	44.0
6	53.4	52.8
7	62.3	61.6
8	71.2	70.4
9	80.1	79.2

n\d	87	86
1	8.7	8.6
2	17.4	17.2
3	26.1	25.8
4	34.8	34.4
5	43.5	43.0
6	52.2	51.6
7	60.9	60.2
8	69.6	68.8
9	78.3	77.4

N	0	1	2	3	4	5	6	7	8	9	D
500	69 8970	69 9057	69 9144	69 9231	69 9317	69 9404	69 9491	69 9578	69 9664	69 9751	87
501	9838	9924	70 0011	70 0098	70 0184	70 0271	70 0358	70 0444	70 0531	70 0617	87
502	70 0704	70 0790	0877	0963	1050	1136	1222	1309	1395	1482	86
503	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	86
504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	86
505	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065	86
506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	86
507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	86
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
509	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	85
510	7570	7655	7740	7826	7911	7996	8081	8166	8251	8336	85
511	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	85
512	9270	9355	9440	9524	9609	9694	9779	9863	9948	71 0033	85
513	71 0117	71 0202	71 0287	71 0371	71 0456	71 0540	71 0625	71 0710	71 0794	0879	85
514	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	84
515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566	84
516	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	84
517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	84
518	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	84
519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	84
520	6003	6087	6170	6254	6337	6421	6504	6588	6671	6754	83
521	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587	83
522	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	83
523	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
524	9331	9414	9497	9580	9663	9745	9828	9911	9994	72 0077	83
525	72 0159	72 0242	72 0325	72 0407	72 0490	72 0573	72 0655	72 0738	72 0821	0903	83
526	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	82
527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
528	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
529	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
530	4276	4358	4440	4522	4604	4685	4767	4849	4931	5013	82
531	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
532	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	82
533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
534	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	81
536	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
537	9974	73 0055	73 0136	73 0217	73 0298	73 0378	73 0459	73 0540	73 0621	73 0702	81
538	73 0782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
539	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	81
540	2394	2474	2555	2635	2715	2796	2876	2956	3037	3117	80
541	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919	80
542	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720	80
543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519	80
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	80
545	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	80
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
549	9572	9651	9731	9810	9889	9968	74 0047	74 0126	74 0205	74 0284	79
550	74 0363	74 0442	74 0521	74 0600	74 0678	74 0757	0836	0915	0994	1073	79

0	1	2	3	4	5	6	7	8	9	D
74 0363	74 0442	74 0521	74 0600	74 0678	74 0757	74 0836	74 0915	74 0994	74 1073	79
1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	79
3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
8188	8266	8343	8421	8498	8576	8653	8731	8808	8885	77
8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	77
9736	9814	9891	9968	75 0045	75 0123	75 0200	75 0277	75 0354	75 0431	77
75 0508	75 0586	75 0663	75 0740	0817	0894	0971	1048	1125	1202	77
1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
5875	5951	6027	6103	6180	6256	6332	6408	6484	6560	76
6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	76
8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
9668	9743	9819	9894	9970	76 0045	76 0121	76 0196	76 0272	76 0347	75
76 0422	76 0498	76 0573	76 0649	76 0724	0799	0875	0950	1025	1101	75
1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	75
1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
2679	2754	2829	2904	2978	3053	3128	3203	3278	3353	75
3428	3503	3578	3653	3727	3802	3877	3952	4027	4101	75
4176	4251	4326	4400	4475	4550	4624	4699	4774	4848	75
4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	75
5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
9377	9451	9525	9599	9673	9746	9820	9894	9968	77 0042	74
77 0115	77 0189	77 0263	77 0336	77 0410	77 0484	77 0557	77 0631	77 0705	0778	74
0852	0926	0999	1073	1146	1220	1293	1367	1440	1514	74
1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
8151	8224	[illegible]	8368	8441	8513	8585	8658	8730	8802	72

Proportional Parts

n\d	79	78
1	7.9	7.8
2	15.8	15.6
3	23.7	23.4
4	31.6	31.2
5	39.5	39.0
6	47.4	46.8
7	55.3	54.6
8	63.2	62.4
9	71.1	70.2

n\d	77	76
1	7.7	7.6
2	15.4	15.2
3	23.1	22.8
4	30.8	30.4
5	38.5	38.0
6	46.2	45.6
7	53.9	53.2
8	61.6	60.8
9	69.3	68.4

n\d	75	74
1	7.5	7.4
2	15.0	14.8
3	22.5	22.2
4	30.0	29.6
5	37.5	37.0
6	45.0	44.4
7	52.5	51.8
8	60.0	59.2
9	67.5	66.6

n\d	73
1	7.3
2	14.6
3	21.9
4	29.2
5	36.5
6	43.8
7	51.1
8	58.4
9	65.7

n\d	72
1	7.2
2	14.4
3	21.6
4	28.8
5	36.0
6	43.2
7	50.4
8	57.6
9	64.8

N	0	1	2	3	4	5	6	7	8	9	D
600	77 8151	77 8224	77 8296	77 8368	77 8441	77 8513	77 8585	77 8658	77 8730	77 8802	72
601	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	72
602	9596	9669	9741	9813	9885	9957	78 0029	78 0101	78 0173	78 0245	72
603	78 0317	78 0389	78 0461	78 0533	78 0605	78 0677	0749	0821	0893	0965	72
604	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
606	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
608	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
610	5330	5401	5472	5543	5615	5686	5757	5828	5899	5970	71
611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
612	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
613	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
614	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
616	9581	9651	9722	9792	9863	9933	79 0004	79 0074	79 0144	79 0215	70
617	79 0285	79 0356	79 0426	79 0496	79 0567	79 0637	0707	0778	0848	0918	70
618	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
620	2392	2462	2532	2602	2672	2742	2812	2882	2952	3022	70
621	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	70
622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	70
623	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	70
624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
629	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
630	9341	9409	9478	9547	9616	9685	9754	9823	9892	9961	69
631	80 0029	80 0098	80 0167	80 0236	80 0305	80 0373	80 0442	80 0511	80 0580	80 0648	69
632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	68
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
639	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	68
640	6180	6248	6316	6384	6451	6519	6587	6655	6723	6790	68
641	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	68
642	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
643	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
644	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
645	9560	9627	9694	9762	9829	9896	9964	81 0031	81 0098	81 0165	67
646	81 0233	81 0300	81 0367	81 0434	81 0501	81 0569	81 0636	0703	0770	0837	67
647	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
649	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
650	2913	2980	3047	3114	3181	3247	3314	3381	3448	3514	67

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81 2913	81 2980	81 3047	81 3114	81 3181	81 3247	81 3314	81 3381	81 3448	81 3514	67
3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	67
4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	66
8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	66
8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
9544	9610	9676	9741	9807	9873	9939	82 0004	82 0070	82 0136	66
82 0201	82 0267	82 0333	82 0399	82 0464	82 0530	82 0595	0661	0727	0792	66
0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	66
1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	65
2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	65
2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	65
3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
6075	6140	6204	6269	6334	6399	6464	6528	6593	6658	65
6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	65
7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	65
8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	64
8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	64
9947	83 0011	83 0075	83 0139	83 0204	83 0268	83 0332	83 0396	83 0460	83 0525	64
83 0589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
2509	2573	2637	2700	2764	2828	2892	2956	3020	3083	64
3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	64
3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	64
4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	64
5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	63
5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	63
7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	63
8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
8849	8912	8975	9038	9101	9164	9227	9289	9352	9415	63
9478	9541	9604	9667	9729	9792	9855	9918	9981	84 0043	63
84 0106	84 0169	84 0232	84 0294	84 0357	84 0420	84 0482	84 0545	84 0608	0671	63
0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	63
1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	63
1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	62
5098	5160	5222	5284	5346	5408	5470	5532	5594	5656	62

Proportional Parts

n\d	67
1	6.7
2	13.4
3	20.1
4	26.8
5	33.5
6	40.2
7	46.9
8	53.6
9	60.3

n\d	66
1	6.6
2	13.2
3	19.8
4	26.4
5	33.0
6	39.6
7	46.2
8	52.8
9	59.4

n\d	65
1	6.5
2	13.0
3	19.5
4	26.0
5	32.5
6	39.0
7	45.5
8	52.0
9	58.5

n\d	64
1	6.4
2	12.8
3	19.2
4	25.6
5	32.0
6	38.4
7	44.8
8	51.2
9	57.6

n\d	63
1	6.3
2	12.6
3	18.9
4	25.2
5	31.5
6	37.8
7	44.1
8	50.4
9	56.7

n\d	62
1	6.2
2	12.4
3	18.6
4	24.8
5	31.0
6	37.2
7	43.4
8	49.6
9	55.8

N	0	1	2	3	4	5	6	7	8	9	D
700	84 5098	84 5160	84 5222	84 5284	84 5346	84 5408	84 5470	84 5532	84 5594	84 5656	62
701	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	62
702	6337	6399	6461	6523	5685	6646	6708	6770	6832	6894	62
703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
704	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	62
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
708	85 0033	85 0095	85 0156	85 0217	85 0279	85 0340	85 0401	85 0462	85 0524	85 0585	61
709	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
710	1258	1320	1381	1442	1503	1564	1625	1686	1747	1809	61
711	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	61
712	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	61
713	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	61
714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	61
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	60
720	7332	7393	7453	7513	7574	7634	7694	7755	7815	7875	60
721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	60
722	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	60
723	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	60
724	9739	9799	9859	9918	9978	86 0038	86 0098	86 0158	86 0218	86 0278	60
725	86 0338	86 0398	86 0458	86 0518	86 0578	0637	0697	0757	0817	0877	60
726	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	60
727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
728	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
729	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
730	3323	3382	3442	3501	3561	3620	3680	3739	3799	3858	59
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	59
732	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	59
733	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	59
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
737	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	59
738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	59
739	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	59
740	9232	9290	9349	9408	9466	9525	9584	9642	9701	9760	59
741	9818	9877	9935	9994	87 0053	87 0111	87 0170	87 0228	87 0287	87 0345	59
742	87 0404	87 0462	87 0521	87 0579	0638	0696	0755	0813	0872	0930	58
743	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
744	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
745	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	58
746	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
747	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
748	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
749	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
750	5061	5119	5177	5235	5293	5351	5409	5466	5524	5582	58

Six-Place Logarithms of Numbers 750-800

0	1	2	3	4	5	6	7	8	9	D
87 5061	87 5119	87 5177	87 5235	87 5293	87 5351	87 5409	87 5466	87 5524	87 5582	58
5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	58
6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	58
6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	58
7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	58
7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	57
8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	57
9669	9726	9784	9841	9898	9956	88 0013	88 0070	88 0127	88 0185	57
88 0242	88 0299	88 0356	88 0413	88 0471	88 0528	0585	0642	0699	0756	57
0814	0871	0928	0985	1042	1099	1156	1213	1271	1328	57
1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	57
1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	57
3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	57
3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	57
4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	57
4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
6491	6547	6604	6660	6716	6773	6829	6885	6942	6998	56
7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	56
7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	56
8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	56
8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
9862	9918	9974	89 0030	89 0086	89 0141	89 0197	89 0253	89 0309	89 0365	56
89 0421	89 0477	89 0533	0589	0645	0700	0756	0812	0868	0924	56
0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	56
1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
2095	2150	2206	2262	2317	2373	2429	2484	2540	2595	56
2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	56
3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	56
3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	55
5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
7627	7682	7737	7792	7847	7902	7957	8012	8067	8122	55
8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	55
9821	9875	9930	9985	90 0039	90 0094	90 0149	90 0203	90 0258	90 0312	55
90 0367	90 0422	90 0476	90 0531	0586	0640	0695	0749	0804	0859	55
0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	55
1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
3090	3144	3199	3253	3307	3361	3416	3470	3524	3578	54

Proportional Parts

n\d	58
1	5.8
2	11.6
3	17.4
4	23.2
5	29.0
6	34.8
7	40.6
8	46.4
9	52.2

n\d	57
1	5.7
2	11.4
3	17.1
4	22.8
5	28.5
6	34.2
7	39.9
8	45.6
9	51.3

n\d	56
1	5.6
2	11.2
3	16.8
4	22.4
5	28.0
6	33.6
7	39.2
8	44.8
9	50.4

n\d	55
1	5.5
2	11.0
3	16.5
4	22.0
5	27.5
6	33.0
7	38.5
8	44.0
9	49.5

n\d	54
1	5.4
2	10.8
3	16.2
4	21.6
5	27.0
6	32.4
7	37.8
8	43.2
9	48.6

Six-Place Logarithms of Numbers 800-850

N	0	1	2	3	4	5	6	7	8	9	D
800	90 3090	90 3144	90 3199	90 3253	90 3307	90 3361	90 3416	90 3470	90 3524	90 3578	54
801	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	54
802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
810	8485	8539	8592	8646	8699	8753	8807	8860	8914	8967	54
811	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	54
812	9556	9610	9663	9716	9770	9823	9877	9930	9984	91 0037	53
813	91 0091	91 0144	91 0197	91 0251	91 0304	91 0358	91 0411	91 0464	91 0518	0571	53
814	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	53
820	3814	3867	3920	3973	4026	4079	4132	4184	4237	4290	53
821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	53
822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	9078	9130	9183	9235	9287	9340	9392	9444	9496	9549	52
831	9601	9653	9706	9758	9810	9862	9914	9967	92 0019	92 0071	52
832	92 0123	92 0176	92 0228	92 0280	92 0332	92 0384	92 0436	92 0489	0541	0593	52
833	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
834	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	52
837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
839	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
840	4279	4331	4383	4434	4486	4538	4589	4641	4693	4744	52
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	52
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
847	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
849	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
850	9419	9470	9521	9572	9623	9674	9725	9776	9827	9879	51

0	1	2	3	4	5	6	7	8	9	D
92 9419	92 9470	92 9521	92 9572	92 9623	92 9674	92 9725	92 9776	92 9827	92 9879	51
9930	9981	93 0032	93 0083	93 0134	93 0185	93 0236	93 0287	93 0338	93 0389	51
93 0440	93 0491	0542	0592	0643	0694	0745	0796	0847	0898	51
0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	51
3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	51
4498	4549	4599	4650	4700	4751	4801	4852	4902	4953	50
5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	50
5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	50
6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	50
6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
7016	7066	7117	7167	7217	7267	7317	7367	7418	7468	50
7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	50
9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
9519	9569	9619	9669	9719	9769	9819	9869	9918	9968	50
94 0018	94 0068	94 0118	94 0168	94 0218	94 0267	94 0317	94 0367	94 0417	94 0467	50
0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	50
1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	50
1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	50
2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	49
3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	49
4483	4532	4581	4631	4680	4729	4779	4828	4877	4927	49
4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	49
5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	49
7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
9390	9439	9488	9536	9585	9634	9683	9731	9780	9829	49
9878	9926	9975	95 0024	95 0073	95 0121	95 0170	95 0219	95 0267	95 0316	49
95 0365	95 0414	95 0462	0511	0560	0608	0657	0706	0754	0803	49
0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
4243	4291	4339	4387	4435	4484	4532	4580	4628	4677	48

Proportional Parts

n\d	52
1	5.2
2	10.4
3	15.6
4	20.8
5	26.0
6	31.2
7	36.4
8	41.6
9	46.8

n\d	51
1	5.1
2	10.2
3	15.3
4	20.4
5	25.5
6	30.6
7	35.7
8	40.8
9	45.9

n\d	50
1	5.0
2	10.0
3	15.0
4	20.0
5	25.0
6	30.0
7	35.0
8	40.0
9	45.0

n\d	49
1	4.9
2	9.8
3	14.7
4	19.6
5	24.5
6	29.4
7	34.3
8	39.2
9	44.1

n\d	48
1	4.8
2	9.6
3	14.4
4	19.2
5	24.0
6	28.8
7	33.6
8	38.4
9	43.2

Six-Place Logarithms of Numbers 900-950

N	0	1	2	3	4	5	6	7	8	9	D
900	95 4243	95 4291	95 4339	95 4387	95 4435	95 4484	95 4532	95 4580	95 4628	95 4677	48
901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	48
902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	48
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	48
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
910	9041	9089	9137	9185	9232	9280	9328	9375	9423	9471	48
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	48
912	9995	96 0042	96 0090	96 0138	96 0185	96 0233	96 0280	96 0328	96 0376	96 0423	48
913	96 0471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	48
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
920	3788	3835	3882	3929	3977	4024	4071	4118	4165	4212	47
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	8483	8530	8576	8623	8670	8716	8763	8810	8856	8903	47
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
933	9882	9928	9975	97 0021	97 0068	97 0114	97 0161	97 0207	97 0254	97 0300	47
934	97 0347	97 0393	97 0440	0486	0533	0579	0626	0672	0719	0765	46
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
940	3128	3174	3220	3266	3313	3359	3405	3451	3497	3543	46
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	7724	7769	7815	7861	7906	7952	7998	8043	8089	8135	46

Six-Place Logarithms of Numbers 950-1000

N	0	1	2	3	4	5	6	7	8	9	D
950	97 7724	97 7769	97 7815	97 7861	97 7906	97 7952	97 7998	97 8043	97 8089	97 8135	46
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	98 0003	98 0049	98 0094	98 0140	98 0185	98 0231	98 0276	98 0322	98 0367	98 0412	45
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	2271	2316	2362	2407	2452	2497	2543	2588	2633	2678	45
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	6772	6817	6861	6906	6951	6996	7040	7085	7130	7175	45
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
977	9895	9939	9983	99 0028	99 0072	99 0117	99 0161	99 0206	99 0250	99 0294	44
978	99 0339	99 0383	99 0428	0472	0516	0561	0605	0650	0694	0738	44
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	1226	1270	1315	1359	1403	1448	1492	1536	1580	1625	44
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	5635	5679	5723	5767	5811	5854	5898	5942	5986	6030	44
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
000	00 0000	00 0043	00 0087	00 0130	00 0174	00 0217	00 0260	00 0304	00 0347	00 0391	43

Proportional Parts

n\d	46
1	4.6
2	9.2
3	13.8
4	18.4
5	23.0
6	27.6
7	32.2
8	36.8
9	41.4

n\d	45
1	4.5
2	9.0
3	13.5
4	18.0
5	22.5
6	27.0
7	31.5
8	36.0
9	40.5

n\d	44
1	4.4
2	8.8
3	13.2
4	17.6
5	22.0
6	26.4
7	30.8
8	35.2
9	39.6

n\d	43
1	4.3
2	8.6
3	12.9
4	17.2
5	21.5
6	25.8
7	30.1
8	34.4
9	38.7

Six-Place Logarithms of Numbers 900-950

N	0	1	2	3	4	5	6	7	8	9	D
900	95 4243	95 4291	95 4339	95 4387	95 4435	95 4484	95 4532	95 4580	95 4628	95 4677	48
901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	48
902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	48
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	48
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
910	9041	9089	9137	9185	9232	9280	9328	9375	9423	9471	48
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	48
912	9995	96 0042	96 0090	96 0138	96 0185	96 0233	96 0280	96 0328	96 0375	96 0423	48
913	96 0471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	48
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
920	3788	3835	3882	3929	3977	4024	4071	4118	4165	4212	47
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	8483	8530	8576	8623	8670	8716	8763	8810	8856	8903	47
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
933	9882	9928	9975	97 0021	97 0068	97 0114	97 0161	97 0207	97 0254	97 0300	47
934	97 0347	97 0393	97 0440	0486	0533	0579	0626	0672	0719	0765	46
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
940	3128	3174	3220	3266	3313	3359	3405	3451	3497	3543	46
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
	7724	7769	7815	7861	7906	7952	7998	8043	8089	8135	46

Six-Place Logarithms of Numbers 950-1000

N	0	1	2	3	4	5	6	7	8	9	D
950	97 7724	97 7769	97 7815	97 7861	97 7906	97 7952	97 7998	97 8043	97 8089	97 8135	46
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	98 0003	98 0049	98 0094	98 0140	98 0185	98 0231	98 0276	98 0322	98 0367	98 0412	45
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	2271	2316	2362	2407	2452	2497	2543	2588	2633	2678	45
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	6772	6817	6861	6906	6951	6996	7040	7085	7130	7175	45
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
977	9895	9939	9983	99 0028	99 0072	99 0117	99 0161	99 0206	99 0250	99 0294	44
978	99 0339	99 0383	99 0428	0472	0516	0561	0605	0650	0694	0738	44
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	1226	1270	1315	1359	1403	1448	1492	1536	1580	1625	44
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	5635	5679	5723	5767	5811	5854	5898	5942	5986	6030	44
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
1000	00 0000	00 0043	00 0087	00 0130	00 0174	00 0217	00 0260	00 0304	00 0347	00 0391	43

Proportional Parts

n\d	46
1	4.6
2	9.2
3	13.8
4	18.4
5	23.0
6	27.6
7	32.2
8	36.8
9	41.4

n\d	45
1	4.5
2	9.0
3	13.5
4	18.0
5	22.5
6	27.0
7	31.5
8	36.0
9	40.5

n\d	44
1	4.4
2	8.8
3	13.2
4	17.6
5	22.0
6	26.4
7	30.8
8	35.2
9	39.6

n\d	43
1	4.3
2	8.6
3	12.9
4	17.2
5	21.5
6	25.8
7	30.1
8	34.4
9	38.7

RATE

1/6%

.00166666 *per period*

ANNUALLY
If compounded *annually* nominal annual rate is **1/6%**

EMIANNUALLY
If compounded *semiannually* nominal annual rate is **1/3%**

QUARTERLY
If compounded *quarterly* nominal annual rate is **2/3%**

MONTHLY
If compounded *monthly* nominal annual rate is **2%**

$i = .00166666$
$j_{(2)} = .00333333$
$j_{(4)} = .00666666$
$j_{(12)} = .02$

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.001 666 6667	1.000 000 0000	1.000 000 0000
2	1.003 336 1111	2.001 666 6667	.499 583 6803
3	1.005 008 3380	3.005 002 7778	.332 778 3945
4	1.006 683 3519	4.010 011 1157	.249 375 8673
5	1.008 361 1574	5.016 694 4676	.199 334 4435
6	1.010 041 7594	6.025 055 6250	.165 973 5714
7	1.011 725 1623	7.035 097 3844	.142 144 4431
8	1.013 411 3709	8.046 822 5467	.124 272 6547
9	1.015 100 3899	9.060 233 9176	.110 372 4263
10	1.016 792 2238	10.075 334 3075	.099 252 2897
11	1.018 486 8776	11.092 126 5313	.090 154 0383
12	1.020 184 3557	12.110 613 4089	.082 572 2006
13	1.021 884 6629	13.130 797 7646	.076 156 8351
14	1.023 587 8040	14.152 682 4275	.070 657 9834
15	1.025 293 7837	15.176 270 2316	.065 892 3428
16	1.027 002 6067	16.201 564 0153	.061 722 4361
17	1.028 714 2777	17.228 566 6220	.058 043 1339
18	1.030 428 8015	18.257 280 8997	.054 772 6688
19	1.032 146 1828	19.287 709 7012	.051 846 4875
20	1.033 866 4265	20.319 855 8840	.049 212 9475
21	1.035 589 5372	21.353 722 3105	.046 830 2428
22	1.037 315 5197	22.389 311 8477	.044 664 1686
23	1.039 044 3789	23.426 627 3674	.042 686 4689
24	1.040 776 1196	24.465 671 7464	.040 873 5967
25	1.042 510 7464	25.506 447 8659	.039 205 7728
26	1.044 248 2644	26.548 958 6124	.037 666 2608
27	1.045 988 6781	27.593 206 8767	.036 240 8039
28	1.047 731 9926	28.639 195 5549	.034 917 1819
29	1.049 478 2126	29.686 927 5475	.033 684 8601
30	1.051 227 3429	30.736 405 7600	.032 534 7084
31	1.052 979 3885	31.787 633 1030	.031 458 7751
32	1.054 734 3542	32.840 612 4915	.030 450 1020
33	1.056 492 2447	33.895 346 8456	.029 502 5746
34	1.058 253 0652	34.951 839 0904	.028 610 7978
35	1.060 016 8203	36.010 092 1555	.027 769 9928
36	1.061 783 5150	37.070 108 9758	.026 975 9121
37	1.063 553 1542	38.131 892 4907	.026 224 7671
38	1.065 325 7427	39.195 445 6449	.025 513 1683
39	1.067 101 2856	40.260 771 3876	.024 838 0735
40	1.068 879 7878	41.327 872 6733	.024 196 7451
41	1.070 661 2541	42.396 752 4611	.023 586 7122
42	1.072 445 6895	43.467 413 7252	.023 005 7396
43	1.074 233 0990	44.539 859 4047	.022 451 7997
44	1.076 023 4875	45.614 092 5037	.021 923 0493
45	1.077 816 8600	46.690 115 9912	.021 417 8093
46	1.079 613 2214	47.767 932 8512	.020 934 5463
47	1.081 412 5768	48.847 546 0726	.020 471 8575
48	1.083 214 9311	49.928 958 6494	.020 028 4570
49	1.085 020 2893	51.012 173 5805	.019 603 1639
50	1.086 828 6564	52.097 193 8698	.019 194 8918
51	1.088 640 0375	53.184 022 5262	.018 802 6594
52	1.090 454 4376	54.272 662 5638	.018 425 4826
53	1.092 271 8617	55.363 117 0014	.018 062 5668
54	1.094 092 3148	56.455 388 8631	.017 713 1009
55	1.095 915 8020	57.549 481 1778	.017 376 3513
56	1.097 742 3283	58.645 396 9798	.017 051 6366
57	1.099 571 8988	59.743 139 3081	.016 738 3236
58	1.101 404 5187	60.842 711 2069	.016 435 8225
59	1.103 240 1929	61.944 115 7256	.016 143 5834
60	1.105 078 9265	63.047 355 9185	.015 861 0934
n	$s = (1+i)^n$	$s_{\overline{n}\vert} = \frac{(1+i)^n - 1}{i}$	$\frac{1}{s_{\overline{n}\vert}} = \frac{i}{(1+i)^n - 1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today. Periodic payment necessary to pay off a loan of $1.*	PERIODS
.998 336 1065	.998 336 1065	1.001 666 6667	1
.996 674 9815	1.995 011 0880	.501 250 3469	2
.995 016 6205	2.990 027 7085	.334 445 0612	3
.993 361 0188	3.983 388 7273	.251 042 5340	4
.991 708 1718	4.975 096 8991	.201 001 1102	5
.990 058 0750	5.965 154 9742	.167 640 2381	6
.988 410 7238	6.953 565 6980	.143 811 1098	7
.986 766 1136	7.940 331 8116	.125 939 3214	8
.985 124 2399	8.925 456 0516	.112 039 0929	9
.983 485 0981	9.908 941 1496	.100 918 9564	10
.981 848 6836	10.890 789 8333	.091 820 7050	11
.980 214 9920	11.871 004 8252	.084 238 8673	12
.978 584 0186	12.849 588 8438	.077 823 5018	13
.976 955 7590	13.826 544 6028	.072 324 6501	14
.975 330 2086	14.801 874 8114	.067 559 0094	15
.973 707 3630	15.775 582 1745	.063 389 1028	16
.972 087 2177	16.747 669 3922	.059 709 8006	17
.970 469 7681	17.718 139 1602	.056 439 3355	18
.968 855 0097	18.686 994 1700	.053 513 1542	19
.967 242 9382	19.654 237 1081	.050 879 6141	20
.965 633 5489	20.619 870 6570	.048 496 9094	21
.964 026 8375	21.583 897 4945	.046 330 8353	22
.962 422 7995	22.546 320 2940	.044 353 1355	23
.960 821 4305	23.507 141 7245	.042 540 2634	24
.959 222 7259	24.466 364 4504	.040 872 4395	25
.957 626 6814	25.423 991 1319	.039 332 9275	26
.956 033 2926	26.380 024 4245	.037 907 4706	27
.954 442 5550	27.334 466 9795	.036 583 8485	28
.952 854 4643	28.287 321 4438	.035 351 5267	29
.951 269 0159	29.238 590 4597	.034 201 3751	30
.949 686 2056	30.188 276 6652	.033 125 4417	31
.948 106 0288	31.136 382 6941	.032 116 7687	32
.946 528 4814	32.082 911 1754	.031 169 2413	33
.944 953 5588	33.027 864 7342	.030 277 4644	34
.943 381 2567	33.971 245 9909	.029 436 6595	35
.941 811 5707	34.913 057 5616	.028 642 5787	36
.940 244 4966	35.853 302 0582	.027 891 4338	37
.938 680 0299	36.791 982 0881	.027 179 8349	38
.937 118 1662	37.729 100 2543	.026 504 7402	39
.935 558 9014	38.664 659 1557	.025 863 4118	40
.934 002 2310	39.598 661 3867	.025 253 3789	41
.932 448 1508	40.531 109 5375	.024 672 4062	42
.930 896 6563	41.462 006 1938	.024 118 4663	43
.929 347 7434	42.391 353 9373	.023 589 7160	44
.927 801 4078	43.319 155 3450	.023 084 4760	45
.926 257 6450	44.245 412 9901	.022 601 2129	46
.924 716 4509	45.170 129 4410	.022 138 5241	47
.923 177 8212	46.093 307 2622	.021 695 1236	48
.921 641 7516	47.014 949 0139	.021 269 8306	49
.920 108 2379	47.935 057 2518	.020 861 5585	50
.918 577 2758	48.853 634 5276	.020 469 3061	51
.917 048 8610	49.770 683 3886	.020 092 1493	52
.915 522 9894	50.686 206 3780	.019 729 2335	53
.913 999 6566	51.600 206 0346	.019 379 7676	54
.912 478 8585	52.512 684 8931	.019 043 0179	55
.910 960 5909	53.423 645 4839	.018 718 3033	56
.909 444 8494	54.333 090 3334	.018 404 9903	57
.907 931 6301	55.241 021 9634	.018 102 4891	58
.906 420 9285	56.147 442 8920	.017 810 2501	59
.904 912 7406	57.052 355 6326	.017 527 7601	60
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\vert} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\vert}} = \frac{i}{1-v^n}$	n

RATE

1/6%

.0016666$\dot{6}$

per period

ANNUALLY
If compounded *annually* nominal annual rate is

1/6%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

1/3%

QUARTERLY
If compounded *quarterly* nominal annual rate is

2/3%

MONTHLY
If compounded *monthly* nominal annual rate is

2%

$i = .00166666$
$j_{(2)} = .00333333$
$j_{(4)} = .00666666$
$j_{(12)} = .02$

RATE

1/6%

00166666

per period

ANNUALLY
If compounded *annually* nominal annual rate is
1/6%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is
1/3%

QUARTERLY
If compounded *quarterly* nominal annual rate is
2/3%

MONTHLY
If compounded *monthly* nominal annual rate is
2%

$i = .00166666$
$j_{(2)} = .00333333$
$j_{(4)} = .00666666$
$j_{(12)} = .02$

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
61	1.106 920 7247	64.152 434 8450	.015 587 8729
62	1.108 765 5926	65.259 355 5698	.015 323 4734
63	1.110 613 5353	66.368 121 1624	.015 067 4749
64	1.112 464 5578	67.478 734 6977	.014 819 4836
65	1.114 318 6654	68.591 199 2555	.014 579 1298
66	1.116 175 8632	69.705 517 9209	.014 346 0666
67	1.118 036 1563	70.821 693 7841	.014 119 9673
68	1.119 899 5499	71.939 729 9404	.013 900 5248
69	1.121 766 0492	73.059 629 4903	.013 687 4496
70	1.123 635 6592	74.181 395 5395	.013 480 4690
71	1.125 508 3853	75.305 031 1987	.013 279 3252
72	1.127 384 2326	76.430 539 5840	.013 083 7752
73	1.129 263 2064	77.557 923 8167	.012 893 5891
74	1.131 145 3117	78.687 187 0230	.012 708 5494
75	1.133 030 5539	79.818 332 3347	.012 528 4502
76	1.134 928 9381	80.951 362 8886	.012 353 0965
77	1.136 810 4697	82.086 281 8268	.012 182 3035
78	1.138 705 1538	83.223 092 2965	.012 015 8957
79	1.140 602 9958	84.361 797 4503	.011 853 7067
80	1.142 504 0007	85.502 400 4461	.011 695 5781
81	1.144 408 1741	86.644 904 4468	.011 541 3596
82	1.146 315 5210	87.789 312 6209	.011 390 9082
83	1.148 226 0469	88.935 628 1419	.011 244 0877
84	1.150 139 7570	90.083 854 1888	.011 100 7684
85	1.152 056 6566	91.233 993 9458	.010 960 8267
86	1.153 976 7510	92.386 050 6024	.010 824 1449
87	1.155 900 0456	93.540 027 3534	.010 690 6105
88	1.157 826 5457	94.695 927 3990	.010 560 1162
89	1.159 756 2566	95.853 753 9446	.010 432 5596
90	1.161 689 1837	97.013 510 2012	.010 307 8427
91	1.163 625 3323	98.175 199 3849	.010 185 8719
92	1.165 564 7079	99.338 824 7172	.010 066 5576
93	1.167 507 3157	100.504 389 4251	.009 949 8142
94	1.169 453 1612	101.671 896 7408	.009 835 5596
95	1.171 402 2498	102.841 349 9020	.009 723 7152
96	1.173 354 5869	104.012 752 1518	.009 614 2058
97	1.175 310 1779	105.186 106 7388	.009 506 9590
98	1.177 269 0282	106.361 416 9166	.009 401 9056
99	1.179 231 1432	107.538 685 9448	.009 298 9792
100	1.181 196 5285	108.717 917 0881	.009 198 1159
101	1.183 165 1894	109.899 113 6166	.009 099 2545
102	1.185 137 1313	111.082 278 8059	.009 002 3360
103	1.187 112 3599	112.267 415 9373	.008 907 3040
104	1.189 090 8805	113.454 528 2972	.008 814 1039
105	1.191 072 6986	114.643 619 1777	.008 722 6835
106	1.193 057 8198	115.834 691 8763	.008 632 9923
107	1.195 046 2495	117.027 749 6961	.008 544 9819
108	1.197 037 9932	118.222 795 9456	.008 458 6056
109	1.199 033 0566	119.419 833 9388	.008 373 8184
110	1.201 031 4450	120.618 866 9954	.008 290 5770
111	1.203 033 1641	121.819 898 4404	.008 208 8395
112	1.205 038 2193	123.022 931 6044	.008 128 5658
113	1.207 046 6164	124.227 969 8238	.008 049 7170
114	1.209 058 3607	125.435 016 4402	.007 972 2555
115	1.211 073 4580	126.644 074 8009	.007 896 1452
116	1.213 091 9138	127.855 148 2589	.007 821 3511
117	1.215 113 7336	129.068 240 1727	.007 747 8394
118	1.217 138 9232	130.283 353 9063	.007 675 5777
119	1.219 167 4880	131.500 492 8295	.007 604 5342
120	1.221 199 4339	132.719 660 3175	.007 534 6787
n	$s=(1+i)^n$	$s_{\overline{n}\vert} = \frac{(1+i)^n-1}{i}$	$\frac{1}{s_{\overline{n}\vert}} = \frac{i}{(1+i)^n-1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today.* *Periodic payment necessary to pay off a loan of $1.*	PERIODS
.903 407 0622	57.955 762 6947	.017 254 5396	61
.901 903 8890	58.857 666 5838	.016 990 1401	62
.900 403 2170	59.758 069 8008	.016 734 1416	63
.898 905 0419	60.656 974 8427	.016 486 1502	64
.897 409 3597	61.554 384 2024	.016 245 7965	65
.895 916 1661	62.450 300 3684	.016 012 7332	66
.894 425 4570	63.344 725 8254	.015 786 6340	67
.892 937 2282	64.237 663 0536	.015 567 1915	68
.891 451 4758	65.129 114 5294	.015 354 1163	69
.889 968 1955	66.019 082 7249	.015 147 1356	70
.888 487 3832	66.907 570 1080	.014 945 9919	71
.887 009 0348	67.794 579 1428	.014 750 4419	72
.885 533 1462	68.680 112 2890	.014 560 2558	73
.884 059 7133	69.564 172 0023	.014 375 2160	74
.882 588 7321	70.446 760 7344	.014 195 1168	75
.881 120 1984	71.327 880 9328	.014 019 7632	76
.879 654 1083	72.207 535 0411	.013 848 9702	77
.878 190 4575	73.085 725 4986	.013 682 5624	78
.876 729 2421	73.962 454 7407	.013 520 3733	79
.875 270 4580	74.837 725 1987	.013 362 2447	80
.873 814 1012	75.711 539 2999	.013 208 0263	81
.872 360 1676	76.583 899 4674	.013 057 5749	82
.870 908 6531	77.454 808 1206	.012 910 7543	83
.869 459 5539	78.324 267 6744	.012 767 4350	84
.868 012 8658	79.192 280 5402	.012 627 4934	85
.866 568 5848	80.058 849 1250	.012 490 8116	86
.865 126 7069	80.923 975 8319	.012 357 2772	87
.863 687 2282	81.787 663 0602	.012 226 7829	88
.862 250 1447	82.649 913 2048	.012 099 2263	89
.860 815 4522	93.510 728 6571	.011 974 5093	90
.859 383 1470	84.370 111 8041	.011 852 5385	91
.857 953 2250	85.228 065 0290	.011 733 2243	92
.856 525 6821	86.084 590 7112	.011 616 4809	93
.855 100 5146	86.939 691 2258	.011 502 2263	94
.853 677 7184	87.793 368 9442	.011 390 3819	95
.852 257 2896	88.645 626 2338	.011 280 8724	96
.850 839 2242	89.496 465 4581	.011 173 6256	97
.849 423 5184	90.345 888 9764	.011 068 5723	98
.848 010 1681	91.193 899 1445	.010 965 6458	99
.846 599 1695	92.040 498 3140	.010 864 7826	100
.845 190 5186	92.885 688 8326	.010 765 9211	101
.843 784 2116	93.729 473 0442	.010 669 0027	102
.842 380 2445	94.571 853 2887	.010 573 9706	103
.840 978 6135	95.412 831 9022	.010 480 7706	104
.839 579 3146	96.252 411 2169	.010 389 3501	105
.838 182 3441	97.090 593 5609	.010 299 6589	106
.836 787 6979	97.927 381 2588	.010 211 6485	107
.835 395 3723	98.762 776 6311	.010 125 2722	108
.834 005 3633	99.596 781 9945	.010 040 4850	109
.832 617 6672	100.429 399 6617	.009 957 2436	110
.831 232 2801	101.260 631 9418	.009 875 5062	111
.829 849 1981	102.090 481 1399	.009 795 2325	112
.828 468 4174	102.918 949 5573	.009 716 3837	113
.827 089 9342	103.746 039 4915	.009 638 9222	114
.825 713 7446	104.571 753 2361	.009 562 8118	115
.824 339 8449	105.396 093 0809	.009 488 0177	116
.822 968 2311	106.219 061 3121	.009 414 5061	117
.821 598 8996	107.040 660 2117	.009 342 2443	118
.820 231 8466	107.860 892 0583	.009 271 2009	119
.818 867 0681	108.679 759 1264	.009 201 3454	120
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\rvert} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rvert}} = \frac{i}{1-v^n}$	n

RATE

1/6%

.00166666

per period

ANNUALLY

If compounded *annually* nominal annual rate is

1/6%

SEMIANNUALLY

If compounded *semiannually* nominal annual rate is

1/3%

QUARTERLY

If compounded *quarterly* nominal annual rate is

2/3%

MONTHLY

If compounded *monthly* nominal annual rate is

2%

$i = .00166666$
$j_{(2)} = .00333333$
$j_{(4)} = .00666666$
$j_{(12)} = .02$

RATE

$\frac{1}{4}\%$

.0025

per period

ANNUALLY
If compounded *annually* nominal annual rate is

$\frac{1}{4}\%$

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

$\frac{1}{2}\%$

QUARTERLY
If compounded *quarterly* nominal annual rate is

1%

MONTHLY
If compounded *monthly* nominal annual rate is

3%

$i = .0025$
$j_{(2)} = .005$
$j_{(4)} = .01$
$j_{(12)} = .03$

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.002 500 0000	1.000 000 0000	1.000 000 0000
2	1.005 006 2500	2.002 500 0000	.499 375 7803
3	1.007 518 7656	3.007 506 2500	.332 501 3872
4	1.010 037 5625	4.015 025 0156	.249 064 4507
5	1.012 562 6564	5.025 062 5782	.199 002 4969
6	1.015 094 0631	6.037 625 2346	.165 628 0344
7	1.017 631 7982	7.052 719 2977	.141 789 2812
8	1.020 175 8777	8.070 351 0959	.123 910 3464
9	1.022 726 3174	9.090 526 9737	.110 004 6238
10	1.025 283 1332	10.113 253 2911	.098 880 1498
11	1.027 846 3411	11.138 536 4243	.089 778 4019
12	1.030 415 9569	12.166 382 7654	.082 193 6988
13	1.032 991 9968	13.196 798 7223	.075 775 9530
14	1.035 574 4768	14.229 790 7191	.070 275 1024
15	1.038 163 4130	15.265 365 1959	.065 507 7679
16	1.040 758 8215	16.303 528 6089	.061 336 4152
17	1.043 360 7186	17.344 287 4304	.057 655 8711
18	1.045 969 1204	18.387 648 1490	.054 384 3341
19	1.048 584 0432	19.433 617 2694	.051 457 2242
20	1.051 205 5033	20.482 201 3126	.048 822 8772
21	1.053 833 5170	21.533 406 8158	.046 439 4700
22	1.056 468 1008	22.587 240 3329	.044 272 7835
23	1.059 109 2711	23.643 708 4337	.042 294 5496
24	1.061 757 0443	24.702 817 7048	.040 481 2120
25	1.064 411 4369	25.764 574 7491	.038 812 9829
26	1.067 072 4655	26.828 986 1859	.037 273 1192
27	1.069 740 1466	27.896 058 6514	.035 847 3580
28	1.072 414 4970	28.965 798 7980	.034 523 4739
29	1.075 095 5332	30.038 213 2950	.033 290 9281
30	1.077 783 2721	31.113 308 8283	.032 140 5867
31	1.080 477 7303	32.191 092 1003	.031 064 4944
32	1.083 178 9246	33.271 569 8306	.030 055 6903
33	1.085 886 8719	34.354 748 7551	.029 108 0574
34	1.088 601 5891	35.440 635 6270	.028 216 1982
35	1.091 323 0930	36.529 237 2161	.027 375 3321
36	1.094 051 4008	37.620 560 3091	.026 581 2096
37	1.096 786 5293	38.714 611 7099	.025 830 0408
38	1.099 528 4956	39.811 398 2392	.025 118 4345
39	1.102 277 3168	40.910 926 7348	.024 443 3475
40	1.105 033 0101	42.013 204 0516	.023 802 0409
41	1.107 795 5927	43.118 237 0618	.023 192 0428
42	1.110 565 0816	44.226 032 6544	.022 611 1170
43	1.113 341 4943	45.336 597 7360	.022 057 2352
44	1.116 124 8481	46.449 939 2304	.021 528 5535
45	1.118 915 1602	47.566 064 0785	.021 023 3918
46	1.121 712 4481	48.684 979 2387	.020 540 2162
47	1.124 516 7292	49.806 691 6868	.020 077 6234
48	1.127 328 0210	50.931 208 4160	.019 634 3270
49	1.130 146 3411	52.058 536 4370	.019 209 1455
50	1.132 971 7069	53.188 682 7781	.018 800 9920
51	1.135 804 1362	54.321 654 4851	.018 408 8649
52	1.138 643 6466	55.457 458 6213	.018 031 8396
53	1.141 490 2557	56.596 102 2678	.017 669 0613
54	1.144 343 9813	57.737 592 5235	.017 319 7384
55	1.147 204 8413	58.881 936 5048	.016 983 1371
56	1.150 072 8534	60.029 141 3461	.016 658 5758
57	1.152 948 0355	61.179 214 1994	.016 345 4208
58	1.155 830 4056	62.332 162 2349	.016 043 0822
59	1.158 719 9816	63.487 992 6405	.015 751 0099
60	1.161 616 7816	64.646 712 6221	.015 468 6907
n	$s=(1+i)^n$	$s_{\overline{n}\|}=\frac{(1+i)^n-1}{i}$	$\frac{1}{s_{\overline{n}\|}}=\frac{i}{(1+i)^n-1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today. Periodic payment necessary to pay off a loan of $1.*	PERIODS
.997 506 2344	.997 506 2344	1.002 500 0000	1
.995 018 6877	1.992 524 9221	.501 875 7803	2
.992 537 3443	2.985 062 2664	.335 001 3872	3
.990 062 1889	3.975 124 4553	.251 564 4507	4
.987 593 2058	4.962 717 6612	.201 502 4969	5
.985 130 3799	5.947 848 0410	.168 128 0344	6
.982 673 6957	6.930 521 7367	.144 289 2812	7
.980 223 1378	7.910 744 8745	.126 410 3464	8
.977 778 6911	8.888 523 5656	.112 504 6238	9
.975 340 3402	9.863 863 9058	.101 380 1498	10
.972 908 0701	10.836 771 9759	.092 278 4019	11
.970 481 8654	11.807 253 8413	.084 693 6988	12
.968 061 7111	12.775 315 5524	.078 275 9530	13
.965 647 5921	13.740 963 1446	.072 775 1024	14
.963 239 4934	14.704 202 6380	.068 007 7679	15
.960 837 3999	15.665 040 0379	.063 836 4152	16
.958 441 2967	16.623 481 3345	.060 155 8711	17
.956 051 1687	17.579 532 5033	.056 884 3341	18
.953 667 0012	18.533 199 5045	.053 957 2242	19
.951 288 7793	19.484 488 2838	.051 322 8772	20
.948 916 4881	20.433 404 7719	.048 939 4700	21
.946 550 1128	21.379 954 8847	.046 772 7835	22
.944 189 6387	22.324 144 5234	.044 794 5496	23
.941 835 0511	23.265 979 5744	.042 981 2120	24
.939 486 3352	24.205 465 9096	.041 312 9829	25
.937 143 4765	25.142 609 3862	.039 773 1192	26
.934 806 4604	26.077 415 8466	.038 347 3580	27
.932 475 2722	27.009 891 1188	.037 023 4739	28
.930 149 8975	27.940 041 0162	.035 790 9281	29
.927 830 3217	28.867 871 3379	.034 640 5867	30
.925 516 5303	29.793 387 8682	.033 564 4944	31
.923 208 5091	30.716 596 3773	.032 555 6903	32
.920 906 2434	31.637 502 6207	.031 608 0574	33
.918 609 7192	32.556 112 3399	.030 716 1982	34
.916 318 9218	33.472 431 2617	.029 875 3321	35
.914 033 8373	34.386 465 0990	.029 081 2096	36
.911 754 4511	35.298 219 5501	.028 330 0408	37
.909 480 7493	36.207 700 2993	.027 618 4345	38
.907 212 7175	37.114 913 0168	.026 943 3475	39
.904 950 3416	38.019 863 3584	.026 302 0409	40
.902 693 6076	38.922 556 9660	.025 692 0428	41
.900 442 5013	39.822 999 4673	.025 111 1170	42
.898 197 0088	40.721 196 4761	.024 557 2352	43
.895 957 1160	41.617 153 5921	.024 028 5535	44
.893 722 8090	42.510 876 4011	.023 523 3918	45
.891 494 0738	43.402 370 4750	.023 040 2162	46
.889 270 8966	44.291 641 3715	.022 577 6234	47
.887 053 2634	45.178 694 6349	.022 134 3270	48
.884 841 1605	46.063 535 7955	.021 709 1455	49
.882 634 5741	46.946 170 3695	.021 300 9920	50
.880 433 4904	47.826 603 8599	.020 908 8649	51
.878 237 8956	48.704 841 7555	.020 531 8396	52
.876 047 7762	49.580 889 5317	.020 169 0613	53
.873 863 1184	50.454 752 6500	.019 819 7384	54
.871 683 9086	51.326 436 5586	.019 483 1371	55
.869 510 1333	52.195 946 6919	.019 158 5758	56
.867 341 7788	53.063 288 4707	.018 845 4208	57
.865 178 8317	53.928 467 3025	.018 543 0822	58
.863 021 2785	54.791 488 5810	.018 251 0099	59
.860 869 1058	55.652 357 6868	.017 968 6907	60
$v^n=\frac{1}{(1+i)^n}$	$a_{\overline{n}\rceil}=\frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rceil}}=\frac{i}{1-v^n}$	n

RATE

$\frac{1}{4}$%

.0025

per period

ANNUALLY
If compounded *annually* nominal annual rate is

$\frac{1}{4}$%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

$\frac{1}{2}$%

QUARTERLY
If compounded *quarterly* nominal annual rate is

1%

MONTHLY
If compounded *monthly* nominal annual rate is

3%

$i = .0025$
$j_{(2)} = .005$
$j_{(4)} = .01$
$j_{(12)} = .03$

RATE

1/4%

.0025

per period

ANNUALLY
If compounded *annually* nominal annual rate is **1/4%**

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is **1/2%**

QUARTERLY
If compounded *quarterly* nominal annual rate is **1%**

MONTHLY
If compounded *monthly* nominal annual rate is **3%**

$i = .0025$
$j_{(2)} = .005$
$j_{(4)} = .01$
$j_{(12)} = .03$

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
61	1.164 520 8235	65.808 329 4037	.015 195 6448
62	1.167 432 1256	66.972 850 2272	.014 931 4237
63	1.170 350 7059	68.140 282 3527	.014 675 6069
64	1.173 276 5826	69.310 633 0586	.014 427 8007
65	1.176 209 7741	70.483 909 6413	.014 187 6352
66	1.179 150 2985	71.660 119 4154	.013 954 7632
67	1.182 098 1743	72.839 269 7139	.013 728 8581
68	1.185 053 4197	74.021 367 8882	.013 509 6125
69	1.188 016 0533	75.206 421 3079	.013 296 7369
70	1.190 986 0934	76.394 437 3612	.013 089 9583
71	1.193 963 5586	77.585 423 4546	.012 889 0190
72	1.196 948 4675	78.779 387 0132	.012 693 6758
73	1.199 940 8387	79.976 335 4808	.012 503 6987
74	1.202 940 6908	81.176 276 3195	.012 318 8701
75	1.205 948 0425	82.379 217 0103	.012 138 9840
76	1.208 962 9126	83.585 165 0528	.011 963 8455
77	1.211 985 3199	84.794 127 9654	.011 793 2695
78	1.215 015 2832	86.006 113 2853	.011 627 0805
79	1.218 052 8214	87.221 128 5685	.011 465 1119
80	1.221 097 9535	88.439 181 3900	.011 307 2055
81	1.224 150 6984	89.660 279 3434	.011 153 2108
82	1.227 211 0751	90.884 430 0418	.011 002 9848
83	1.230 279 1028	92.111 641 1169	.010 856 3911
84	1.233 354 8005	93.341 920 2197	.010 713 3001
85	1.236 438 1876	94.575 275 0202	.010 573 5881
86	1.239 529 2830	95.811 713 2078	.010 437 1372
87	1.242 628 1062	97.051 242 4908	.010 303 8351
88	1.245 734 6765	98.293 870 5970	.010 173 5743
89	1.248 849 0132	99.539 605 2735	.010 046 2524
90	1.251 971 1357	100.788 454 2867	.009 921 7714
91	1.255 101 0636	102.040 425 4224	.009 800 0375
92	1.258 238 8162	103.295 526 4860	.009 680 9614
93	1.261 384 4133	104.553 765 3022	.009 564 4571
94	1.264 537 8743	105.815 149 7155	.009 450 4426
95	1.267 699 2190	107.079 687 5898	.009 338 8393
96	1.270 868 4670	108.347 386 8087	.009 229 5719
97	1.274 045 6382	109.618 255 2757	.009 122 5681
98	1.277 230 7523	110.892 300 9139	.009 017 7586
99	1.280 423 8292	112.169 531 6662	.008 915 0769
100	1.283 624 8887	113.449 955 4954	.008 814 4592
101	1.286 833 9510	114.733 580 3841	.008 715 8441
102	1.290 051 0358	116.020 414 3351	.008 619 1728
103	1.293 276 1634	117.310 465 3709	.008 524 3887
104	1.296 509 3538	118.603 741 5344	.008 431 4372
105	1.299 750 6272	119.900 250 8882	.008 340 2661
106	1.303 000 0038	121.200 001 5154	.008 250 8250
107	1.306 257 5038	122.503 001 5192	.008 163 0653
108	1.309 523 1476	123.809 259 0230	.008 076 9404
109	1.312 796 9554	125.118 782 1706	.007 992 4052
110	1.316 078 9478	126.431 579 1260	.007 909 4164
111	1.319 369 1452	127.747 658 0738	.007 827 9322
112	1.322 667 5680	129.067 027 2190	.007 747 9122
113	1.325 974 2370	130.380 694 7870	.007 669 3177
114	1.329 289 1726	131.715 669 0240	.007 592 1112
115	1.332 612 3955	133.044 958 1966	.007 516 2563
116	1.335 943 9265	134.377 570 5920	.007 441 7181
117	1.339 283 7863	135.713 514 5185	.007 368 4629
118	1.342 631 9958	137.052 798 3048	.007 296 4581
119	1.345 988 5758	138.395 430 3006	.007 225 6721
120	1.349 353 5472	139.741 418 8763	.007 156 0745
n	$s = (1+i)^n$	$s_{\overline{n}\rvert} = \frac{(1+i)^n - 1}{i}$	$\frac{1}{s_{\overline{n}\rvert}} = \frac{i}{(1+i)^n - 1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today. Periodic payment necessary to pay off a loan of $1.*	PERIODS
.858 722 3000	56.511 079 9868	.017 695 6448	61
.856 580 8479	57.367 660 8348	.017 431 4237	62
.854 444 7361	58.222 105 5708	.017 175 6069	63
.852 313 9512	59.074 419 5220	.016 927 8007	64
.850 188 4800	59.924 608 0020	.016 687 6352	65
.848 068 3092	60.772 676 3112	.016 454 7632	66
.845 953 4257	61.618 629 7369	.016 228 8581	67
.843 843 8161	62.462 473 5530	.016 009 6125	68
.841 739 4674	63.304 213 0205	.015 796 7369	69
.839 640 3665	64.143 853 3870	.015 589 9583	70
.837 546 5003	64.981 399 8873	.015 389 0190	71
.835 457 8556	65.816 857 7429	.015 193 6758	72
.833 374 4196	66.650 232 1625	.015 003 6987	73
.831 296 1791	67.481 528 3417	.014 818 8701	74
.829 223 1213	68.310 751 4630	.014 638 9840	75
.827 155 2333	69.137 906 6963	.014 463 8455	76
.825 092 5020	69.962 999 1983	.014 293 2695	77
.823 034 9147	70.786 034 1130	.014 127 0805	78
.820 982 4586	71.607 016 5716	.013 965 1119	79
.818 935 1208	72.425 951 6923	.013 807 2055	80
.816 892 8885	73.242 844 5809	.013 653 2108	81
.814 855 7492	74.057 700 3300	.013 502 9848	82
.812 823 6900	74.870 524 0200	.013 356 3911	83
.810 796 6982	75.681 320 7182	.013 213 3001	84
.808 774 7613	76.490 095 4795	.013 073 5881	85
.806 757 8666	77.296 853 3461	.012 937 1372	86
.804 746 0016	78.101 599 3478	.012 803 8351	87
.802 739 1537	78.904 338 5015	.012 673 5743	88
.800 737 3105	79.705 075 8120	.012 546 2524	89
.798 740 4593	80.503 816 2713	.012 421 7714	90
.796 748 5879	81.300 564 8592	.012 300 0375	91
.794 761 6836	82.095 326 5428	.012 180 9614	92
.792 779 7343	82.888 106 2771	.012 064 4571	93
.790 802 7275	83.678 909 0046	.011 950 4426	94
.788 830 6509	84.467 739 6555	.011 838 8393	95
.786 863 4921	85.254 603 1476	.011 729 5719	96
.784 901 2390	86.039 504 3866	.011 622 5681	97
.782 943 8793	86.822 448 2660	.011 517 7586	98
.780 991 4008	87.603 439 6668	.011 415 0769	99
.779 043 7914	88.382 483 4581	.011 314 4592	100
.777 101 0388	89.159 584 4969	.011 215 8441	101
.775 163 1309	89.934 747 6278	.011 119 1728	102
.773 230 0558	90.707 977 6836	.011 024 3887	103
.771 301 8013	91.479 279 4849	.010 931 4372	104
.769 378 3554	92.248 657 8403	.010 840 2661	105
.767 459 7061	93.016 117 5464	.010 750 8250	106
.765 545 8415	93.781 663 3880	.010 663 0653	107
.763 636 7497	94.545 300 1376	.010 576 9404	108
.761 732 4186	95.307 032 5562	.010 492 4052	109
.759 832 8365	96.066 865 3928	.010 409 4164	110
.757 937 9915	96.824 803 3843	.010 327 9322	111
.756 047 8719	97.580 851 2562	.010 247 9122	112
.754 162 4657	98.335 013 7218	.010 169 3177	113
.752 281 7613	99.087 295 4831	.010 092 1112	114
.750 405 7469	99.837 701 2301	.010 016 2563	115
.748 534 4109	100.586 235 6410	.009 941 7181	116
.746 667 7415	101.332 903 3825	.009 868 4629	117
.744 805 7272	102.077 709 1097	.009 796 4581	118
.742 948 3563	102.820 657 4661	.009 725 6721	119
.741 095 6173	103.561 753 0834	.009 656 0745	120
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\rvert} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rvert}} = \frac{i}{1-v^n}$	n

RATE

1/4%

.0025

per period

ANNUALLY
If compounded *annually* nominal annual rate is
1/4%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is
1/2%

QUARTERLY
If compounded *quarterly* nominal annual rate is
1%

MONTHLY
If compounded *monthly* nominal annual rate is
3%

$i = .0025$
$j_{(2)} = .005$
$j_{(4)} = .01$
$j_{(12)} = .03$

RATE

1/3%

00333333

per period

ANNUALLY
If compounded *annually* nominal annual rate is

1/3%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

2/3%

QUARTERLY
If compounded *quarterly* nominal annual rate is

1 1/3%

MONTHLY
If compounded *monthly* nominal annual rate is

4%

$i = .00333333$
$j_{(2)} = .00666666$
$j_{(4)} = .01333333$
$j_{(12)} = .04$

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.003 333 3333	1.000 000 0000	1.000 000 0000
2	1.006 677 7778	2.003 333 3333	.499 168 0532
3	1.010 033 3704	3.010 011 1111	.332 224 6872
4	1.013 400 1483	4.020 044 4815	.248 753 4664
5	1.016 778 1488	5.033 444 6298	.198 671 1037
6	1.020 167 4093	6.050 222 7785	.165 283 1700
7	1.023 567 9673	7.070 390 1878	.141 434 9100
8	1.026 979 8605	8.093 958 1551	.123 548 9461
9	1.030 403 1267	9.120 938 0156	.109 637 8463
10	1.033 837 8038	10.151 341 1423	.098 509 1513
11	1.037 283 9298	11.185 178 9461	.089 404 0234
12	1.040 741 5429	12.222 462 8759	.081 816 5709
13	1.044 210 6814	13.263 204 4189	.075 396 5609
14	1.047 691 3837	14.307 415 1003	.069 893 8273
15	1.051 183 6883	15.355 106 4839	.065 124 9147
16	1.054 687 6339	16.406 290 1722	.060 952 2317
17	1.058 203 2594	17.460 977 8061	.057 270 5613
18	1.061 730 6036	18.519 181 0655	.053 998 0681
19	1.065 269 7056	19.580 911 6690	.051 070 1451
20	1.068 820 6046	20.646 181 3746	.048 435 1068
21	1.072 383 3399	21.715 001 9792	.046 051 1125
22	1.075 957 9511	22.787 385 3191	.043 883 9290
23	1.079 544 4776	23.863 343 2702	.041 905 2766
24	1.083 142 9592	24.942 887 7477	.040 091 5888
25	1.086 753 4357	26.026 030 7069	.038 423 0700
26	1.090 375 9471	27.112 784 1426	.036 882 9698
27	1.094 010 5336	28.203 160 0897	.035 457 0196
28	1.097 657 2354	29.297 170 6233	.034 132 9889
29	1.101 316 0929	30.394 827 8588	.032 900 3344
30	1.104 987 1465	31.496 143 9516	.031 749 9184
31	1.108 670 4370	32.601 131 0981	.030 673 7824
32	1.112 366 0051	33.709 801 5351	.029 664 9625
33	1.116 073 8918	34.822 167 5402	.028 717 3393
34	1.119 794 1381	35.938 241 4320	.027 825 5129
35	1.123 526 7852	37.058 035 5701	.026 984 7007
36	1.127 271 8745	38.181 562 3554	.026 190 6517
37	1.131 029 4474	39.308 834 2299	.025 439 5741
38	1.134 799 5456	40.439 863 6773	.024 728 0754
39	1.138 582 2107	41.574 663 2229	.024 053 1113
40	1.142 377 4848	42.713 245 4337	.023 411 9414
41	1.146 185 4097	43.855 622 9184	.022 802 0932
42	1.150 006 0278	45.001 808 3282	.022 221 3293
43	1.153 839 3812	46.151 814 3559	.021 667 6205
44	1.157 685 5125	47.305 653 7371	.021 139 1223
45	1.161 544 4642	48.463 339 2496	.020 634 1539
46	1.165 416 2790	49.624 883 7137	.020 151 1807
47	1.169 301 0000	50.790 299 9928	.019 688 7988
48	1.173 198 6700	51.959 600 9928	.019 245 7213
49	1.177 109 3322	53.132 799 6627	.018 820 7662
50	1.181 033 0300	54.309 908 9949	.018 412 8462
51	1.184 969 8067	55.490 942 0249	.018 020 9592
52	1.188 919 7061	56.675 911 8317	.017 644 1802
53	1.192 882 7718	57.864 831 5378	.017 281 6540
54	1.196 859 0477	59.057 714 3096	.016 932 5889
55	1.200 848 5779	60.254 573 3573	.016 596 2506
56	1.204 851 4065	61.455 421 9351	.016 271 9573
57	1.208 867 5778	62.660 273 3416	.015 959 0750
58	1.212 897 1364	63.869 140 9194	.015 657 0135
59	1.216 940 1269	65.082 038 0558	.015 365 2226
60	1.220 996 5939	66.298 978 1826	.015 083 1887
n	$s=(1+i)^n$	$s_{\overline{n}\rceil} = \frac{(1+i)^n-1}{i}$	$\frac{1}{s_{\overline{n}\rceil}} = \frac{i}{(1+i)^n-1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today. Periodic payment necessary to pay off a loan of $1.*	PERIODS
.996 677 7409	.996 677 7409	1.003 333 3333	1
.993 366 5191	1.990 044 2600	.502 501 3866	2
.990 066 2981	2.980 110 5581	.335 558 0206	3
.986 777 0413	3.966 887 5995	.252 086 7998	4
.983 498 7123	4.950 386 3118	.202 004 4370	5
.980 231 2747	5.930 617 5865	.168 616 5033	6
.976 974 6924	6.907 592 2789	.144 768 2434	7
.973 728 9293	7.881 321 2082	.126 882 2795	8
.970 493 9495	8.851 815 1577	.112 971 1796	9
.967 269 7171	9.819 084 8747	.101 842 4846	10
.964 056 1964	10.783 141 0712	.092 737 3567	11
.960 853 3519	11.743 994 4231	.085 149 9042	12
.957 661 1481	12.701 655 5712	.078 729 8943	13
.954 479 5496	13.656 135 1208	.073 227 1606	14
.951 308 5212	14.607 443 6420	.068 458 2480	15
.948 148 0278	15.555 591 6698	.064 285 5650	16
.944 998 0343	16.500 589 7041	.060 603 8946	17
.941 858 5060	17.442 448 2100	.057 331 4014	18
.938 729 4079	18.381 177 6180	.054 403 4784	19
.935 610 7056	19.316 788 3236	.051 768 4401	20
.932 502 3644	20.249 290 6879	.049 384 4459	21
.929 404 3499	21.178 695 0378	.047 217 2624	22
.926 316 6278	22.105 011 6656	.045 238 6099	23
.923 239 1639	23.028 250 8295	.043 424 9222	24
.920 171 9242	23.948 422 7537	.041 756 4033	25
.917 114 8746	24.865 537 6282	.040 216 3032	26
.914 067 9813	25.779 605 6095	.038 790 3529	27
.911 031 2106	26.690 636 8201	.037 466 3222	28
.908 004 5288	27.598 641 3490	.036 233 6677	29
.904 987 9025	28.503 629 2515	.035 083 2517	30
.901 981 2982	29.405 610 5496	.034 007 1157	31
.898 984 6826	30.304 595 2322	.032 998 2959	32
.895 998 0225	31.200 593 2547	.032 050 6726	33
.893 021 2849	32.093 614 5395	.031 158 8462	34
.890 054 4367	32.983 668 9763	.030 318 0341	35
.887 097 4453	33.870 766 4215	.029 523 9850	36
.884 150 2777	34.754 916 6992	.028 772 9074	37
.881 212 9013	35.636 129 6005	.028 061 4088	38
.878 285 2837	36.514 414 8843	.027 386 4446	39
.875 367 3924	37.389 782 2767	.026 745 2748	40
.872 459 1951	38.262 241 4718	.026 135 4265	41
.869 560 6596	39.131 802 1313	.025 554 6626	42
.866 671 7537	39.998 473 8850	.025 000 9539	43
.863 792 4456	40.862 266 3306	.024 472 4556	44
.860 922 7032	41.723 189 0338	.023 967 4872	45
.858 062 4949	42.581 251 5287	.023 484 5141	46
.855 211 7889	43.436 463 3177	.023 022 1322	47
.852 370 5538	44.288 833 8714	.022 579 0546	48
.849 538 7579	45.138 372 6293	.022 154 0995	49
.846 716 3700	45.985 088 9993	.021 746 1795	50
.843 903 3588	46.828 992 3582	.021 354 2925	51
.841 099 6932	47.670 092 0513	.020 977 5135	52
.838 305 3420	48.508 397 3933	.020 614 9874	53
.835 520 2744	49.343 917 6678	.020 265 9223	54
.832 744 4596	50.176 662 1274	.019 929 5839	55
.829 977 8667	51.006 639 9940	.019 605 2906	56
.827 220 4651	51.833 860 4592	.019 292 4083	57
.824 472 2244	52.658 332 6836	.018 990 3468	58
.821 733 1140	53.480 065 7976	.018 698 5559	59
.819 003 1037	54.299 068 9012	.018 416 5221	60
$v^n=\frac{1}{(1+i)^n}$	$a_{\overline{n}\rvert}=\frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rvert}}=\frac{i}{1-v^n}$	n

RATE

1/3%

.00333333̇

per period

ANNUALLY
If compounded *annually* nominal annual rate is

1/3%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

2/3%

QUARTERLY
If compounded *quarterly* nominal annual rate is

1 1/3%

MONTHLY
If compounded *monthly* nominal annual rate is

4%

i = .00333333
$j_{(2)}$ = .00666666
$j_{(4)}$ = .01333333
$j_{(12)}$ = .04

RATE

$\frac{1}{3}\%$

00333333

per period

ANNUALLY
If compounded *annually* nominal annual rate is

$\frac{1}{3}\%$

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

$\frac{2}{3}\%$

QUARTERLY
If compounded *quarterly* nominal annual rate is

$1\frac{1}{3}\%$

MONTHLY
If compounded *monthly* nominal annual rate is

4%

$i = .00333333$
$j_{(2)} = .00666666$
$j_{(4)} = .01333333$
$j_{(12)} = .04$

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
61	1.225 066 5826	67.519 974 7766	.014 810 4321
62	1.229 150 1379	68.745 041 3592	.014 546 5037
63	1.233 247 3050	69.974 191 4970	.014 290 9833
64	1.237 358 1293	71.207 438 8020	.014 043 4766
65	1.241 482 6564	72.444 796 9314	.013 803 6138
66	1.245 620 9320	73.686 279 5878	.013 571 0475
67	1.249 773 0017	74.931 900 5198	.013 345 4509
68	1.253 938 9117	76.181 673 5215	.013 126 5166
69	1.258 118 7081	77.435 612 4332	.012 913 9548
70	1.262 312 4371	78.693 731 1413	.012 707 4925
71	1.266 520 1453	79.956 043 5785	.012 506 8720
72	1.270 741 8791	81.222 563 7237	.012 311 8497
73	1.274 977 6853	82.493 305 6028	.012 122 1958
74	1.279 227 6110	83.768 283 2882	.011 937 6924
75	1.283 491 7030	85.047 510 8991	.011 758 1337
76	1.287 770 0087	86.331 002 6021	.011 583 3243
77	1.292 062 5754	87.618 772 6108	.011 413 0793
78	1.296 369 4506	88.910 835 1862	.011 247 2231
79	1.300 690 6821	90.207 204 6368	.011 085 5891
80	1.305 026 3177	91.507 895 3189	.010 928 0188
81	1.309 376 4055	92.812 921 6366	.010 774 3618
82	1.313 740 9935	94.122 298 0421	.010 624 4750
83	1.318 120 1301	95.436 039 0356	.010 478 2220
84	1.322 513 8639	96.754 159 1657	.010 335 4730
85	1.326 922 2434	98.076 673 0296	.010 196 1044
86	1.331 345 3176	99.403 595 2730	.010 059 9983
87	1.335 783 1353	100.734 940 5906	.009 927 0421
88	1.340 235 7458	102.070 723 7259	.009 797 1285
89	1.344 703 1982	103.410 959 4716	.009 670 1549
90	1.349 185 5422	104.755 662 6699	.009 546 0233
91	1.353 682 8274	106.104 848 2121	.009 424 6400
92	1.358 195 1035	107.458 531 0395	.009 305 9154
93	1.362 722 4205	108.816 726 1429	.009 189 7637
94	1.367 264 8285	110.179 448 5634	.009 076 1028
95	1.371 822 3780	111.546 713 3920	.008 964 8540
96	1.376 395 1192	112.918 535 7699	.008 855 9420
97	1.380 983 1030	114.294 930 8892	.008 749 2944
98	1.385 586 3800	115.675 913 9921	.008 644 8420
99	1.390 205 0012	117.061 500 3721	.008 542 5182
100	1.394 839 0179	118.451 705 3733	.008 442 2592
101	1.399 488 4813	119.846 544 3913	.008 344 0036
102	1.404 153 4429	121.246 032 8726	.008 247 6925
103	1.408 833 9544	122.650 186 3155	.008 153 2693
104	1.413 530 0676	124.059 020 2699	.008 060 6795
105	1.418 241 8345	125.472 550 3374	.007 969 8707
106	1.422 969 3072	126.890 792 1719	.007 880 7925
107	1.427 712 5383	128.313 761 4791	.007 793 3963
108	1.432 471 5801	129.741 474 0174	.007 707 6356
109	1.437 246 4853	131.173 945 5974	.007 623 4651
110	1.442 037 3069	132.611 192 0828	.007 540 8416
111	1.446 844 0980	134.053 229 3897	.007 459 7233
112	1.451 666 9116	135.500 073 4877	.007 380 0698
113	1.456 505 8013	136.951 740 3993	.007 301 8422
114	1.461 360 8207	138.408 246 2006	.007 225 0030
115	1.466 232 0234	139.869 607 0213	.007 149 5160
116	1.471 119 4635	141.335 839 0447	.007 075 3463
117	1.476 023 1950	142.806 958 5082	.007 002 4599
118	1.480 943 2723	144.282 981 7032	.006 930 8243
119	1.485 879 7499	145.763 924 9756	.006 860 4080
120	1.490 832 6824	147.249 804 7255	.006 791 1805
n	$s=(1+i)^n$	$s_{\overline{n}\rvert}=\frac{(1+i)^n-1}{i}$	$\frac{1}{s_{\overline{n}\rvert}}=\frac{i}{(1+i)^n-1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today. Periodic payment necessary to pay off a loan of $1.*	PERIODS
.816 282 1631	55.115 351 0644	.018 143 7654	61
.813 570 2622	55.928 921 3266	.017 879 8371	62
.810 867 3710	56.739 788 6976	.017 624 3166	63
.808 173 4595	57.547 962 1571	.017 376 8099	64
.805 488 4978	58.353 450 6549	.017 136 9472	65
.802 812 4563	59.156 263 1112	.016 904 3808	66
.800 145 3053	59.956 408 4165	.016 678 7842	67
.797 487 0152	60.753 895 4317	.016 459 8499	68
.794 837 5567	61.548 732 9884	.016 247 2881	69
.792 196 9004	62.340 929 8888	.016 040 8259	70
.789 565 0170	63.130 494 9058	.015 840 2053	71
.786 941 8774	63.917 436 7831	.015 645 1831	72
.784 327 4525	64.701 764 2357	.015 455 5291	73
.781 721 7135	65.483 485 9492	.015 271 0257	74
.779 124 6314	66.262 610 5806	.015 091 4670	75
.776 536 1775	67.039 146 7581	.014 916 6576	76
.773 956 3231	67.813 103 0811	.014 746 4126	77
.771 385 0396	68.584 488 1207	.014 580 5564	78
.768 822 2986	69.353 310 4193	.014 418 9224	79
.766 268 0717	70.119 578 4910	.014 261 3521	80
.763 722 3306	70.883 300 8216	.014 107 6952	81
.761 185 0471	71.644 485 8687	.013 957 8083	82
.758 656 1931	72.403 142 0619	.013 811 5553	83
.756 135 7407	73.159 277 8025	.013 668 8063	84
.753 623 6618	73.912 901 4643	.013 529 4378	85
.751 119 9287	74.664 021 3930	.013 393 3316	86
.748 624 5136	75.412 645 9066	.013 260 3755	87
.746 137 3890	76.158 783 2956	.013 130 4619	88
.743 658 5273	76.902 441 8229	.013 003 4883	89
.741 187 9009	77.643 629 7238	.012 879 3567	90
.738 725 4826	78.382 355 2065	.012 757 9734	91
.736 271 2452	79.118 626 4516	.012 639 2487	92
.733 825 1613	79.852 451 6129	.012 523 0970	93
.731 387 2039	80.583 838 8169	.012 409 4361	94
.728 957 3461	81.312 796 1630	.012 298 1873	95
.726 535 5609	82.039 331 7239	.012 189 2753	96
.724 121 8215	82.763 453 5454	.012 082 6277	97
.721 716 1012	83.485 169 6466	.011 978 1753	98
.719 318 3733	84.204 488 0199	.011 875 8516	99
.716 928 6112	84.921 416 6311	.011 775 5925	100
.714 546 7886	85.635 963 4197	.011 677 3370	101
.712 172 8790	86.348 136 2987	.011 581 0259	102
.709 806 8562	87.057 943 1549	.011 486 6026	103
.707 448 6938	87.765 391 8487	.011 394 0128	104
.705 098 3660	88.470 490 2146	.011 303 2040	105
.702 755 8465	89.173 246 0611	.011 214 1258	106
.700 421 1094	89.873 667 1705	.011 126 7297	107
.698 094 1290	90.571 761 2995	.011 040 9689	108
.695 774 8794	91.267 536 1789	.010 956 7985	109
.693 463 3350	91.960 999 5139	.010 874 1750	110
.691 159 4701	92.652 158 9840	.010 793 0566	111
.688 863 2592	93.341 022 2431	.010 713 4031	112
.686 574 6769	94.027 596 9201	.010 635 1756	113
.684 293 6979	94.711 890 6180	.010 558 3364	114
.682 020 2970	95.393 910 9150	.010 482 8494	115
.679 754 4488	96.073 665 3638	.010 408 6796	116
.677 496 1284	96.751 161 4921	.010 335 7932	117
.675 245 3107	97.426 406 8028	.010 264 1577	118
.673 001 9708	98.099 408 7735	.010 193 7414	119
.670 766 0838	98.770 174 8573	.010 124 5138	120
$v^n = \dfrac{1}{(1+i)^n}$	$a_{\overline{n}\rceil} = \dfrac{1-v^n}{i}$	$\dfrac{1}{a_{\overline{n}\rceil}} = \dfrac{i}{1-v^n}$	n

RATE

$1/3$%

00333333

per period

ANNUALLY
If compounded *annually* nominal annual rate is

$1/3$%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

$2/3$%

QUARTERLY
If compounded *quarterly* nominal annual rate is

$1\,1/3$%

MONTHLY
If compounded *monthly* nominal annual rate is

4%

$i = .0033333\dot{3}$
$j_{(2)} = .0066666\dot{6}$
$j_{(4)} = .0133333\dot{3}$
$j_{(12)} = .04$

RATE

3/8%

00375

per period

ANNUALLY
If compounded *annually* nominal annual rate is

3/8%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

3/4%

QUARTERLY
If compounded *quarterly* nominal annual rate is

1½%

MONTHLY
If compounded *monthly* nominal annual rate is

4½%

$i = .00375$
$j_{(2)} = .0075$
$j_{(4)} = .015$
$j_{(12)} = .045$

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.003 750 0000	1.000 000 0000	1.000 000 0000
2	1.007 514 0625	2.003 750 0000	.499 064 2545
3	1.011 292 2402	3.011 264 0625	.332 086 4525
4	1.015 084 5861	4.022 556 3027	.248 598 1363
5	1.018 891 1533	5.037 640 8889	.198 505 6144
6	1.022 711 9952	6.056 532 0422	.165 110 9898
7	1.026 547 1651	7.079 244 0374	.141 258 0206
8	1.030 396 7170	8.105 791 2025	.123 368 5861
9	1.034 260 7047	9.136 187 9195	.109 454 8414
10	1.038 139 1823	10.170 448 6242	.098 324 0796
11	1.042 032 2043	11.208 587 8065	.089 217 3053
12	1.045 939 8250	12.250 620 0108	.081 628 5216
13	1.049 862 0994	13.296 559 8359	.075 207 4230
14	1.053 799 0823	14.346 421 9352	.069 703 7913
15	1.057 750 8288	15.400 221 0175	.064 934 1330
16	1.061 717 3944	16.457 971 8463	.060 760 8282
17	1.065 698 8347	17.519 689 2407	.057 078 6380
18	1.069 695 2053	18.585 388 0754	.053 805 7099
19	1.073 706 5623	19.655 083 2807	.050 877 4237
20	1.077 732 9619	20.728 789 8430	.048 242 0830
21	1.081 774 4605	21.806 522 8049	.045 857 8385
22	1.085 831 1147	22.888 297 2654	.043 690 4497
23	1.089 902 9814	23.974 128 3802	.041 711 6311
24	1.093 990 1176	25.064 031 3616	.039 897 8116
25	1.098 092 5805	26.158 021 4792	.038 229 1910
26	1.102 210 4277	27.256 114 0597	.036 689 0158
27	1.106 343 7168	28.358 324 4875	.035 263 0142
28	1.110 492 5058	29.464 668 2043	.033 938 9534
29	1.114 656 8527	30.575 160 7101	.032 706 2876
30	1.118 836 8159	31.689 817 5627	.031 555 8775
31	1.123 032 4539	32.808 654 3786	.030 479 7627
32	1.127 243 8256	33.931 686 8325	.029 470 9781
33	1.131 470 9900	35.058 930 6581	.028 523 4028
34	1.135 714 0062	36.190 401 6481	.027 631 6359
35	1.139 972 9337	37.326 115 6543	.026 790 8938
36	1.144 247 8322	38.466 088 5880	.025 996 9245
37	1.148 538 7616	39.610 336 4202	.025 245 9355
38	1.152 845 7819	40.758 875 1818	.024 534 5338
39	1.157 168 9536	41.911 720 9637	.023 859 6740
40	1.161 508 3372	43.068 889 9173	.023 218 6156
41	1.165 863 9935	44.230 398 2545	.022 608 8853
42	1.170 235 9834	45.396 262 2479	.022 028 2453
43	1.174 624 3684	46.566 498 2314	.021 474 6661
44	1.179 029 2097	47.741 122 5997	.020 946 3026
45	1.183 450 5693	48.920.151 8095	.020 441 4738
46	1.187 888 5089	50.103 602 3788	.019 958 6447
47	1.192 343 0908	51.291 490 8877	.019 496 4113
48	1.196 814 3774	52.483 833 9785	.019 053 4861
49	1.201 302 4313	53.680 648 3559	.018 628 6871
50	1.205 807 3155	54.881 950 7873	.018 220 9267
51	1.210 329 0929	56.087 758 1027	.017 829 2026
52	1.214 867 8270	57.298 087 1956	.017 452 5896
53	1.219 423 5813	58.512 955 0226	.017 090 2324
54	1.223 996 4198	59.732 378 6039	.016 741 3390
55	1.228 586 4063	60.956 375 0237	.016 405 1750
56	1.233 193 6054	62.184 961 4300	.016 081 0585
57	1.237 818 0814	63.418 155 0354	.015 768 3553
58	1.242 459 8992	64.655 973 1168	.015 466 4751
59	1.247 119 1238	65.898 433 0160	.015 174 8677
60	1.251 795 8205	67.145 552 1398	.014 893 0192
n	$s = (1+i)^n$	$s_{\overline{n}\mid} = \frac{(1+i)^n - 1}{i}$	$\frac{1}{s_{\overline{n}\mid}} = \frac{i}{(1+i)^n - 1}$

PRESENT WORTH OF 1 *What \$1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What \$1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth \$1 today.* *Periodic payment necessary to pay off a loan of \$1.*	PERIODS
.996 264 0100	.996 264 0100	1.003 750 0000	1
.992 541 9775	1.988 805 9875	.502 814 2545	2
.988 833 8506	2.977 639 8381	.335 836 4525	3
.985 139 5772	3.962 779 4153	.252 348 1363	4
.981 459 1055	4.944 238 5209	.202 255 6144	5
.977 792 3841	5.922 030 9050	.168 860 9898	6
.974 139 3615	6.896 170 2665	.145 008 0206	7
.970 499 9866	7.866 670 2530	.127 118 5861	8
.966 874 2083	8.833 544 4613	.113 204 8414	9
.963 261 9759	9.796 806 4371	.102 074 0796	10
.959 663 2387	10.756 469 6759	.092 967 3053	11
.956 077 9464	11.712 547 6223	.085 378 5216	12
.952 506 0487	12.665 053 6710	.078 957 4230	13
.948 947 4956	13.614 001 1666	.073 453 7913	14
.945 402 2372	14.559 403 4039	.068 684 1330	15
.941 870 2239	15.501 273 6278	.064 510 8282	16
.938 351 4061	16.439 625 0339	.060 828 6380	17
.934 845 7346	17.374 470 7685	.057 555 7099	18
.931 353 1603	18.305 823 9288	.054 627 4237	19
.927 873 6341	19.233 697 5629	.051 992 0830	20
.924 407 1075	20.158 104 6704	.049 607 8385	21
.920 953 5317	21.079 058 2021	.047 440 4497	22
.917 512 8585	21.996 571 0607	.045 461 6311	23
.914 085 0396	22.910 656 1003	.043 647 8116	24
.910 670 0270	23.821 326 1273	.041 979 1910	25
.907 267 7729	24.728 593 9002	.040 439 0158	26
.903 878 2295	25.632 472 1297	.039 013 0142	27
.900 501 3495	26.532 973 4792	.037 688 9534	28
.897 137 0854	27.430 110 5645	.036 456 2876	29
.893 785 3902	28.323 895 9547	.035 305 8775	30
.890 446 2169	29.214 342 1716	.034 229 7627	31
.887 119 5187	30.101 461 6902	.033 220 9781	32
.883 805 2490	30.985 266 9392	.032 273 4028	33
.880 503 3614	31.865 770 3006	.031 381 6359	34
.877 213 8096	32.742 984 1102	.030 540 8938	35
.873 936 5475	33.616 920 6577	.029 746 9245	36
.870 671 5293	34.487 592 1870	.028 995 9355	37
.867 418 7091	35.355 010 8961	.028 284 5338	38
.864 178 0415	36.219 188 9376	.027 609 6740	39
.860 949 4809	37.080 138 4185	.026 968 6156	40
.857 732 9822	37.937 871 4008	.026 358 8853	41
.854 528 5004	38.792 399 9012	.025 778 2453	42
.851 335 9904	39.643 735 8916	.025 224 6661	43
.848 155 4076	40.491 891 2992	.024 696 3026	44
.844 986 7075	41.336 878 0067	.024 191 4738	45
.841 829 8456	42.178 707 8522	.023 708 6447	46
.838 684 7776	43.017 392 6299	.023 246 4113	47
.835 551 4597	43.852 944 0895	.022 803 4861	48
.832 429 8477	44.685 373 9373	.022 378 6871	49
.829 319 8981	45.514 693 8354	.021 970 9267	50
.826 221 5672	46.340 915 4026	.021 579 2026	51
.823 134 8117	47.164 050 2143	.021 202 5896	52
.820 059 5882	47.984 109 8026	.020 840 2324	53
.816 995 8538	48.801 105 6563	.020 491 3390	54
.813 943 5654	49.615 049 2218	.020 155 1750	55
.810 902 6804	50.425 951 9021	.019 831 0585	56
.807 873 1560	51.233 825 0582	.019 518 3553	57
.804 854 9500	52.038 680 0081	.019 216 4751	58
.801 848 0199	52.840 528 0280	.018 924 8677	59
.798 852 3237	53.639 380 3517	.018 643 0192	60
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\rceil} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rceil}} = \frac{i}{1-v^n}$	n

RATE

3/8%

.00375

per period

ANNUALLY

If compounded *annually* nominal annual rate is

3/8%

SEMIANNUALLY

If compounded *semiannually* nominal annual rate is

3/4%

QUARTERLY

If compounded *quarterly* nominal annual rate is

1½%

MONTHLY

If compounded *monthly* nominal annual rate is

4½%

$i = .00375$
$j_{(2)} = .0075$
$j_{(4)} = .015$
$j_{(12)} = .045$

RATE

3/8%

.00375

per period

ANNUALLY
If compounded *annually* nominal annual rate is

3/8%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

3/4%

QUARTERLY
If compounded *quarterly* nominal annual rate is

1½%

MONTHLY
If compounded *monthly* nominal annual rate is

4½%

$i = .00375$
$j_{(2)} = .0075$
$j_{(4)} = .015$
$j_{(12)} = .045$

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
61	1.256 490 0549	68.397 347 9603	.014 620 4499
62	1.261 201 8926	69.653 838 0152	.014 356 7107
63	1.265 931 3997	70.915 039 9077	.014 101 3811
64	1.270 678 6424	72.180 971 3074	.013 854 0668
65	1.275 443 6873	73.451 649 9498	.013 614 3981
66	1.280 226 6011	74.727 093 6371	.013 382 0272
67	1.285 027 4509	76.007 320 2382	.013 156 6275
68	1.289 846 3038	77.292 347 6891	.012 937 8914
69	1.294 683 2275	78.582 193 9929	.012 725 5291
70	1.299 538 2896	79.876 877 2204	.012 519 2676
71	1.304 411 5582	81.176 415 5100	.012 318 8489
72	1.309 303 1015	82.480 827 0682	.012 124 0297
73	1.314 212 9881	83.790 130 1697	.011 934 5799
74	1.319 141 2868	85.104 343 1578	.011 750 2816
75	1.324 088 0667	86.423 484 4446	.011 570 9290
76	1.329 053 3969	87.747 572 5113	.011 396 3267
77	1.334 037 3472	89.076 625 9082	.011 226 2896
78	1.339 039 9872	90.410 663 2554	.011 060 6422
79	1.344 061 3872	91.749 703 2426	.010 899 2178
80	1.349 101 6174	93.093 764 6298	.010 741 8580
81	1.354 160 7484	94.442 866 2471	.010 588 4122
82	1.359 238 8512	95.797 026 9955	.010 438 7373
83	1.364 335 9969	97.156 265 8468	.010 292 6969
84	1.369 452 2569	98.520 601 8437	.010 150 1613
85	1.374 587 7029	99.890 054 1006	.010 011 0067
86	1.379 742 4068	101.264 641 8035	.009 875 1152
87	1.384 916 4408	102.644 384 2103	.009 742 3742
88	1.390 109 8774	104.029 300 6510	.009 612 6764
89	1.395 322 7895	105.419 410 5285	.009 485 9191
90	1.400 555 2499	106.814 733 3180	.009 362 0044
91	1.405 807 3321	108.215 288 5679	.009 240 8385
92	1.411 079 1096	109.621 095 9000	.009 122 3317
93	1.416 370 6563	111.032 175 0097	.009 006 3984
94	1.421 682 0462	112.448 545 6659	.008 892 9563
95	1.427 013 3539	113.870 227 7122	.008 781 9268
96	1.432 364 6540	115.297 241 0661	.008 673 2344
97	1.437 736 0215	116.729 605 7201	.008 566 8070
98	1.443 127 5315	118.167 341 7416	.008 462 5751
99	1.448 539 2598	119.610 469 2731	.008 360 4722
100	1.453 971 2820	121.059 008 5329	.008 260 4344
101	1.459 423 6743	122.512 979 8149	.008 162 4004
102	1.464 896 5131	123.972 403 4892	.008 066 3113
103	1.470 389 8750	125.437 300 0023	.007 972 1104
104	1.475 903 8370	126.907 689 8773	.007 879 7432
105	1.481 438 4764	128.383 593 7143	.007 789 1573
106	1.486 993 8707	129.865 032 1907	.007 700 3023
107	1.492 570 0977	131.352 026 0614	.007 613 1296
108	1.498 167 2356	132.844 596 1592	.007 527 5926
109	1.503 785 3627	134.342 763 3948	.007 443 6462
110	1.509 424 5578	135.846 548 7575	.007 361 2470
111	1.515 084 8999	137.355 973 3153	.007 280 3532
112	1.520 766 4683	138.871 058 2153	.007 200 9245
113	1.526 469 3426	140.391 824 6836	.007 122 9219
114	1.532 193 6026	141.918 294 0262	.007 046 3079
115	1.537 939 3286	143.450 487 6287	.006 971 0464
116	1.543 706 6011	144.988 426 9574	.006 897 1022
117	1.549 495 5008	146.532 133 5584	.006 824 4417
118	1.555 306 1090	148.081 629 0593	.006 753 0321
119	1.561 138 5069	149.636 935 1683	.006 682 8420
120	1.566 992 7763	151.198 073 6751	.006 613 8409
n	$s=(1+i)^n$	$s_{\overline{n}\vert}=\dfrac{(1+i)^n-1}{i}$	$\dfrac{1}{s_{\overline{n}\vert}}=\dfrac{i}{(1+i)^n-1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today.* *Periodic payment necessary to pay off a loan of $1.*	PERIODS
.795 867 8194	54.435 248 1711	.018 370 4499	61
.792 894 4651	55.228 142 6362	.018 106 7107	62
.789 932 2193	56.018 074 8555	.017 851 3811	63
.786 981 0404	56.805 055 8959	.017 604 0668	64
.784 040 8871	57.589 096 7829	.017 364 3981	65
.781 111 7181	58.370 208 5011	.017 132 0272	66
.778 193 4925	59.148 401 9936	.016 906 6275	67
.775 286 1694	59.923 688 1630	.016 687 8914	68
.772 389 7080	60.696 077 8709	.016 475 5291	69
.769 504 0677	61.465 581 9387	.016 269 2676	70
.766 629 2082	62.232 211 1469	.016 068 8489	71
.763 765 0891	62.995 976 2360	.015 874 0297	72
.760 911 6704	63.756 887 9063	.015 684 5799	73
.758 068 9119	64.514 956 8183	.015 500 2816	74
.755 236 7740	65.270 193 5923	.015 320 9290	75
.752 415 2170	66.022 608 8093	.015 146 3267	76
.749 604 2012	66.772 213 0105	.014 976 2896	77
.746 803 6874	67.519 016 6979	.014 810 6422	78
.744 013 6362	68.263 030 3341	.014 649 2178	79
.741 234 0087	69.004 264 3428	.014 491 8580	80
.738 464 7658	69.742 729 1087	.014 338 4122	81
.735 705 8688	70.478 434 9775	.014 188 7373	82
.732 957 2790	71.211 392 2565	.014 042 6969	83
.730 218 9579	71.941 611 2145	.013 900 1613	84
.727 490 8672	72.669 102 0817	.013 761 0067	85
.724 772 9686	73.393 875 0502	.013 625 1152	86
.722 065 2240	74.115 940 2742	.013 492 3742	87
.719 367 5955	74.835 307 8697	.013 362 6764	88
.716 680 0453	75.551 987 9150	.013 235 9191	89
.714 002 5358	76.265 990 4508	.013 112 0044	90
.711 335 0294	76.977 325 4803	.012 990 8385	91
.708 677 4889	77.686 002 9691	.012 872 3317	92
.706 029 8768	78.392 032 8460	.012 756 3984	93
.703 392 1562	79.095 425 0022	.012 642 9563	94
.700 764 2902	79.796 189 2924	.012 531 9268	95
.698 146 2417	80.494 335 5341	.012 423 2344	96
.695 537 9743	81.189 873 5085	.012 316 8070	97
.692 939 4514	81.882 812 9599	.012 212 5751	98
.690 350 6365	82.573 163 5964	.012 110 4722	99
.687 771 4934	83.260 935 0898	.012 010 4344	100
.685 201 9860	83.946 137 0758	.011 912 4004	101
.682 642 0782	84.628 779 1539	.011 816 3113	102
.680 091 7342	85.308 870 8881	.011 722 1104	103
.677 550 9182	85.986 421 8063	.011 629 7432	104
.675 019 5947	86.661 441 4011	.011 539 1573	105
.672 497 7283	87.333 939 1293	.011 450 3023	106
.669 985 2835	88.003 924 4128	.011 363 1296	107
.667 482 2251	88.671 406 6379	.011 277 5926	108
.664 988 5182	89.336 395 1561	.011 193 6462	109
.662 504 1277	89.998 899 2837	.011 111 2470	110
.660 029 0189	90.658 928 3026	.011 030 3532	111
.657 563 1570	91.316 491 4596	.010 950 9245	112
.655 106 5076	91.971 597 9673	.010 872 9219	113
.652 659 0362	92.624 257 0035	.010 796 3079	114
.650 220 7086	93.274 477 7121	.010 721 0464	115
.647 791 4905	93.922 269 2026	.010 647 1022	116
.645 371 3479	94.567 640 5505	.010 574 4417	117
.642 960 2470	95.210 600 7975	.010 503 0321	118
.640 558 1539	95.851 158 9514	.010 432 8420	119
.638 165 0351	96.489 323 9865	.010 363 8409	120
$v^n=\frac{1}{(1+i)^n}$	$a_{\overline{n}\rvert}=\frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rvert}}=\frac{i}{1-v^n}$	n

RATE

3/8%

.00375

per period

ANNUALLY
If compounded *annually* nominal annual rate is

3/8%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

3/4%

QUARTERLY
If compounded *quarterly* nominal annual rate is

1 1/2%

MONTHLY
If compounded *monthly* nominal annual rate is

4 1/2%

$i = .00375$
$j_{(2)} = .0075$
$j_{(4)} = .015$
$j_{(12)} = .045$

RATE

½%

.005 *per period*

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.005 000 0000	1.000 000 0000	1.000 000 0000
2	1.010 025 0000	2.005 000 0000	.498 753 1172
3	1.015 075 1250	3.015 025 0000	.331 672 2084
4	1.020 150 5006	4.030 100 1250	.248 132 7930
5	1.025 251 2531	5.050 250 6256	.198 009 9750
6	1.030 377 5094	6.075 501 8788	.164 595 4556
7	1.035 529 3969	7.105 879 3881	.140 728 5355
8	1.040 707 0439	8.141 408 7851	.122 828 8649
9	1.045 910 5791	9.182 115 8290	.108 907 3606
10	1.051 140 1320	10.228 026 4082	.097 770 5727
11	1.056 395 8327	11.279 166 5402	.088 659 0331
12	1.061 677 8119	12.335 562 3729	.081 066 4297
13	1.066 986 2009	13.397 240 1848	.074 642 2387
14	1.072 321 1319	14.464 226 3857	.069 136 0860
15	1.077 682 7376	15.536 547 5176	.064 364 3640
16	1.083 071 1513	16.614 230 2552	.060 189 3669
17	1.088 486 5070	17.697 301 4065	.056 505 7902
18	1.093 928 9396	18.785 787 9135	.053 231 7305
19	1.099 398 5843	19.879 716 8531	.050 302 5273
20	1.104 895 5772	20.979 115 4373	.047 666 4520
21	1.110 420 0551	22.084 011 0145	.045 281 6293
22	1.115 972 1553	23.194 431 0696	.043 113 7973
23	1.121 552 0161	24.310 403 2250	.041 134 6530
24	1.127 159 7762	25.431 955 2411	.039 320 6103
25	1.132 795 5751	26.559 115 0173	.037 651 8570
26	1.138 459 5530	27.691 910 5924	.036 111 6289
27	1.144 151 8507	28.830 370 1453	.034 685 6456
28	1.149 872 6100	29.974 521 9961	.033 361 6663
29	1.155 621 9730	31.124 394 6060	.032 129 1390
30	1.161 400 0829	32.280 016 5791	.030 978 9184
31	1.167 207 0833	33.441 416 6620	.029 903 0394
32	1.173 043 1187	34.608 623 7453	.028 894 5324
33	1.178 908 3343	35.781 666 8640	.027 947 2727
34	1.184 802 8760	36.960 575 1989	.027 055 8560
35	1.190 726 8904	38.145 378 0743	.026 215 4958
36	1.196 680 5248	39.336 104 9647	.025 421 9375
37	1.202 663 9274	40.532 785 4895	.024 671 3861
38	1.208 677 2471	41.735 449 4170	.023 960 4464
39	1.214 720 6333	42.944 126 6640	.023 286 0714
40	1.220 794 2965	44.158 847 2974	.022 645 5186
41	1.226 898 2077	45.379 641 5338	.022 036 3133
42	1.233 032 6987	46.606 539 7415	.021 456 2163
43	1.239 197 8622	47.839 572 4402	.020 903 1969
44	1.245 393 8515	49.078 770 3024	.020 375 4086
45	1.251 620 8208	50.324 164 1539	.019 871 1696
46	1.257 878 9249	51.575 784 9747	.019 388 9439
47	1.264 168 3195	52.833 663 8996	.018 927 3264
48	1.270 489 1611	54.097 832 2191	.018 485 0290
49	1.276 841 6069	55.368 321 3802	.018 060 8690
50	1.283 225 8149	56.645 162 9871	.017 653 7580
51	1.289 641 9440	57.928 388 8020	.017 262 6931
52	1.296 090 1537	59.218 030 7460	.016 886 7486
53	1.302 570 6045	60.514 120 8997	.016 525 0686
54	1.309 083 4575	61.816 691 5042	.016 176 8606
55	1.315 628 8748	63.125 774 9618	.015 841 3897
56	1.322 207 0192	64.441 403 8366	.015 517 9735
57	1.328 818 0543	65.763 610 8558	.015 205 9777
58	1.335 462 1446	67.092 428 9100	.014 904 8114
59	1.342 139 4553	68.427 891 0546	.014 613 9240
60	1.348 850 1525	69.770 030 5099	.014 332 8015
n	$s=(1+i)^n$	$s_{\overline{n}\rvert}=\dfrac{(1+i)^n-1}{i}$	$\dfrac{1}{s_{\overline{n}\rvert}}=\dfrac{i}{(1+i)^n-1}$

ANNUALLY
If compounded *annually* nominal annual rate is

½%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

1%

QUARTERLY
If compounded *quarterly* nominal annual rate is

2%

MONTHLY
If compounded *monthly* nominal annual rate is

6%

$i = .005$
$j_{(2)} = .01$
$j_{(4)} = .02$
$j_{(12)} = .06$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today. Periodic payment necessary to pay off a loan of $1.*	PERIODS
.995 024 8756	.995 024 8756	1.005 000 0000	1
.990 074 5031	1.985 099 3787	.503 753 1172	2
.985 148 7593	2.970 248 1380	.336 672 2084	3
.980 247 5217	3.950 495 6597	.253 132 7930	4
.975 370 6684	4.925 866 3281	.203 009 9750	5
.970 518 0780	5.896 384 4061	.169 595 4556	6
.965 689 6298	6.862 074 0359	.145 728 5355	7
.960 885 2038	7.822 959 2397	.127 828 8649	8
.956 104 6804	8.779 063 9201	.113 907 3606	9
.951 347 9407	9.730 411 8608	.102 770 5727	10
.946 614 8664	10.677 026 7272	.093 659 0331	11
.941 905 3397	11.618 932 0668	.086 066 4297	12
.937 219 2434	12.556 151 5103	.079 642 2387	13
.932 556 4611	13.488 707 7714	.074 136 0860	14
.927 916 8768	14.416 624 6482	.069 364 3640	15
.923 300 3749	15.339 925 0231	.065 189 3669	16
.918 706 8407	16.258 631 8637	.061 505 7902	17
.914 136 1599	17.172 768 0236	.058 231 7305	18
.909 588 2188	18.082 356 2424	.055 302 5273	19
.905 062 9043	18.987 419 1467	.052 666 4520	20
.900 560 1037	19.887 979 2504	.050 281 6293	21
.896 079 7052	20.784 058 9556	.048 113 7973	22
.891 621 5972	21.675 680 5529	.046 134 6530	23
.887 185 6689	22.562 866 2218	.044 320 6103	24
.882 771 8098	23.445 638 0316	.042 651 8570	25
.878 379 9103	24.324 017 9419	.041 111 6289	26
.874 009 8610	25.198 027 8029	.039 685 6456	27
.869 661 5532	26.067 689 3561	.038 361 6663	28
.865 334 8788	26.933 024 2349	.037 129 1390	29
.861 029 7302	27.794 053 9651	.035 978 9184	30
.856 746 0002	28.650 799 9653	.034 903 0394	31
.852 483 5823	29.503 283 5475	.033 894 5324	32
.848 242 3704	30.351 525 9179	.032 947 2727	33
.844 022 2591	31.195 548 1771	.032 055 8560	34
.839 823 1434	32.035 371 3205	.031 215 4958	35
.835 644 9188	32.871 016 2393	.030 421 9375	36
.831 487 4814	33.702 503 7207	.029 671 3861	37
.827 350 7278	34.529 854 4484	.028 960 4464	38
.823 234 5550	35.353 089 0034	.028 286 0714	39
.819 138 8607	36.172 227 8641	.027 645 5186	40
.815 063 5430	36.987 291 4070	.027 036 3133	41
.811 008 5005	37.798 299 9075	.026 456 2163	42
.806 973 6323	38.605 273 5398	.025 903 1969	43
.802 958 8381	39.408 232 3779	.025 375 4086	44
.798 964 0180	40.207 196 3959	.024 871 1696	45
.794 989 0727	41.002 185 4686	.024 388 9439	46
.791 033 9031	41.793 219 3717	.023 927 3264	47
.787 098 4111	42.580 317 7828	.023 485 0290	48
.783 182 4986	43.363 500 2814	.023 060 8690	49
.779 286 0683	44.142 786 3497	.022 653 7580	50
.775 409 0231	44.918 195 3728	.022 262 6931	51
.771 551 2668	45.689 746 6396	.021 886 7486	52
.767 712 7033	46.457 459 3429	.021 525 0686	53
.763 893 2371	47.221 352 5800	.021 176 8606	54
.760 092 7732	47.981 445 3532	.020 841 3897	55
.756 311 2171	48.737 756 5704	.020 517 9735	56
.752 548 4748	49.490 305 0452	.020 205 9777	57
.748 804 4525	50.239 109 4977	.019 904 8114	58
.745 079 0572	50.984 188 5549	.019 613 9240	59
.741 372 1962	51.725 560 7511	.019 332 8015	60
$v^n=\frac{1}{(1+i)^n}$	$a_{\overline{n}\rvert}=\frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rvert}}=\frac{i}{1-v^n}$	n

RATE

$\frac{1}{2}\%$

.005

per period

ANNUALLY
If compounded *annually* nominal annual rate is

$\frac{1}{2}\%$

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

1%

QUARTERLY
If compounded *quarterly* nominal annual rate is

2%

MONTHLY
If compounded *monthly* nominal annual rate is

6%

$i = .005$
$j_{(2)} = .01$
$j_{(4)} = .02$
$j_{(12)} = .06$

RATE

1/2%

.005

per period

ANNUALLY
If compounded *annually* nominal annual rate is **1/2%**

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is **1%**

QUARTERLY
If compounded *quarterly* nominal annual rate is **2%**

MONTHLY
If compounded *monthly* nominal annual rate is **6%**

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
61	1.355 594 4093	71.118 880 6624	.014 060 9697
62	1.362 372 3753	72.474 475 0657	.013 797 9613
63	1.369 184 2372	73.836 847 4411	.013 543 3735
64	1.376 030 1584	75.206 031 6783	.013 296 8058
65	1.382 910 3092	76.582 061 8366	.013 057 8882
66	1.389 824 8607	77.964 972 1458	.012 826 2728
67	1.396 773 9850	79.354 797 0066	.012 601 6326
68	1.403 757 8550	80.751 570 9916	.012 383 6600
69	1.410 776 6442	82.155 328 8466	.012 172 0650
70	1.417 830 5275	83.566 105 4908	.011 966 5742
71	1.424 919 6801	84.983 936 0182	.011 766 9297
72	1.432 044 2785	86.408 855 6983	.011 572 8879
73	1.439 204 4999	87.840 899 9768	.011 384 2185
74	1.446 400 5224	89.280 104 4767	.011 200 7037
75	1.453 632 5250	90.726 504 9991	.011 022 1374
76	1.460 900 6876	92.180 137 5241	.010 848 3240
77	1.468 205 1911	93.641 038 2117	.010 679 0785
78	1.475 546 2170	95.109 243 4028	.010 514 2252
79	1.482 923 9481	96.584 789 6198	.010 353 5971
80	1.490 338 5678	98.067 713 5679	.010 197 0359
81	1.497 790 2607	99.558 052 1357	.010 044 3910
82	1.505 279 2120	101.055 842 3964	.009 895 5189
83	1.512 805 6080	102.561 121 6084	.009 750 2834
84	1.520 369 6361	104.073 927 2164	.009 608 5545
85	1.527 971 4843	105.594 296 8525	.009 470 2084
86	1.535 611 3417	107.122 268 3368	.009 335 1272
87	1.543 289 3984	108.657 879 6784	.009 203 1982
88	1.551 005 8454	110.201 169 0768	.009 074 3139
89	1.558 760 8746	111.752 174 9222	.008 948 3717
90	1.566 554 6790	113.310 935 7968	.008 825 2735
91	1.574 387 4524	114.877 490 4758	.008 704 9255
92	1.582 259 3896	116.451 877 9282	.008 587 2381
93	1.590 170 6866	118.034 137 3178	.008 472 1253
94	1.598 121 5400	119.624 308 0044	.008 359 5050
95	1.606 112 1477	121.222 429 5445	.008 249 2984
96	1.614 142 7085	122.828 541 6922	.008 141 4302
97	1.622 213 4220	124.442 684 4006	.008 035 8279
98	1.630 324 4891	126.064 897 8226	.007 932 4222
99	1.638 476 1116	127.695 222 3118	.007 831 1466
100	1.646 668 4921	129.333 698 4233	.007 731 9369
101	1.654 901 8346	130.980 366 9154	.007 634 7320
102	1.663 176 3438	132.635 268 7500	.007 539 4728
103	1.671 492 2255	134.298 445 0938	.007 446 1026
104	1.679 849 6866	135.969 937 3192	.007 354 5669
105	1.688 248 9350	137.649 787 0058	.007 264 8133
106	1.696 690 1797	139.338 035 9408	.007 176 7913
107	1.705 173 6306	141.034 726 1206	.007 090 4523
108	1.713 699 4988	142.739 899 7512	.007 005 7496
109	1.722 267 9962	144.453 599 2499	.006 922 6382
110	1.730 879 3362	146.175 867 2462	.006 841 0745
111	1.739 533 7329	147.906 746 5824	.006 761 0168
112	1.748 231 4016	149.646 280 3153	.006 682 4247
113	1.756 972 5586	151.394 511 7169	.006 605 2593
114	1.765 757 4214	153.151 484 2755	.006 529 4829
115	1.774 586 2085	154.917 241 6968	.006 455 0594
116	1.783 459 1995	156.691 827 9053	.006 381 9538
117	1.792 376 4352	158.475 287 0449	.006 310 1321
118	1.801 338 3174	160.267 663 4801	.006 239 5619
119	1.810 345 0090	162.069 001 7975	.006 170 2114
120	1.819 396 7340	163.879 346 8065	.006 102 0502
n	$s=(1+i)^n$	$s_{\overline{n}\rceil}=\frac{(1+i)^n-1}{i}$	$\frac{1}{s_{\overline{n}\rceil}}=\frac{i}{(1+i)^n-1}$

$i = .005$
$j_{(2)} = .01$
$j_{(4)} = .02$
$j_{(12)} = .06$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today.* *Periodic payment necessary to pay off a loan of $1.*	PERIODS
.737 683 7774	52.463 244 5285	.019 060 9637	61
.734 013 7088	53.197 258 2373	.018 797 9613	62
.730 361 8993	53.927 620 1366	.018 543 3735	63
.726 728 2580	54.654 348 3946	.018 296 8058	64
.723 112 6946	55.377 461 0892	.018 057 8882	65
.719 515 1190	56.096 976 2082	.017 826 2728	66
.715 935 4418	56.812 911 6499	.017 601 6326	67
.712 373 5739	57.525 285 2238	.017 383 6600	68
.708 829 4267	58.234 114 6505	.017 172 0650	69
.705 302 9122	58.939 417 5627	.016 966 5742	70
.701 793 9425	59.641 211 5052	.016 766 9297	71
.698 302 4303	60.339 513 9355	.016 572 8879	72
.694 828 2889	61.034 342 2244	.016 384 2185	73
.691 371 4317	61.725 713 6561	.016 200 7037	74
.687 931 7729	62.413 645 4290	.016 022 1374	75
.684 509 2267	63.098 154 6557	.015 848 3240	76
.681 103 7082	63.779 258 3639	.015 679 0785	77
.677 715 1325	64.456 973 4964	.015 514 2252	78
.674 343 4154	65.131 316 9118	.015 353 5971	79
.670 988 4731	65.802 305 3849	.015 197 0359	80
.667 650 2220	66.469 955 6069	.015 044 3910	81
.664 328 5791	67.134 284 1859	.014 895 5189	82
.661 023 4618	67.795 307 6477	.014 750 2834	83
.657 734 7878	68.453 042 4355	.014 608 5545	84
.654 462 4754	69.107 504 9110	.014 470 2084	85
.651 206 4432	69.758 711 3542	.014 335 1272	86
.647 966 6102	70.406 677 9644	.014 203 1982	87
.644 742 8957	71.051 420 8601	.014 074 3139	88
.641 535 2196	71.692 956 0797	.013 948 3717	89
.638 343 5021	72.331 299 5818	.013 825 2735	90
.635 167 6638	72.966 467 2455	.013 704 9255	91
.632 007 6256	73.598 474 8712	.013 587 2381	92
.628 863 3091	74.227 338 1803	.013 472 1253	93
.625 734 6359	74.853 072 8162	.013 359 5050	94
.622 621 5283	75.475 694 3445	.013 249 2984	95
.619 523 9087	76.095 218 2532	.013 141 4302	96
.616 441 7002	76.711 659 9535	.013 035 8279	97
.613 374 8261	77.325 034 7796	.012 932 4222	98
.610 323 2101	77.935 357 9896	.012 831 1466	99
.607 286 7762	78.542 644 7658	.012 731 9369	100
.604 265 4489	79.146 910 2147	.012 634 7320	101
.601 259 1532	79.748 169 3679	.012 539 4728	102
.598 267 8141	80.346 437 1820	.012 446 1026	103
.595 291 3573	80.941 728 5393	.012 354 5669	104
.592 329 7088	81.534 058 2480	.012 264 8133	105
.589 382 7948	82.123 441 0428	.012 176 7913	106
.586 450 5421	82.709 891 5849	.012 090 4523	107
.583 532 8777	83.293 424 4626	.012 005 7496	108
.580 629 7290	83.874 054 1916	.011 922 6382	109
.577 741 0239	84.451 795 2155	.011 841 0745	110
.574 866 6905	85.026 661 9060	.011 761 0168	111
.572 006 6572	85.598 668 5632	.011 682 4247	112
.569 160 8529	86.167 829 4161	.011 605 2593	113
.566 329 2069	86.734 158 6230	.011 529 4829	114
.563 511 6486	87.297 670 2716	.011 455 0594	115
.560 708 1081	87.858 378 3797	.011 381 9538	116
.557 918 5155	88.416 296 8953	.011 310 1321	117
.555 142 8015	88.971 439 6968	.011 239 5619	118
.552 380 8970	89.523 820 5938	.011 170 2114	119
.549 632 7334	90.073 453 3272	.011 102 0502	120
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\rceil} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rceil}} = \frac{i}{1-v^n}$	n

RATE

½%

.005

per period

ANNUALLY

If compounded *annually* nominal annual rate is

½%

SEMIANNUALLY

If compounded *semiannually* nominal annual rate is

1%

QUARTERLY

If compounded *quarterly* nominal annual rate is

2%

MONTHLY

If compounded *monthly* nominal annual rate is

6%

$i = .005$
$j_{(2)} = .01$
$j_{(4)} = .02$
$j_{(12)} = .06$

RATE

3/4%

.0075

per period

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.007 500 0000	1.000 000 0000	1.000 000 0000
2	1.015 056 2500	2.007 500 0000	.498 132 0050
3	1.022 669 1719	3.022 556 2500	.330 845 7866
4	1.030 339 1907	4.045 225 4219	.247 205 0123
5	1.038 066 7346	5.075 564 6125	.197 022 4155
6	1.045 852 2351	6.113 631 3471	.163 568 9074
7	1.053 696 1269	7.159 483 5822	.139 674 8786
8	1.061 598 8478	8.213 179 7091	.121 755 5241
9	1.069 560 8392	9.274 778 5569	.107 819 2858
10	1.077 582 5455	10.344 339 3961	.096 671 2287
11	1.085 664 4146	11.421 921 9416	.087 550 9398
12	1.093 806 8977	12.507 586 3561	.079 951 4768
13	1.102 010 4494	13.601 393 2538	.073 521 8798
14	1.110 275 5278	14.703 403 7032	.068 011 4632
15	1.118 602 5942	15.813 679 2310	.063 236 3908
16	1.126 992 1137	16.932 281 8252	.059 058 7855
17	1.135 444 5545	18.059 273 9389	.055 373 2118
18	1.143 960 3887	19.194 718 4934	.052 097 6643
19	1.152 540 0916	20.338 678 8821	.049 167 4020
20	1.161 184 1423	21.491 218 9738	.046 530 6319
21	1.169 893 0234	22.652 403 1161	.044 145 4266
22	1.178 667 2210	23.822 296 1394	.041 977 4817
23	1.187 507 2252	25.000 963 3605	.039 998 4587
24	1.196 413 5294	26.188 470 5857	.038 184 7423
25	1.205 386 6309	27.384 884 1151	.036 516 4956
26	1.214 427 0306	28.590 270 7459	.034 976 9335
27	1.223 535 2333	29.804 697 7765	.033 551 7578
28	1.232 711 7476	31.028 233 0099	.032 228 7125
29	1.241 957 0857	32.260 944 7574	.030 997 2323
30	1.251 271 7638	33.502 901 8431	.029 848 1608
31	1.260 656 3021	34.754 173 6069	.028 773 5226
32	1.270 111 2243	36.014 829 9090	.027 766 3397
33	1.279 637 0585	37.284 941 1333	.026 820 4795
34	1.289 234 3364	38.564 578 1918	.025 930 5313
35	1.298 903 5940	39.853 812 5282	.025 091 7023
36	1.308 645 3709	41.152 716 1222	.024 299 7327
37	1.318 460 2112	42.461 361 4931	.023 550 8228
38	1.328 348 6628	43.779 821 7043	.022 841 5732
39	1.338 311 2778	45.108 170 3671	.022 168 9329
40	1.348 348 6123	46.446 481 6449	.021 530 1561
41	1.358 461 2269	47.794 830 2572	.020 922 7650
42	1.368 649 6861	49.153 291 4841	.020 344 5175
43	1.378 914 5588	50.521 941 1703	.019 793 3804
44	1.389 256 4180	51.900 855 7290	.019 267 5051
45	1.399 675 8411	53.290 112 1470	.018 765 2073
46	1.410 173 4099	54.689 787 9881	.018 284 9493
47	1.420 749 7105	56.099 961 3980	.017 825 3242
48	1.431 405 3333	57.520 711 1085	.017 385 0424
49	1.442 140 8733	58.952 116 4418	.016 962 9194
50	1.452 956 9299	60.394 257 3151	.016 557 8657
51	1.463 854 1068	61.847 214 2450	.016 168 8770
52	1.474 833 0126	63.311 068 3518	.015 795 0265
53	1.485 894 2602	64.785 901 3645	.015 435 4571
54	1.497 038 4672	66.271 795 6247	.015 089 3754
55	1.508 266 2557	67.768 834 0919	.014 756 0455
56	1.519 578 2526	69.277 100 3476	.014 434 7843
57	1.530 975 0895	70.796 678 6002	.014 124 9564
58	1.542 457 4027	72.327 653 6897	.013 825 9704
59	1.554 025 8332	73.870 111 0923	.013 537 2749
60	1.565 681 0269	75.424 136 9255	.013 258 3552
n	$s=(1+i)^n$	$s_{\overline{n}\|}=\frac{(1+i)^n-1}{i}$	$\frac{1}{s_{\overline{n}\|}}=\frac{i}{(1+i)^n-1}$

ANNUALLY
If compounded *annually* nominal annual rate is

3/4%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

1½%

QUARTERLY
If compounded *quarterly* nominal annual rate is

3%

MONTHLY
If compounded *monthly* nominal annual rate is

9%

$i = .0075$
$j_{(2)} = .015$
$j_{(4)} = .03$
$j_{(12)} = .09$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today.* *Periodic payment necessary to pay off a loan of $1.*	PERIODS
.992 555 8313	.992 555 8313	1.007 500 0000	1
.985 167 0782	1.977 722 9094	.505 632 0050	2
.977 833 3282	2.955 556 2377	.338 345 7866	3
.970 554 1719	3.926 110 4096	.254 705 0123	4
.963 329 2029	4.889 439 6125	.204 522 4155	5
.956 158 0178	5.845 597 6303	.171 068 9074	6
.949 040 2162	6.794 637 8464	.147 174 8786	7
.941 975 4006	7.736 613 2471	.129 255 5241	8
.934 963 1768	8.671 576 4239	.115 319 2858	9
.928 003 1532	9.599 579 5771	.104 171 2287	10
.921 094 9411	10.520 674 5182	.095 050 9398	11
.914 238 1550	11.434 912 6731	.087 451 4768	12
.907 432 4119	12.342 345 0850	.081 021 8798	13
.900 677 3319	13.243 022 4169	.075 511 4632	14
.893 972 5378	14.136 994 9547	.070 736 3908	15
.887 317 6554	15.024 312 6101	.066 558 7855	16
.880 712 3131	15.905 024 9232	.062 873 2118	17
.874 156 1420	16.779 181 0652	.059 597 6643	18
.867 648 7762	17.646 829 8414	.056 667 4020	19
.861 189 8523	18.508 019 6937	.054 030 6319	20
.854 779 0097	19.362 798 7034	.051 645 4266	21
.848 415 8905	20.211 214 5940	.049 477 4817	22
.842 100 1395	21.053 314 7335	.047 498 4587	23
.835 831 4040	21.889 146 1374	.045 684 7423	24
.829 609 3340	22.718 755 4714	.044 016 4956	25
.823 433 5821	23.542 189 0535	.042 476 9335	26
.817 303 8036	24.359 492 8571	.041 051 7578	27
.811 219 6562	25.170 712 5132	.039 728 7125	28
.805 180 8001	25.975 893 3134	.038 497 2323	29
.799 186 8984	26.775 080 2118	.037 348 1608	30
.793 237 6163	27.568 317 8281	.036 273 5226	31
.787 332 6216	28.355 650 4497	.035 266 3397	32
.781 471 5847	29.137 122 0344	.034 320 4795	33
.775 654 1784	29.912 776 2128	.033 430 5313	34
.769 880 0778	30.682 656 2907	.032 591 7023	35
.764 148 9606	31.446 805 2513	.031 799 7327	36
.758 460 5068	32.205 265 7581	.031 050 8228	37
.752 814 3988	32.958 080 1569	.030 341 5732	38
.747 210 3214	33.705 290 4783	.029 668 9329	39
.741 647 9617	34.446 938 4400	.029 030 1561	40
.736 127 0091	35.183 065 4492	.028 422 7650	41
.730 647 1555	35.913 712 6046	.027 844 5175	42
.725 208 0948	36.638 920 6994	.027 293 3804	43
.719 809 5233	37.358 730 2227	.026 767 5051	44
.714 451 1398	38.073 181 3625	.026 265 2073	45
.709 132 6449	38.782 314 0074	.025 784 9493	46
.703 853 7419	39.486 167 7493	.025 325 3242	47
.698 614 1359	40.184 781 8852	.024 885 0424	48
.693 413 5344	40.878 195 4195	.024 462 9194	49
.688 251 6470	41.566 447 0665	.024 057 8657	50
.683 128 1856	42.249 575 2521	.023 668 8770	51
.678 042 8641	42.927 618 1163	.023 295 0265	52
.672 995 3986	43.600 613 5149	.022 935 4571	53
.667 985 5073	44.268 599 0222	.022 589 3754	54
.663 012 9105	44.931 611 9327	.022 256 0455	55
.658 077 3305	45.589 689 2633	.021 934 7843	56
.653 178 4918	46.242 867 7551	.021 624 9564	57
.648 316 1209	46.891 183 8760	.021 325 9704	58
.643 489 9463	47.534 673 8224	.021 037 2749	59
.638 699 6986	48.173 373 5210	.020 758 3552	60
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\rvert} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rvert}} = \frac{i}{1-v^n}$	n

RATE

3/4%

.0075

per period

ANNUALLY
If compounded *annually* nominal annual rate is

3/4%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

1½%

QUARTERLY
If compounded *quarterly* nominal annual rate is

3%

MONTHLY
If compounded *monthly* nominal annual rate is

9%

$i = .0075$
$j_{(2)} = .015$
$j_{(4)} = .03$
$j_{(12)} = .09$

RATE

1%

.01

per period

ANNUALLY
If compounded *annually* nominal annual rate is
1%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is
2%

QUARTERLY
If compounded *quarterly* nominal annual rate is
4%

MONTHLY
If compounded *monthly* nominal annual rate is
12%

$i = .01$
$j_{(2)} = .02$
$j_{(4)} = .04$
$j_{(12)} = .12$

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.010 000 0000	1.000 000 0000	1.000 000 0000
2	1.020 100 0000	2.010 000 0000	.497 512 4378
3	1.030 301 0000	3.030 100 0000	.330 022 1115
4	1.040 604 0100	4.060 401 0000	.246 281 0939
5	1.051 010 0501	5.101 005 0100	.196 039 7996
6	1.061 520 1506	6.152 015 0601	.162 548 3667
7	1.072 135 3521	7.213 535 2107	.138 628 2829
8	1.082 856 7056	8.285 670 5628	.120 690 2920
9	1.093 685 2727	9.368 527 2684	.106 740 3628
10	1.104 622 1254	10.462 212 5411	.095 582 0766
11	1.115 668 3467	11.566 834 6665	.086 454 0757
12	1.126 825 0301	12.682 503 0132	.078 848 7887
13	1.138 093 2804	13.809 328 0433	.072 414 8197
14	1.149 474 2132	14.947 421 3238	.066 901 1717
15	1.160 968 9554	16.096 895 5370	.062 123 7802
16	1.172 578 6449	17.257 864 4924	.057 944 5968
17	1.184 304 4314	18.430 443 1373	.054 258 0551
18	1.196 147 4757	19.614 747 5687	.050 982 0479
19	1.208 108 9504	20.810 895 0444	.048 051 7536
20	1.220 190 0399	22.019 003 9948	.045 415 3149
21	1.232 391 9403	23.239 194 0347	.043 030 7522
22	1.244 715 8598	24.471 585 9751	.040 863 7185
23	1.257 163 0183	25.716 301 8348	.038 885 8401
24	1.269 734 6485	26.973 464 8532	.037 073 4722
25	1.282 431 9950	28.243 199 5017	.035 406 7534
26	1.295 256 3150	29.525 631 4967	.033 868 8776
27	1.308 208 8781	30.820 887 8117	.032 445 5287
28	1.321 290 9669	32.129 096 6898	.031 124 4356
29	1.334 503 8766	33.450 387 6567	.029 895 0198
30	1.347 848 9153	34.784 891 5333	.028 748 1132
31	1.361 327 4045	36.132 740 4486	.027 675 7309
32	1.374 940 6785	37.494 067 8531	.026 670 8857
33	1.388 690 0853	38.869 008 5316	.025 727 4378
34	1.402 576 9862	40.257 698 6170	.024 839 9694
35	1.416 602 7560	41.660 275 6031	.024 003 6818
36	1.430 768 7836	43.076 878 3592	.023 214 3098
37	1.445 076 4714	44.507 647 1427	.022 468 0491
38	1.459 527 2361	45.952 723 6142	.021 761 4958
39	1.474 122 5085	47.412 250 8503	.021 091 5951
40	1.488 863 7336	48.886 373 3588	.020 455 5980
41	1.503 752 3709	50.375 237 0924	.019 851 0232
42	1.518 789 8946	51.878 989 4633	.019 275 6260
43	1.533 977 7936	53.397 779 3580	.018 727 3705
44	1.549 317 5715	54.931 757 1515	.018 204 4058
45	1.564 810 7472	56.481 074 7231	.017 705 0455
46	1.580 458 8547	58.045 885 4703	.017 227 7499
47	1.596 263 4432	59.626 344 3250	.016 771 1103
48	1.612 226 0777	61.222 607 7682	.016 333 8354
49	1.628 348 3385	62.834 833 8459	.015 914 7393
50	1.644 631 8218	64.463 182 1844	.015 512 7309
51	1.661 078 1401	66.107 814 0062	.015 126 8048
52	1.677 688 9215	67.768 892 1463	.014 756 0329
53	1.694 465 8107	69.446 581 0678	.014 399 5570
54	1.711 410 4688	71.141 046 8784	.014 056 5826
55	1.728 524 5735	72.852 457 3472	.013 726 3730
56	1.745 809 8192	74.580 981 9207	.013 408 2440
57	1.763 267 9174	76.326 791 7399	.013 101 5595
58	1.780 900 5966	78.090 059 6573	.012 805 7272
59	1.798 709 6025	79.870 960 2539	.012 520 1950
60	1.816 696 6986	81.669 669 8564	.012 244 4477
n	$s = (1+i)^n$	$s_{\overline{n}\rvert} = \frac{(1+i)^n - 1}{i}$	$\frac{1}{s_{\overline{n}\rvert}} = \frac{i}{(1+i)^n - 1}$

RATE

1%

.01

per period

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today.* *Periodic payment necessary to pay off a loan of $1.*	PERIODS
.990 099 0099	.990 099 0099	1.010 000 0000	1
.980 296 0494	1.970 395 0593	.507 512 4378	2
.970 590 1479	2.940 985 2072	.340 022 1115	3
.960 980 3445	3.901 965 5517	.256 281 0939	4
.951 465 6876	4.853 431 2393	.206 039 7996	5
.942 045 2353	5.795 476 4746	.172 548 3667	6
.932 718 0547	6.728 194 5293	.148 628 2829	7
.923 483 2225	7.651 677 7518	.130 690 2920	8
.914 339 8242	8.566 017 5760	.116 740 3628	9
.905 286 9547	9.471 304 5307	.105 582 0766	10
.896 323 7175	10.367 628 2482	.096 454 0757	11
.887 449 2253	11.255 077 4735	.088 848 7887	12
.878 662 5993	12.133 740 0728	.082 414 8197	13
.869 962 9696	13.003 703 0423	.076 901 1717	14
.861 349 4748	13.865 052 5172	.072 123 7802	15
.852 821 2622	14.717 873 7794	.067 944 5968	16
.844 377 4873	15.562 251 2667	.064 258 0551	17
.836 017 3142	16.398 268 5809	.060 982 0479	18
.827 739 9150	17.226 008 4959	.058 051 7536	19
.819 544 4703	18.045 552 9663	.055 415 3149	20
.811 430 1687	18.856 983 1349	.053 030 7522	21
.803 396 2066	19.660 379 3415	.050 863 7185	22
.795 441 7887	20.455 821 1302	.048 885 8401	23
.787 566 1274	21.243 387 2576	.047 073 4722	24
.779 768 4430	22.023 155 7006	.045 406 7534	25
.772 047 9634	22.795 203 6640	.043 868 8776	26
.764 403 9241	23.559 607 5881	.042 445 5287	27
.756 835 5684	24.316 443 1565	.041 124 4356	28
.749 342 1470	25.065 785 3035	.039 895 0198	29
.741 922 9178	25.807 708 2213	.038 748 1132	30
.734 577 1463	26.542 285 3676	.037 675 7309	31
.727 304 1053	27.269 589 4729	.036 670 8857	32
.720 103 0745	27.989 692 5474	.035 727 4378	33
.712 973 3411	28.702 665 8885	.034 839 9694	34
.705 914 1991	29.408 580 0876	.034 003 6818	35
.698 924 9496	30.107 505 0373	.033 214 3098	36
.692 004 9006	30.799 509 9379	.032 468 0491	37
.685 153 3670	31.484 663 3048	.031 761 4958	38
.678 369 6702	32.163 032 9751	.031 091 5951	39
.671 653 1389	32.834 686 1140	.030 455 5980	40
.665 003 1078	33.499 689 2217	.029 851 0232	41
.658 418 9186	34.158 108 1403	.029 275 6260	42
.651 899 9194	34.810 008 0597	.028 727 3705	43
.645 445 4648	35.455 453 5245	.028 204 4058	44
.639 054 9156	36.094 508 4401	.027 705 0455	45
.632 727 6392	36.727 236 0793	.027 227 7499	46
.626 463 0091	37.353 699 0884	.026 771 1103	47
.620 260 4051	37.973 959 4935	.026 333 8354	48
.614 119 2129	38.588 078 7064	.025 914 7393	49
.608 038 8247	39.196 117 5311	.025 512 7309	50
.602 018 6383	39.798 136 1694	.025 126 8048	51
.596 058 0577	40.394 194 2271	.024 756 0329	52
.590 156 4928	40.984 350 7199	.024 399 5570	53
.584 313 3592	41.568 664 0791	.024 056 5826	54
.578 528 0784	42.147 192 1576	.023 726 3730	55
.572 800 0776	42.719 992 2352	.023 408 2440	56
.567 128 7898	43.287 121 0250	.023 101 5595	57
.561 513 6532	43.848 634 6782	.022 805 7272	58
.555 954 1121	44.404 588 7903	.022 520 1950	59
.550 449 6159	44.955 038 4062	.022 244 4477	60
$v^n=\frac{1}{(1+i)^n}$	$a_{\overline{n}\vert}=\frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\vert}}=\frac{i}{1-v^n}$	n

ANNUALLY

If compounded *annually* nominal annual rate is

1%

SEMIANNUALLY

If compounded *semiannually* nominal annual rate is

2%

QUARTERLY

If compounded *quarterly* nominal annual rate is

4%

MONTHLY

If compounded *monthly* nominal annual rate is

12%

$i = .01$
$j_{(2)} = .02$
$j_{(4)} = .04$
$j_{(12)} = .12$

RATE

1¼%

.0125

per period

ANNUALLY
If compounded *annually* nominal annual rate is

1¼%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

2½%

QUARTERLY
If compounded *quarterly* nominal annual rate is

5%

MONTHLY
If compounded *monthly* nominal annual rate is

15%

$i = .0125$
$j_{(2)} = .025$
$j_{(4)} = .05$
$j_{(12)} = .15$

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.012 500 0000	1.000 000 0000	1.000 000 0000
2	1.025 156 2500	2.012 500 0000	.496 894 4099
3	1.037 970 7031	3.037 656 2500	.329 201 1728
4	1.050 945 3369	4.075 626 9531	.245 361 0233
5	1.064 082 1536	5.126 572 2900	.195 062 1084
6	1.077 383 1805	6.190 654 4437	.161 533 8102
7	1.090 850 4703	7.268 037 6242	.137 588 7209
8	1.104 486 1012	8.358 888 0945	.119 633 1365
9	1.118 292 1774	9.463 374 1957	.105 670 5546
10	1.132 270 8297	10.581 666 3731	.094 503 0740
11	1.146 424 2150	11.713 937 2028	.085 368 3935
12	1.160 754 5177	12.860 361 4178	.077 758 3123
13	1.175 263 9492	14.021 115 9356	.071 320 9993
14	1.189 954 7486	15.196 379 8848	.065 805 1462
15	1.204 829 1829	16.386 334 6333	.061 026 4603
16	1.219 889 5477	17.591 163 8162	.056 846 7221
17	1.235 138 1670	18.811 053 3639	.053 160 2341
18	1.250 577 3941	20.046 191 5310	.049 884 7873
19	1.266 209 6116	21.296 768 9251	.046 955 4797
20	1.282 037 2317	22.562 978 5367	.044 320 3896
21	1.298 062 6971	23.845 015 7684	.041 937 4854
22	1.314 288 4808	25.143 078 4655	.039 772 3772
23	1.330 717 0868	26.457 366 9463	.037 796 6561
24	1.347 351 0504	27.788 084 0331	.035 986 6480
25	1.364 192 9385	29.135 435 0836	.034 322 4667
26	1.381 245 3503	30.499 628 0221	.032 787 2851
27	1.398 510 9172	31.880 873 3724	.031 366 7693
28	1.415 992 3036	33.279 384 2895	.030 048 6329
29	1.433 692 2074	34.695 376 5932	.028 822 2841
30	1.451 613 3600	36.129 068 8006	.027 678 5434
31	1.469 758 5270	37.580 682 1606	.026 609 4159
32	1.488 130 5086	39.050 440 6876	.025 607 9056
33	1.506 732 1400	40.538 571 1962	.024 667 8650
34	1.525 566 2917	42.045 303 3361	.023 783 8693
35	1.544 635 8703	43.570 869 6278	.022 951 1141
36	1.563 943 8187	45.115 505 4982	.022 165 3285
37	1.583 493 1165	46.679 449 3169	.021 422 7035
38	1.603 286 7804	48.262 942 4334	.020 719 8308
39	1.623 327 8652	49.866 229 2138	.020 053 6519
40	1.643 619 4635	51.489 557 0790	.019 421 4139
41	1.664 164 7068	53.133 176 5424	.018 820 6327
42	1.684 966 7656	54.797 341 2492	.018 249 0606
43	1.706 028 8502	56.482 308 0148	.017 704 6589
44	1.727 354 2108	58.188 336 8650	.017 185 5745
45	1.748 946 1384	59.915 691 0758	.016 690 1188
46	1.770 807 9652	61.664 637 2143	.016 216 7499
47	1.792 943 0647	63.435 445 1795	.015 764 0574
48	1.815 354 8531	65.228 388 2442	.015 330 7483
49	1.838 046 7887	67.043 743 0973	.014 915 6350
50	1.861 022 3736	68.881 789 8860	.014 517 6251
51	1.884 285 1532	70.742 812 2596	.014 135 7117
52	1.907 838 7177	72.627 097 4128	.013 768 9655
53	1.931 686 7016	74.534 936 1305	.013 416 5272
54	1.955 832 7854	76.466 622 8321	.013 077 6012
55	1.980 280 6952	78.422 455 6175	.012 751 4497
56	2.005 034 2039	80.402 736 3127	.012 437 3877
57	2.030 097 1315	82.407 770 5166	.012 134 7780
58	2.055 473 3456	84.437 867 6481	.011 843 0276
59	2.081 166 7624	86.493 340 9937	.011 561 5837
60	2.107 181 3470	88.574 507 7561	.011 289 9301
n	$s=(1+i)^n$	$s_{\overline{n}\vert}=\frac{(1+i)^n-1}{i}$	$\frac{1}{s_{\overline{n}\vert}}=\frac{i}{(1+i)^n-1}$

RATE

$1\frac{1}{4}\%$

.0125

per period

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today. Periodic payment necessary to pay off a loan of $1.*	PERIODS
.987 654 3210	.987 654 3210	1.012 500 0000	1
.975 461 0578	1.963 115 3788	.509 394 4099	2
.963 418 3287	2.926 533 7074	.341 701 1728	3
.951 524 2752	3.878 057 9826	.257 861 0233	4
.939 777 0619	4.817 835 0446	.207 562 1084	5
.928 174 8760	5.746 009 9206	.174 033 8102	6
.916 715 9269	6.662 725 8475	.150 088 7209	7
.905 398 4463	7.568 124 2938	.132 133 1365	8
.894 220 6877	8.462 344 9815	.118 170 5546	9
.883 180 9262	9.345 525 9077	.107 003 0740	10
.872 277 4579	10.217 803 3656	.097 868 3935	11
.861 508 6004	11.079 311 9660	.090 258 3123	12
.850 872 6918	11.930 184 6578	.083 820 9993	13
.840 368 0906	12.770 552 7485	.078 305 1462	14
.829 993 1759	13.600 545 9244	.073 526 4603	15
.819 746 3466	14.420 292 2710	.069 346 7221	16
.809 626 0213	15.229 918 2924	.065 660 2341	17
.799 630 6384	16.029 548 9307	.062 384 7873	18
.789 758 6552	16.819 307 5859	.059 455 4797	19
.780 008 5483	17.599 316 1342	.056 820 3896	20
.770 378 8132	18.369 694 9474	.054 437 4854	21
.760 867 9636	19.130 562 9110	.052 272 3772	22
.751 474 5320	19.882 037 4430	.050 296 6561	23
.742 197 0686	20.624 234 5116	.048 486 6480	24
.733 034 1418	21.357 268 6534	.046 822 4667	25
.723 984 3376	22.081 252 9910	.045 287 2851	26
.715 046 2594	22.796 299 2504	.043 866 7693	27
.706 218 5278	23.502 517 7782	.042 548 6329	28
.697 499 7805	24.200 017 5587	.041 322 2841	29
.688 888 6721	24.888 906 2308	.040 178 5434	30
.680 383 8737	25.569 290 1045	.039 109 4159	31
.671 984 0728	26.241 274 1773	.038 107 9056	32
.663 687 9731	26.904 962 1504	.037 167 8650	33
.655 494 2944	27.560 456 4448	.036 283 8693	34
.647 401 7723	28.207 858 2171	.035 451 1141	35
.639 409 1578	28.847 267 3749	.034 665 3285	36
.631 515 2176	29.478 782 5925	.033 922 7035	37
.623 718 7334	30.102 501 3259	.033 219 8308	38
.616 018 5021	30.718 519 8281	.032 553 6519	39
.608 413 3355	31.326 933 1635	.031 921 4139	40
.600 902 0597	31.927 835 2233	.031 320 6327	41
.593 483 5158	32.521 318 7390	.030 749 0606	42
.586 156 5588	33.107 475 2978	.030 204 6589	43
.578 920 0581	33.686 395 3558	.029 685 5745	44
.571 772 8968	34.258 168 2527	.029 190 1188	45
.564 713 9722	34.822 882 2249	.028 716 7499	46
.557 742 1948	35.380 624 4196	.028 264 0574	47
.550 856 4886	35.931 480 9083	.027 830 7483	48
.544 055 7913	36.475 536 6995	.027 415 6350	49
.537 339 0531	37.012 875 7526	.027 017 6251	50
.530 705 2376	37.543 580 9902	.026 635 7117	51
.524 153 3211	38.067 734 3114	.026 268 9655	52
.517 682 2925	38.585 416 6038	.025 916 5272	53
.511 291 1530	39.096 707 7568	.025 577 6012	54
.504 978 9166	39.601 686 6734	.025 251 4497	55
.498 744 6090	40.100 431 2824	.024 937 3877	56
.492 587 2681	40.593 018 5505	.024 634 7780	57
.486 505 9438	41.079 524 4943	.024 343 0276	58
.480 499 6976	41.560 024 1919	.024 061 5837	59
.474 567 6026	42.034 591 7945	.023 789 9301	60
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\rceil} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rceil}} = \frac{i}{1-v^n}$	n

ANNUALLY
If compounded *annually* nominal annual rate is
$1\frac{1}{4}\%$

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is
$2\frac{1}{2}\%$

QUARTERLY
If compounded *quarterly* nominal annual rate is
5%

MONTHLY
If compounded *monthly* nominal annual rate is
15%

$i = .0125$
$j_{(2)} = .025$
$j_{(4)} = .05$
$j_{(12)} = .15$

RATE

1½%

.015

per period

ANNUALLY
If compounded *annually* nominal annual rate is

1½%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

3%

QUARTERLY
If compounded *quarterly* nominal annual rate is

6%

MONTHLY
If compounded *monthly* nominal annual rate is

18%

$i = .015$
$j_{(2)} = .03$
$j_{(4)} = .06$
$j_{(12)} = .18$

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.015 000 0000	1.000 000 0000	1.000 000 0000
2	1.030 225 0000	2.015 000 0000	.496 277 9156
3	1.045 678 3750	3.045 225 0000	.328 382 9602
4	1.061 363 5506	4.090 903 3750	.244 444 7860
5	1.077 284 0039	5.152 266 9256	.194 089 3231
6	1.093 443 2639	6.229 550 9295	.160 525 2146
7	1.109 844 9129	7.322 994 1935	.136 556 1645
8	1.126 492 5866	8.432 839 1064	.118 584 0246
9	1.143 389 9754	9.559 331 6929	.104 609 8234
10	1.160 540 8250	10.702 721 6683	.093 434 1779
11	1.177 948 9374	11.863 262 4934	.084 293 8442
12	1.195 618 1715	13.041 211 4308	.076 679 9929
13	1.213 552 4440	14.236 829 6022	.070 240 3574
14	1.231 755 7307	15.450 382 0463	.064 723 3186
15	1.250 232 0667	16.682 137 7770	.059 944 3557
16	1.268 985 5477	17.932 369 8436	.055 765 0778
17	1.288 020 3309	19.201 355 3913	.052 079 6569
18	1.307 340 6358	20.489 375 7221	.048 805 7818
19	1.326 950 7454	21.796 716 3580	.045 878 4701
20	1.346 855 0066	23.123 667 1033	.043 245 7359
21	1.367 057 8316	24.470 522 1099	.040 865 4950
22	1.387 563 6991	25.837 579 9415	.038 703 3152
23	1.408 377 1546	27.225 143 6407	.036 730 7520
24	1.429 502 8119	28.633 520 7953	.034 924 1020
25	1.450 945 3541	30.063 023 6072	.033 263 4539
26	1.472 709 5344	31.513 968 9613	.031 731 9599
27	1.494 800 1774	32.986 678 4957	.030 315 2680
28	1.517 222 1801	34.481 478 6732	.029 001 0765
29	1.539 980 5128	35.998 700 8533	.027 778 7802
30	1.563 080 2205	37.538 681 3661	.026 639 1883
31	1.586 526 4238	39.101 761 5865	.025 574 2954
32	1.610 324 3202	40.688 288 0103	.024 577 0970
33	1.634 479 1850	42.298 612 3305	.023 641 4375
34	1.658 996 3727	43.933 091 5155	.022 761 8855
35	1.683 881 3183	45.592 087 8882	.021 933 6303
36	1.709 139 5381	47.275 969 2065	.021 152 3955
37	1.734 776 6312	48.985 108 7446	.020 414 3673
38	1.760 798 2806	50.719 885 3758	.019 716 1329
39	1.787 210 2548	52.480 683 6564	.019 054 6298
40	1.814 018 4087	54.267 893 9113	.018 427 1017
41	1.841 228 6848	56.081 912 3199	.017 831 0610
42	1.868 847 1151	57.923 141 0047	.017 264 2571
43	1.896 879 8218	59.791 988 1198	.016 724 6488
44	1.925 333 0191	61.688 867 9416	.016 210 3801
45	1.954 213 0144	63.614 200 9607	.015 719 7604
46	1.983 526 2096	65.568 413 9751	.015 251 2458
47	2.013 279 1028	67.551 940 1848	.014 803 4238
48	2.043 478 2893	69.565 219 2875	.014 374 9996
49	2.074 130 4637	71.608 697 5768	.013 964 7841
50	2.105 242 4206	73.682 828 0405	.013 571 6832
51	2.136 821 0569	75.788 070 4611	.013 194 6887
52	2.168 873 3728	77.924 891 5180	.012 832 8700
53	2.201 406 4734	80.093 764 8908	.012 485 3664
54	2.234 427 5705	82.295 171 3642	.012 151 3812
55	2.267 943 9840	84.529 598 9346	.011 830 1756
56	2.301 963 1438	86.797 542 9186	.011 521 0635
57	2.336 492 5909	89.099 506 0624	.011 225 4068
58	2.371 539 9798	91.435 998 6534	.010 936 6116
59	2.407 113 0795	93.807 538 6332	.010 660 1241
60	2.443 219 7757	96.214 651 7126	.010 393 4274
n	$s = (1+i)^n$	$s_{\overline{n}\vert} = \frac{(1+i)^n - 1}{i}$	$\frac{1}{s_{\overline{n}\vert}} = \frac{i}{(1+i)^n - 1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today.* *Periodic payment necessary to pay off a loan of $1.*	PERIODS
.985 221 6749	.985 221 6749	1.015 000 0000	1
.970 661 7486	1.955 883 4235	.511 277 9156	2
.956 316 9937	2.912 200 4173	.343 382 9602	3
.942 184 2303	3.854 384 6476	.259 444 7860	4
.928 260 3254	4.782 644 9730	.209 089 3231	5
.914 542 1925	5.697 187 1655	.175 525 2146	6
.901 026 7907	6.598 213 9561	.151 556 1645	7
.887 711 1238	7.485 925 0799	.133 584 0246	8
.874 592 2402	8.360 517 3201	.119 609 8234	9
.861 667 2317	9.222 184 5519	.108 434 1779	10
.848 933 2332	10.071 117 7851	.099 293 8442	11
.836 387 4219	10.907 505 2070	.091 679 9929	12
.824 027 0166	11.731 532 2236	.085 240 3574	13
.811 849 2775	12.543 381 5011	.079 723 3186	14
.799 851 5049	13.343 233 0060	.074 944 3557	15
.788 031 0393	14.131 264 0453	.070 765 0778	16
.776 385 2604	14.907 649 3057	.067 079 6569	17
.764 911 5866	15.672 560 8924	.063 805 7818	18
.753 607 4745	16.426 168 3669	.060 878 4701	19
.742 470 4182	17.168 638 7851	.058 245 7359	20
.731 497 9490	17.900 136 7341	.055 865 4950	21
.720 687 6345	18.620 824 3685	.053 703 3152	22
.710 037 0783	19.330 861 4468	.051 730 7520	23
.699 543 9195	20.030 405 3663	.049 924 1020	24
.689 205 8320	20.719 611 1984	.048 263 4539	25
.679 020 5242	21.398 631 7225	.046 731 9599	26
.668 985 7381	22.067 617 4606	.045 315 2680	27
.659 099 2494	22.726 716 7100	.044 001 0765	28
.649 358 8664	23.376 075 5763	.042 778 7802	29
.639 762 4299	24.015 838 0062	.041 639 1883	30
.630 307 8127	24.646 145 8189	.040 574 2954	31
.620 992 9189	25.267 138 7379	.039 577 0970	32
.611 815 6837	25.878 954 4216	.038 641 4375	33
.602 774 0726	26.481 728 4941	.037 761 8855	34
.593 866 0814	27.075 594 5755	.036 933 6303	35
.585 089 7353	27.660 684 3109	.036 152 3955	36
.576 443 0890	28.237 127 3999	.035 414 3673	37
.567 924 2256	28.805 051 6255	.034 716 1329	38
.559 531 2568	29.364 582 8822	.034 054 6298	39
.551 262 3219	29.915 845 2042	.033 427 1017	40
.543 115 5881	30.458 960 7923	.032 831 0610	41
.535 089 2494	30.994 050 0417	.032 264 2571	42
.527 181 5265	31.521 231 5681	.031 724 6488	43
.519 390 6665	32.040 622 2346	.031 210 3804	44
.511 714 9423	32.552 337 1770	.030 719 7604	45
.504 152 6526	33.056 489 8295	.030 251 2458	46
.496 702 1207	33.553 191 9503	.029 803 4238	47
.489 361 6953	34.042 553 6456	.029 374 9996	48
.482 129 7491	34.524 683 3947	.028 964 7841	49
.475 004 6789	34.999 688 0736	.028 571 6832	50
.467 984 9053	35.467 672 9789	.028 194 6887	51
.461 068 8722	35.928 741 8511	.027 832 8700	52
.454 255 0465	36.382 996 8977	.027 485 3664	53
.447 541 9178	36.830 538 8154	.027 151 3812	54
.440 927 9978	37.271 466 8132	.026 830 1756	55
.434 411 8205	37.705 878 6337	.026 521 0635	56
.427 991 9414	38.133 870 5751	.026 223 4068	57
.421 666 9373	38.555 537 5124	.025 936 6116	58
.415 435 4062	38.970 972 9186	.025 660 1241	59
.409 295 9667	39.380 268 8853	.025 393 4274	60
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\rceil} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rceil}} = \frac{i}{1-v^n}$	n

RATE

$1\frac{1}{2}\%$

.015

per period

ANNUALLY
If compounded *annually* nominal annual rate is

$1\frac{1}{2}\%$

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

3%

QUARTERLY
If compounded *quarterly* nominal annual rate is

6%

MONTHLY
If compounded *monthly* nominal annual rate is

18%

$i = .015$
$j_{(2)} = .03$
$j_{(4)} = .06$
$j_{(12)} = .18$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today. Periodic payment necessary to pay off a loan of $1.*	PERIODS
.982 800 9828	.982 800 9828	1.017 500 0000	1
.965 897 7718	1.948 698 7546	.513 162 9492	2
.949 285 2794	2.897 984 0340	.345 067 4635	3
.932 958 5056	3.830 942 5396	.261 032 3673	4
.916 912 5362	4.747 855 0757	.210 621 4246	5
.901 142 5417	5.648 997 6174	.177 022 5565	6
.885 643 7756	6.534 641 3930	.153 030 5857	7
.870 411 5731	7.405 052 9661	.135 042 9233	8
.855 441 3495	8.260 494 3156	.121 058 1306	9
.840 728 5990	9.101 222 9146	.109 875 3442	10
.826 268 8934	9.927 491 8080	.100 730 3778	11
.812 057 8805	10.739 549 6884	.093 113 7738	12
.798 091 2830	11.537 640 9714	.086 672 8305	13
.784 364 8973	12.322 005 8687	.081 155 6179	14
.770 874 5919	13.092 880 4607	.076 377 3872	15
.757 616 3066	13.850 496 7672	.072 199 5764	16
.744 586 0507	14.595 082 8179	.068 516 2265	17
.731 779 9024	15.326 862 7203	.065 244 9244	18
.719 194 0073	16.046 056 7276	.062 320 6073	19
.706 824 5772	16.752 881 3048	.059 691 2246	20
.694 667 8891	17.447 549 1939	.057 314 6399	21
.682 720 2841	18.130 269 4780	.055 156 3782	22
.670 978 1662	18.801 247 6442	.053 187 9596	23
.659 438 0012	19.460 685 6454	.051 385 6510	24
.648 096 3157	20.108 781 9611	.049 729 5163	25
.636 949 6960	20.745 731 6571	.048 202 6865	26
.625 994 7872	21.371 726 4443	.046 790 7917	27
.615 228 2921	21.986 954 7364	.045 481 5145	28
.604 646 9701	22.591 601 7066	.044 264 2365	29
.594 247 6365	23.185 849 3431	.043 129 7549	30
.584 027 1612	23.769 876 5042	.042 070 0545	31
.573 982 4680	24.343 858 9722	.041 078 1216	32
.564 110 5336	24.907 969 5059	.040 147 7928	33
.554 408 3869	25.462 377 8928	.039 273 6297	34
.544 873 1075	26.007 251 0003	.038 450 8151	35
.535 501 8255	26.542 752 8258	.037 675 0673	36
.526 291 7204	27.069 044 5462	.036 942 5673	37
.517 240 0201	27.586 284 5663	.036 249 8979	38
.508 344 0001	28.094 628 5664	.035 593 9926	39
.499 600 9829	28.594 229 5493	.034 972 0911	40
.491 008 3370	29.085 237 8863	.034 381 7026	41
.482 563 4762	29.567 801 3625	.033 820 5735	42
.474 263 8586	30.042 065 2211	.033 286 6596	43
.466 106 9864	30.508 172 2075	.032 778 1026	44
.458 090 4043	30.966 262 6117	.032 293 2093	45
.450 211 6996	31.416 474 3113	.031 830 4336	46
.442 468 5008	31.858 942 8121	.031 388 3611	47
.434 858 4774	32.293 801 2895	.030 965 6950	48
.427 379 3390	32.721 180 6285	.030 561 2445	49
.420 028 8344	33.141 209 4629	.030 173 9139	50
.412 804 7513	33.554 014 2142	.029 802 6935	51
.405 704 9152	33.959 719 1294	.029 446 6511	52
.398 727 1894	34.358 446 3188	.029 104 9249	53
.391 869 4736	34.750 315 7925	.028 776 7169	54
.385 129 7038	35.135 445 4963	.028 461 2871	55
.378 505 8514	35.513 951 3477	.028 157 9481	56
.371 995 9228	35.885 947 2705	.027 866 0611	57
.365 597 9585	36.251 545 2290	.027 585 0310	58
.359 310 0329	36.610 855 2619	.027 314 3032	59
.353 130 2535	36.963 985 5154	.027 053 3598	60
$v^n=\frac{1}{(1+i)^n}$	$a_{\overline{n}\vert}=\frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\vert}}=\frac{i}{1-v^n}$	n

RATE 1¾%

.0175

per period

ANNUALLY
If compounded *annually* nominal annual rate is

1¾%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

3½%

QUARTERLY
If compounded *quarterly* nominal annual rate is

7%

MONTHLY
If compounded *monthly* nominal annual rate is

21%

$i = .0175$
$j_{(2)} = .035$
$j_{(4)} = .07$
$j_{(12)} = .21$

RATE

1¾%

.0175 *per period*

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.017 500 0000	1.000 000 0000	1.000 000 0000
2	1.035 306 2500	2.017 500 0000	.495 662 9492
3	1.053 424 1094	3.052 806 2500	.327 567 4635
4	1.071 859 0313	4.106 230 3594	.243 532 3673
5	1.090 616 5643	5.178 089 3907	.193 121 4246
6	1.109 702 3542	6.268 705 9550	.159 522 5565
7	1.129 122 1454	7.378 408 3092	.135 530 5857
8	1.148 881 7830	8.507 530 4546	.117 542 9233
9	1.168 987 2142	9.656 412 2376	.103 558 1306
10	1.189 444 4904	10.825 399 4517	.092 375 3442
11	1.210 259 7690	12.014 843 9421	.083 230 3778
12	1.231 439 3149	13.225 103 7111	.075 613 7738
13	1.252 989 5030	14.456 543 0261	.069 172 8305
14	1.274 916 8193	15.709 532 5290	.063 655 6179
15	1.297 227 8636	16.984 449 3483	.058 877 3872
16	1.319 929 3512	18.281 677 2119	.054 699 5764
17	1.343 028 1149	19.601 606 5631	.051 016 2265
18	1.366 531 1069	20.944 634 6779	.047 744 9244
19	1.390 445 4012	22.311 165 7848	.044 820 6073
20	1.414 778 1958	23.701 611 1860	.042 191 2246
21	1.439 536 8142	25.116 389 3818	.039 814 6399
22	1.464 728 7084	26.555 926 1960	.037 656 3782
23	1.490 361 4608	28.020 654 9044	.035 687 9596
24	1.516 442 7864	29.511 016 3652	.033 885 6510
25	1.542 980 5352	31.027 459 1516	.032 229 5163
26	1.569 982 6945	32.570 439 6868	.030 702 6865
27	1.597 457 3917	34.140 422 3813	.029 290 7917
28	1.625 412 8960	35.737 879 7730	.027 981 5145
29	1.653 857 6217	37.363 292 6690	.026 764 2365
30	1.682 800 1301	39.017 150 2907	.025 629 7549
31	1.712 249 1324	40.699 950 4208	.024 570 0545
32	1.742 213 4922	42.412 199 5532	.023 578 1216
33	1.772 702 2283	44.154 413 0453	.022 647 7928
34	1.803 724 5173	45.927 115 2736	.021 773 6297
35	1.835 289 6963	47.730 839 7909	.020 950 8151
36	1.867 407 2660	49.566 129 4873	.020 175 0673
37	1.900 086 8932	51.433 536 7533	.019 442 5673
38	1.933 338 4138	53.333 623 6465	.018 749 8979
39	1.967 171 8361	55.266 962 0603	.018 093 9926
40	2.001 597 3432	57.234 133 8963	.017 472 0911
41	2.036 625 2967	59.235 731 2395	.016 881 7026
42	2.072 266 2394	61.272 356 5362	.016 320 5735
43	2.108 530 8986	63.344 622 7756	.015 786 6596
44	2.145 430 1893	65.453 153 6742	.015 278 1026
45	2.182 975 2176	67.598 583 8635	.014 793 2093
46	2.221 177 2839	69.781 559 0811	.014 330 4336
47	2.260 047 8864	72.002 736 3650	.013 888 3611
48	2.299 598 7244	74.262 784 2514	.013 465 6950
49	2.339 841 7021	76.562 382 9758	.013 061 2445
50	2.380 788 9319	78.902 224 6779	.012 673 9139
51	2.422 452 7382	81.283 013 6097	.012 302 6935
52	2.464 845 6611	83.705 466 3479	.011 946 6511
53	2.507 980 4602	86.170 312 0090	.011 604 9249
54	2.551 870 1182	88.678 292 4691	.011 276 7169
55	2.596 527 8453	91.230 162 5874	.010 961 2871
56	2.641 967 0826	93.826 690 4326	.010 657 9481
57	2.688 201 5065	96.468 657 5152	.010 366 0611
58	2.735 245 0329	99.156 859 0217	.010 085 0310
59	2.783 111 8210	101.892 104 0546	.009 814 3032
60	2.831 816 2778	104.675 215 8756	.009 553 3598
n	$s=(1+i)^n$	$s_{\overline{n\vert}}=\frac{(1+i)^n-1}{i}$	$\frac{1}{s_{\overline{n\vert}}}=\frac{i}{(1+i)^n-1}$

ANNUALLY
If compounded *annually* nominal annual rate is

1¾%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

3½%

QUARTERLY
If compounded *quarterly* nominal annual rate is

7%

MONTHLY
If compounded *monthly* nominal annual rate is

21%

$i = .0175$
$j_{(2)} = .035$
$j_{(4)} = .07$
$j_{(12)} = .21$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today. Periodic payment necessary to pay off a loan of $1.*	PERIODS
.982 800 9828	.982 800 9828	1.017 500 0000	1
.965 897 7718	1.948 698 7546	.513 162 9492	2
.949 285 2794	2.897 984 0340	.345 067 4635	3
.932 958 5056	3.830 942 5396	.261 032 3673	4
.916 912 5362	4.747 855 0757	.210 621 4246	5
.901 142 5417	5.648 997 6174	.177 022 5565	6
.885 643 7756	6.534 641 3930	.153 030 5857	7
.870 411 5731	7.405 052 9661	.135 042 9233	8
.855 441 3495	8.260 494 3156	.121 058 1306	9
.840 728 5990	9.101 222 9146	.109 875 3442	10
.826 268 8934	9.927 491 8080	.100 730 3778	11
.812 057 8805	10.739 549 6884	.093 113 7738	12
.798 091 2830	11.537 640 9714	.086 672 8305	13
.784 364 8973	12.322 005 8687	.081 155 6179	14
.770 874 5919	13.092 880 4607	.076 377 3872	15
.757 616 3066	13.850 496 7672	.072 199 5764	16
.744 586 0507	14.595 082 8179	.068 516 2265	17
.731 779 9024	15.326 862 7203	.065 244 9244	18
.719 194 0073	16.046 056 7276	.062 320 6073	19
.706 824 5772	16.752 881 3048	.059 691 2246	20
.694 667 8891	17.447 549 1939	.057 314 6399	21
.682 720 2841	18.130 269 4780	.055 156 3782	22
.670 978 1662	18.801 247 6442	.053 187 9596	23
.659 438 0012	19.460 685 6454	.051 385 6510	24
.648 096 3157	20.108 781 9611	.049 729 5163	25
.636 949 6960	20.745 731 6571	.048 202 6865	26
.625 994 7872	21.371 726 4443	.046 790 7917	27
.615 228 2921	21.986 954 7364	.045 481 5145	28
.604 646 9701	22.591 601 7066	.044 264 2365	29
.594 247 6365	23.185 849 3431	.043 129 7549	30
.584 027 1612	23.769 876 5042	.042 070 0545	31
.573 982 4680	24.343 858 9722	.041 078 1216	32
.564 110 5336	24.907 969 5059	.040 147 7928	33
.554 408 3869	25.462 377 8928	.039 273 6297	34
.544 873 1075	26.007 251 0003	.038 450 8151	35
.535 501 8255	26.542 752 8258	.037 675 0673	36
.526 291 7204	27.069 044 5462	.036 942 5673	37
.517 240 0201	27.586 284 5663	.036 249 8979	38
.508 344 0001	28.094 628 5664	.035 593 9926	39
.499 600 9829	28.594 229 5493	.034 972 0911	40
.491 008 3370	29.085 237 8863	.034 381 7026	41
.482 563 4762	29.567 801 3625	.033 820 5735	42
.474 263 8586	30.042 065 2211	.033 286 6596	43
.466 106 9864	30.508 172 2075	.032 778 1026	44
.458 090 4043	30.966 262 6117	.032 293 2093	45
.450 211 6996	31.416 474 3113	.031 830 4336	46
.442 468 5008	31.858 942 8121	.031 388 3611	47
.434 858 4774	32.293 801 2895	.030 965 6950	48
.427 379 3390	32.721 180 6285	.030 561 2445	49
.420 028 8344	33.141 209 4629	.030 173 9139	50
.412 804 7513	33.554 014 2142	.029 802 6935	51
.405 704 9152	33.959 719 1294	.029 446 6511	52
.398 727 1894	34.358 446 3188	.029 104 9249	53
.391 869 4736	34.750 315 7925	.028 776 7169	54
.385 129 7038	35.135 445 4963	.028 461 2871	55
.378 505 8514	35.513 951 3477	.028 157 9481	56
.371 995 9228	35.885 947 2705	.027 866 0611	57
.365 597 9585	36.251 545 2290	.027 585 0310	58
.359 310 0329	36.610 855 2619	.027 314 3032	59
.353 130 2535	36.963 985 5154	.027 053 3598	60
$v^n=\frac{1}{(1+i)^n}$	$a_{\overline{n}\vert}=\frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\vert}}=\frac{i}{1-v^n}$	n

RATE

1¾%

.0175

per period

ANNUALLY
If compounded
annually
nominal annual rate is

1¾%

SEMIANNUALLY
If compounded
semiannually
nominal annual rate is

3½%

QUARTERLY
If compounded
quarterly
nominal annual rate is

7%

MONTHLY
If compounded
monthly
nominal annual rate is

21%

$i = .0175$
$j_{(2)} = .035$
$j_{(4)} = .07$
$j_{(12)} = .21$

RATE

2%

.02

per period

ANNUALLY
If compounded *annually* nominal annual rate is

2%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

4%

QUARTERLY
If compounded *quarterly* nominal annual rate is

8%

MONTHLY
If compounded *monthly* nominal annual rate is

24%

$i = .02$
$j_{(2)} = .04$
$j_{(4)} = .08$
$j_{(12)} = .24$

PERIODS	AMOUNT OF 1 *How \$1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How \$1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to \$1 at future date*
1	1.020 000 0000	1.000 000 0000	1.000 000 0000
2	1.040 400 0000	2.020 000 0000	.495 049 5050
3	1.061 208 0000	3.060 400 0000	.326 754 6726
4	1.082 432 1600	4.121 608 0000	.242 623 7527
5	1.104 080 8032	5.204 040 1600	.192 158 3941
6	1.126 162 4193	6.308 120 9632	.158 525 8123
7	1.148 685 6676	7.434 283 3825	.134 511 9561
8	1.171 659 3810	8.582 969 0501	.116 509 7991
9	1.195 092 5686	9.754 628 4311	.102 515 4374
10	1.218 994 4200	10.949 720 9997	.091 326 5279
11	1.243 374 3084	12.168 715 4197	.082 177 9428
12	1.268 241 7946	13.412 089 7281	.074 559 5966
13	1.293 606 6305	14.680 331 5227	.068 118 3527
14	1.319 478 7631	15.973 938 1531	.062 601 9702
15	1.345 868 3383	17.293 416 9162	.057 825 4723
16	1.372 785 7051	18.639 285 2545	.053 650 1259
17	1.400 241 4192	20.012 070 9596	.049 969 8408
18	1.428 246 2476	21.412 312 3788	.046 702 1022
19	1.456 811 1725	22.840 558 6264	.043 781 7663
20	1.485 947 3960	24.297 369 7989	.041 156 7181
21	1.515 666 3439	25.783 317 1949	.038 784 7689
22	1.545 979 6708	27.298 983 5388	.036 631 4005
23	1.576 899 2642	28.844 963 2096	.034 668 0976
24	1.608 437 2495	30.421 862 4738	.032 871 0973
25	1.640 605 9945	32.030 299 7232	.031 220 4384
26	1.673 418 1144	33.670 905 7177	.029 699 2308
27	1.706 886 4766	35.344 323 8321	.028 293 0862
28	1.741 024 2062	37.051 210 3087	.026 989 6716
29	1.775 844 6903	38.792 234 5149	.025 778 3552
30	1.811 361 5841	40.568 079 2052	.024 649 9223
31	1.847 588 8158	42.379 440 7893	.023 596 3472
32	1.884 540 5921	44.227 029 6051	.022 610 6073
33	1.922 231 4039	46.111 570 1972	.021 686 5311
34	1.960 676 0320	48.033 801 6011	.020 818 6728
35	1.999 889 5527	49.994 477 6331	.020 002 2092
36	2.039 887 3437	51.994 367 1858	.019 232 8526
37	2.080 685 0906	54.034 254 5295	.018 506 7789
38	2.122 298 7924	56.114 939 6201	.017 820 5663
39	2.164 744 7682	58.237 238 4125	.017 171 1439
40	2.208 039 6636	60.401 983 1807	.016 555 7478
41	2.252 200 4569	62.610 022 8444	.015 971 8836
42	2.297 244 4660	64.862 223 3012	.015 417 2945
43	2.343 189 3553	67.159 467 7673	.014 889 9334
44	2.390 053 1425	69.502 657 1226	.014 387 9391
45	2.437 854 2053	71.892 710 2651	.013 909 6161
46	2.486 611 2894	74.330 564 4704	.013 453 4159
47	2.536 343 5152	76.817 175 7598	.013 017 9220
48	2.587 070 3855	79.353 519 2750	.012 601 8355
49	2.638 811 7932	81.940 589 6605	.012 203 9639
50	2.691 588 0291	84.579 401 4537	.011 823 2097
51	2.745 419 7897	87.270 989 4828	.011 458 5615
52	2.800 328 1854	90.016 409 2724	.011 109 0856
53	2.856 334 7492	92.816 737 4579	.010 773 9189
54	2.913 461 4441	95.673 072 2070	.010 452 2618
55	2.971 730 6730	98.586 533 6512	.010 143 3732
56	3.031 165 2865	101.558 264 3242	.009 846 5645
57	3.091 788 5922	104.589 429 6107	.009 561 1957
58	3.153 624 3641	107.681 218 2029	.009 286 6706
59	3.216 696 8513	110.834 842 5669	.009 022 4335
60	3.281 030 7884	114.051 539 4183	.008 767 9658
n	$s = (1+i)^n$	$s_{\overline{n}\mid} = \frac{(1+i)^n - 1}{i}$	$\frac{1}{s_{\overline{n}\mid}} = \frac{i}{(1+i)^n - 1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today. Periodic payment necessary to pay off a loan of $1.*	PERIODS
.980 392 1569	.980 392 1569	1.020 000 0000	1
.961 168 7812	1.941 560 9381	.515 049 5050	2
.942 322 3345	2.883 883 2726	.346 754 6726	3
.923 845 4260	3.807 728 6987	.262 623 7527	4
.905 730 8098	4.713 459 5085	.212 158 3941	5
.887 971 3822	5.601 430 8907	.178 525 8123	6
.870 560 1786	6.471 991 0693	.154 511 9561	7
.853 490 3712	7.325 481 4405	.136 509 7991	8
.836 755 2659	8.162 236 7064	.122 515 4374	9
.820 348 2999	8.982 585 0062	.111 326 5279	10
.804 263 0391	9.786 848 0453	.102 177 9428	11
.788 493 1756	10.575 341 2209	.094 559 5966	12
.773 032 5251	11.348 373 7460	.088 118 3527	13
.757 875 0246	12.106 248 7706	.082 601 9702	14
.743 014 7300	12.849 263 5006	.077 825 4723	15
.728 445 8137	13.577 709 3143	.073 650 1259	16
.714 162 5625	14.291 871 8768	.069 969 8408	17
.700 159 3750	14.992 031 2517	.066 702 1022	18
.686 430 7598	15.678 462 0115	.063 781 7663	19
.672 971 3331	16.351 433 3446	.061 156 7181	20
.659 775 8168	17.011 209 1614	.058 784 7689	21
.646 839 0361	17.658 048 1974	.056 631 4005	22
.634 155 9177	18.292 204 1151	.054 668 0976	23
.621 721 4879	18.913 925 6031	.052 871 0973	24
.609 530 8705	19.523 456 4736	.051 220 4384	25
.597 579 2848	20.121 035 7584	.049 699 2308	26
.585 862 0440	20.706 897 8024	.048 293 0862	27
.574 374 5529	21.281 272 3553	.046 989 6716	28
.563 112 3068	21.844 384 6620	.045 778 3552	29
.552 070 8890	22.396 455 5510	.044 649 9223	30
.541 245 9696	22.937 701 5206	.043 596 3472	31
.530 633 3035	23.468 334 8241	.042 610 6073	32
.520 228 7289	23.988 563 5530	.041 686 5311	33
.510 028 1656	24.498 591 7187	.040 818 6728	34
.500 027 6134	24.998 619 3320	.040 002 2092	35
.490 223 1504	25.488 842 4824	.039 232 8526	36
.480 610 9317	25.969 453 4141	.038 506 7789	37
.471 187 1880	26.440 640 6021	.037 820 5663	38
.461 948 2235	26.902 588 8256	.037 171 1439	39
.452 890 4152	27.355 479 2407	.036 555 7478	40
.444 010 2110	27.799 489 4517	.035 971 8836	41
.435 304 1284	28.234 793 5801	.035 417 2945	42
.426 768 7533	28.661 562 3334	.034 889 9334	43
.418 400 7386	29.079 963 0720	.034 387 9391	44
.410 196 8025	29.490 159 8745	.033 909 6161	45
.402 153 7280	29.892 313 6025	.033 453 4159	46
.394 268 3607	30.286 581 9632	.033 017 9220	47
.386 537 6086	30.673 119 5718	.032 601 8355	48
.378 958 4398	31.052 078 0115	.032 203 9639	49
.371 527 8821	31.423 605 8937	.031 823 2097	50
.364 243 0217	31.787 848 9153	.031 458 5615	51
.357 101 0017	32.144 949 9170	.031 109 0856	52
.350 099 0212	32.495 048 9382	.030 773 9189	53
.343 234 3345	32.838 283 2728	.030 452 2618	54
.336 504 2496	33.174 787 5223	.030 143 3732	55
.329 906 1270	33.504 693 6494	.029 846 5645	56
.323 437 3794	33.828 131 0288	.029 561 1957	57
.317 095 4700	34.145 226 4988	.029 286 6706	58
.310 877 9118	34.456 104 4106	.029 022 4335	59
.304 782 2665	34.760 886 6770	.028 767 9658	60
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\rceil} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rceil}} = \frac{i}{1-v^n}$	n

RATE

2%

.02

per period

ANNUALLY
If compounded *annually* nominal annual rate is

2%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

4%

QUARTERLY
If compounded *quarterly* nominal annual rate is

8%

MONTHLY
If compounded *monthly* nominal annual rate is

24%

$i = .02$
$j_{(2)} = .04$
$j_{(4)} = .08$
$j_{(12)} = .24$

RATE

2½%

.025

per period

ANNUALLY
If compounded *annually* nominal annual rate is
2½%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is
5%

QUARTERLY
If compounded *quarterly* nominal annual rate is
10%

MONTHLY
If compounded *monthly* nominal annual rate is
30%

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.025 000 0000	1.000 000 0000	1.000 000 0000
2	1.050 625 0000	2.025 000 0000	.493 827 1605
3	1.076 890 6250	3.075 625 0000	.325 137 1672
4	1.103 812 8906	4.152 515 6250	.240 817 8777
5	1.131 408 2129	5.256 328 5156	.190 246 8609
6	1.159 693 4182	6.387 736 7285	.156 549 9711
7	1.188 685 7537	7.547 430 1467	.132 495 4296
8	1.218 402 8975	8.736 115 9004	.114 467 3458
9	1.248 862 9699	9.954 518 7979	.100 456 8900
10	1.280 084 5442	11.203 381 7679	.089 258 7632
11	1.312 086 6578	12.483 466 3121	.080 105 9558
12	1.344 888 8242	13.795 552 9699	.072 487 1270
13	1.378 511 0449	15.140 441 7941	.066 048 2708
14	1.412 973 8210	16.518 952 8390	.060 536 5249
15	1.448 298 1665	17.931 926 6599	.055 766 4561
16	1.484 505 6207	19.380 224 8264	.051 598 9886
17	1.521 618 2612	20.864 730 4471	.047 927 7699
18	1.559 658 7177	22.386 348 7083	.044 670 0805
19	1.598 650 1856	23.946 007 4260	.041 760 6151
20	1.638 616 4403	25.544 657 6116	.039 147 1287
21	1.679 581 8513	27.183 274 0519	.036 787 3273
22	1.721 571 3976	28.862 855 9032	.034 646 6061
23	1.764 610 6825	30.584 427 3008	.032 696 3781
24	1.808 725 9496	32.349 037 9833	.030 912 8204
25	1.853 944 0983	34.157 763 9329	.029 275 9210
26	1.900 292 7008	36.011 708 0312	.027 768 7467
27	1.947 800 0183	37.912 000 7320	.026 376 8722
28	1.996 495 0188	39.859 800 7503	.025 087 9327
29	2.046 407 3942	41.856 295 7690	.023 891 2685
30	2.097 567 5791	43.902 703 1633	.022 777 6407
31	2.150 006 7686	46.000 270 7424	.021 739 0025
32	2.203 756 9378	48.150 277 5109	.020 768 3123
33	2.258 850 8612	50.354 034 4487	.019 859 3819
34	2.315 322 1327	52.612 885 3099	.019 006 7508
35	2.373 205 1861	54.928 207 4426	.018 205 5823
36	2.432 535 3157	57.301 412 6287	.017 451 5767
37	2.493 348 6986	59.733 947 9444	.016 740 8992
38	2.555 682 4161	62.227 296 6430	.016 070 1180
39	2.619 574 4765	64.782 979 0591	.015 436 1534
40	2.685 063 8384	67.402 553 5356	.014 836 2332
41	2.752 190 4343	70.087 617 3740	.014 267 8555
42	2.820 995 1952	72.839 807 8083	.013 728 7567
43	2.891 520 0751	75.660 803 0035	.013 216 8833
44	2.963 808 0770	78.552 323 0786	.012 730 3683
45	3.037 903 2789	81.516 131 1556	.012 267 5106
46	3.113 850 8609	84.554 034 4345	.011 826 7568
47	3.191 697 1324	87.667 885 2954	.011 406 6855
48	3.271 489 5607	90.859 582 4277	.011 005 9938
49	3.353 276 7997	94.131 071 9884	.010 623 4847
50	3.437 108 7197	97.484 348 7881	.010 258 0569
51	3.523 036 4377	100.921 457 5078	.009 908 6956
52	3.611 112 3486	104.444 493 9455	.009 574 4635
53	3.701 390 1574	108.055 606 2942	.009 254 4944
54	3.793 924 9113	111.756 996 4515	.008 947 9856
55	3.888 773 0341	115.550 921 3628	.008 654 1932
56	3.985 992 3599	119.439 694 3969	.008 372 4260
57	4.085 642 1689	123.425 686 7568	.008 102 0412
58	4.187 783 2231	127.511 328 9257	.007 842 4404
59	4.292 477 8037	131.699 112 1489	.007 593 0656
60	4.399 789 7488	135.991 589 9526	.007 353 3959
n	$s=(1+i)^n$	$s_{\overline{n}\vert} = \frac{(1+i)^n-1}{i}$	$\frac{1}{s_{\overline{n}\vert}} = \frac{i}{(1+i)^n-1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today. Periodic payment necessary to pay off a loan of $1.*	PERIODS
.975 609 7561	.975 609 7561	1.025 000 0000	1
.951 814 3962	1.927 424 1523	.518 827 1605	2
.928 599 4109	2.856 023 5632	.350 137 1672	3
.905 950 6448	3.761 974 2080	.265 817 8777	4
.883 854 2876	4.645 828 4956	.215 246 8609	5
.862 296 8660	5.508 125 3616	.181 549 9711	6
.841 265 2351	6.349 390 5967	.157 495 4296	7
.820 746 5708	7.170 137 1675	.139 467 3458	8
.800 728 3618	7.970 865 5292	.125 456 8900	9
.781 198 4017	8.752 063 9310	.114 258 7632	10
.762 144 7822	9.514 208 7131	.105 105 9558	11
.743 555 8850	10.257 764 5982	.097 487 1270	12
.725 420 3757	10.983 184 9738	.091 048 2708	13
.707 727 1958	11.690 912 1696	.085 536 5249	14
.690 465 5568	12.381 377 7264	.080 766 4561	15
.673 624 9335	13.055 002 6599	.076 598 9886	16
.657 195 0571	13.712 197 7170	.072 927 7699	17
.641 165 9093	14.353 363 6264	.069 670 0805	18
.625 527 7164	14.978 891 3428	.066 760 6151	19
.610 270 9429	15.589 162 2856	.064 147 1287	20
.595 386 2857	16.184 548 5714	.061 787 3273	21
.580 864 6690	16.765 413 2404	.059 646 6061	22
.566 697 2380	17.332 110 4784	.057 696 3781	23
.552 875 3542	17.884 985 8326	.055 912 8204	24
.539 390 5894	18.424 376 4220	.054 275 9210	25
.526 234 7214	18.950 611 1434	.052 768 7467	26
.513 399 7282	19.464 010 8717	.051 376 8722	27
.500 877 7836	19.964 888 6553	.050 087 9327	28
.488 661 2523	20.453 549 9076	.048 891 2685	29
.476 742 6852	20.930 292 5928	.047 777 6407	30
.465 114 8148	21.395 407 4076	.046 739 0025	31
.453 770 5510	21.849 177 9586	.045 768 3123	32
.442 702 9766	22.291 880 9352	.044 859 3819	33
.431 905 3430	22.723 786 2783	.044 006 7508	34
.421 371 0664	23.145 157 3447	.043 205 5823	35
.411 093 7233	23.556 251 0680	.042 451 5767	36
.401 067 0471	23.957 318 1151	.041 740 8992	37
.391 284 9240	24.348 603 0391	.041 070 1180	38
.381 741 3893	24.730 344 4284	.040 436 1534	39
.372 430 6237	25.102 775 0521	.039 836 2332	40
.363 346 9499	25.466 122 0020	.039 267 8555	41
.354 484 8292	25.820 606 8313	.038 728 7567	42
.345 838 8578	26.166 445 6890	.038 216 8833	43
.337 403 7637	26.503 849 4527	.037 730 3683	44
.329 174 4036	26.833 023 8563	.037 267 5106	45
.321 145 7596	27.154 169 6159	.036 826 7568	46
.313 312 9362	27.467 482 5521	.036 406 6855	47
.305 671 1573	27.773 153 7094	.036 005 9938	48
.298 215 7632	28.071 369 4726	.035 623 4847	49
.290 942 2080	28.362 311 6805	.035 258 0569	50
.283 846 0566	28.646 157 7371	.034 908 6956	51
.276 922 9820	28.923 080 7191	.034 574 4635	52
.270 168 7629	29.193 249 4821	.034 254 4944	53
.263 579 2809	29.456 828 7630	.033 947 9856	54
.257 150 5180	29.713 979 2810	.033 654 1932	55
.250 878 5541	29.964 857 8351	.033 372 4260	56
.244 759 5650	30.209 617 4001	.033 102 0412	57
.238 789 8195	30.448 407 2196	.032 842 4404	58
.232 965 6776	30.681 372 8972	.032 593 0656	59
.227 283 5879	30.908 656 4851	.032 353 3959	60
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\rvert} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rvert}} = \frac{i}{1-v^n}$	n

RATE

2½%

.025

per period

ANNUALLY

If compounded *annually* nominal annual rate is

2½%

SEMIANNUALLY

If compounded *semiannually* nominal annual rate is

5%

QUARTERLY

If compounded *quarterly* nominal annual rate is

10%

MONTHLY

If compounded *monthly* nominal annual rate is

30%

$i = .025$
$j_{(2)} = .05$
$j_{(4)} = .1$
$j_{(12)} = .3$

RATE

3%

.03

per period

ANNUALLY If compounded *annually* nominal annual rate is **3%**

SEMIANNUALLY If compounded *semiannually* nominal annual rate is **6%**

QUARTERLY If compounded *quarterly* nominal annual rate is **12%**

MONTHLY If compounded *monthly* nominal annual rate is **36%**

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.030 000 0000	1.000 000 0000	1.000 000 0000
2	1.060 900 0000	2.030 000 0000	.492 610 8374
3	1.092 727 0000	3.090 900 0000	.323 530 3633
4	1.125 508 8100	4.183 627 0000	.239 027 0452
5	1.159 274 0743	5.309 135 8100	.188 354 5714
6	1.194 052 2965	6.468 409 8843	.154 597 5005
7	1.229 873 8654	7.662 462 1808	.130 506 3538
8	1.266 770 0814	8.892 336 0463	.112 456 3888
9	1.304 773 1838	10.159 106 1276	.098 433 8570
10	1.343 916 3793	11.463 879 3115	.087 230 5066
11	1.384 233 8707	12.807 795 6908	.078 077 4478
12	1.425 760 8868	14.192 029 5615	.070 462 0855
13	1.468 533 7135	15.617 790 4484	.064 029 5440
14	1.512 589 7249	17.086 324 1618	.058 526 3390
15	1.557 967 4166	18.598 913 8867	.053 766 5805
16	1.604 706 4391	20.156 881 3033	.049 610 8493
17	1.652 847 6323	21.761 587 7424	.045 952 5294
18	1.702 433 0612	23.414 435 3747	.042 708 6959
19	1.753 506 0531	25.116 868 4359	.039 813 8806
20	1.806 111 2347	26.870 374 4890	.037 215 7076
21	1.860 294 5717	28.676 485 7236	.034 871 7765
22	1.916 103 4089	30.536 780 2954	.032 747 3948
23	1.973 586 5111	32.452 883 7042	.030 813 9027
24	2.032 794 1065	34.426 470 2153	.029 047 4159
25	2.093 777 9297	36.459 264 3218	.027 427 8710
26	2.156 591 2675	38.553 042 2515	.025 938 2903
27	2.221 289 0056	40.709 633 5190	.024 564 2103
28	2.287 927 6757	42.930 922 5246	.023 293 2334
29	2.356 565 5060	45.218 850 2003	.022 114 6711
30	2.427 262 4712	47.575 415 7063	.021 019 2593
31	2.500 080 3453	50.002 678 1775	.019 998 9288
32	2.575 082 7557	52.502 758 5228	.019 046 6183
33	2.652 335 2384	55.077 841 2785	.018 156 1219
34	2.731 905 2955	57.730 176 5169	.017 321 9633
35	2.813 862 4544	60.462 081 8124	.016 539 2916
36	2.898 278 3280	63.275 944 2668	.015 803 7942
37	2.985 226 6778	66.174 222 5948	.015 111 6244
38	3.074 783 4782	69.159 449 2726	.014 459 3401
39	3.167 026 9825	72.234 232 7508	.013 843 8516
40	3.262 037 7920	75.401 259 7333	.013 262 3779
41	3.359 898 9258	78.663 297 5253	.012 712 4089
42	3.460 695 8935	82.023 196 4511	.012 191 6731
43	3.564 516 7703	85.483 892 3446	.011 698 1103
44	3.671 452 2734	89.048 409 1149	.011 229 8469
45	3.781 595 8417	92.719 861 3884	.010 785 1757
46	3.895 043 7169	96.501 457 2300	.010 362 5378
47	4.011 895 0284	100.396 500 9469	.009 960 5065
48	4.132 251 8793	104.408 395 9753	.009 577 7738
49	4.256 219 4356	108.540 647 8546	.009 213 1383
50	4.383 906 0187	112.796 867 2902	.008 865 4944
51	4.515 423 1993	117.180 773 3089	.008 533 8232
52	4.650 885 8952	121.696 196 5082	.008 217 1837
53	4.790 412 4721	126.347 082 4035	.007 914 7059
54	4.934 124 8463	131.137 494 8756	.007 625 5841
55	5.082 148 5917	136.071 619 7218	.007 349 0710
56	5.234 613 0494	141.153 768 3135	.007 084 4726
57	5.391 651 4409	146.388 381 3629	.006 831 1452
58	5.553 400 9841	151.780 032 8038	.006 588 4819
59	5.720 003 0136	157.333 433 7879	.006 355 9281
60	5.891 603 1040	163.053 436 8015	.006 132 9587
n	$s=(1+i)^n$	$s_{\overline{n}\vert}=\frac{(1+i)^n-1}{i}$	$\frac{1}{s_{\overline{n}\vert}}=\frac{i}{(1+i)^n-1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today. Periodic payment necessary to pay off a loan of $1.*	PERIODS
.970 873 7864	.970 873 7864	1.030 000 0000	1
.942 595 9091	1.913 469 6955	.522 610 8374	2
.915 141 6594	2.828 611 3549	.353 530 3633	3
.888 487 0479	3.717 098 4028	.269 027 0452	4
.862 608 7844	4.579 707 1872	.218 354 5714	5
.837 484 2567	5.417 191 4439	.184 597 5005	6
.813 091 5113	6.230 282 9552	.160 506 3538	7
.789 409 2343	7.019 692 1895	.142 456 3888	8
.766 416 7323	7.786 108 9219	.128 433 8570	9
.744 093 9149	8.530 202 8368	.117 230 5066	10
.722 421 2766	9.252 624 1134	.108 077 4478	11
.701 379 8802	9.954 003 9936	.100 462 0855	12
.680 951 3400	10.634 955 3336	.094 029 5440	13
.661 117 8058	11.296 073 1394	.088 526 3390	14
.641 861 9474	11.937 935 0868	.083 766 5805	15
.623 166 9392	12.561 102 0260	.079 610 8493	16
.605 016 4458	13.166 118 4718	.075 952 5294	17
.587 394 6076	13.753 513 0795	.072 708 6959	18
.570 286 0268	14.323 799 1063	.069 813 8806	19
.553 675 7542	14.877 474 8605	.067 215 7076	20
.537 549 2759	15.415 024 1364	.064 871 7765	21
.521 892 5009	15.936 916 6372	.062 747 3948	22
.506 691 7484	16.443 608 3857	.060 813 9027	23
.491 933 7363	16.935 542 1220	.059 047 4159	24
.477 605 5693	17.413 147 6913	.057 427 8710	25
.463 694 7274	17.876 842 4187	.055 938 2903	26
.450 189 0558	18.327 031 4745	.054 564 2103	27
.437 076 7532	18.764 108 2277	.053 293 2334	28
.424 346 3623	19.188 454 5900	.052 114 6711	29
.411 986 7595	19.600 441 3495	.051 019 2593	30
.399 987 1452	20.000 428 4946	.049 998 9288	31
.388 337 0341	20.388 765 5288	.049 046 6183	32
.377 026 2467	20.765 791 7755	.048 156 1219	33
.366 044 8997	21.131 836 6752	.047 321 9633	34
.355 383 3978	21.487 220 0731	.046 539 2916	35
.345 032 4251	21.832 252 4981	.045 803 7942	36
.334 982 9369	22.167 235 4351	.045 111 6244	37
.325 226 1524	22.492 461 5874	.044 459 3401	38
.315 753 5460	22.808 215 1334	.043 843 8516	39
.306 556 8408	23.114 771 9742	.043 262 3779	40
.297 628 0008	23.412 399 9750	.042 712 4089	41
.288 959 2240	23.701 359 1990	.042 191 6731	42
.280 542 9360	23.981 902 1349	.041 698 1103	43
.272 371 7825	24.254 273 9174	.041 229 8469	44
.264 438 6238	24.518 712 5412	.040 785 1757	45
.256 736 5279	24.775 449 0691	.040 362 5378	46
.249 258 7650	25.024 707 8341	.039 960 5065	47
.241 998 8009	25.266 706 6350	.039 577 7738	48
.234 950 2922	25.501 656 9272	.039 213 1383	49
.228 107 0798	25.729 764 0070	.038 865 4944	50
.221 463 1843	25.951 227 1913	.038 533 8232	51
.215 012 8003	26.166 239 9915	.038 217 1837	52
.208 750 2915	26.374 990 2830	.037 914 7059	53
.202 670 1859	26.577 660 4690	.037 625 5841	54
.196 767 1708	26.774 427 6398	.037 349 0710	55
.191 036 0882	26.965 463 7279	.037 084 4726	56
.185 471 9303	27.150 935 6582	.036 831 1432	57
.180 069 8352	27.331 005 4934	.036 588 4819	58
.174 825 0827	27.505 830 5761	.036 355 9281	59
.169 733 0900	27.675 563 6661	.036 132 9587	60
$v^n=\frac{1}{(1+i)^n}$	$a_{\overline{n}\rvert}=\frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rvert}}=\frac{i}{1-v^n}$	n

RATE

3%

.03

per period

ANNUALLY
If compounded *annually* nominal annual rate is

3%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is.

6%

QUARTERLY
If compounded *quarterly* nominal annual rate is

12%

MONTHLY
If compounded *monthly* nominal annual rate is

36%

$i = .03$
$j_{(2)} = .06$
$j_{(4)} = .12$
$j_{(12)} = .36$

RATE

3½%

.035

per period

ANNUALLY
If compounded *annually* nominal annual rate is

3½%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

7%

QUARTERLY
If compounded *quarterly* nominal annual rate is

14%

MONTHLY
If compounded *monthly* nominal annual rate is

42%

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.035 000 0000	1.000 000 0000	1.000 000 0000
2	1.071 225 0000	2.035 000 0000	.491 400 4914
3	1.108 717 8750	3.106 225 0000	.321 934 1806
4	1.147 523 0006	4.214 942 8750	.237 251 1395
5	1.187 686 3056	5.362 465 8756	.186 481 3732
6	1.229 255 3263	6.550 152 1813	.152 668 2087
7	1.272 279 2628	7.779 407 5076	.128 544 4938
8	1.316 809 0370	9.051 686 7704	.110 476 6465
9	1.362 897 3533	10.368 495 8073	.096 446 0051
10	1.410 598 7606	11.731 393 1606	.085 241 3679
11	1.459 969 7172	13.141 991 9212	.076 091 9658
12	1.511 068 6573	14.601 961 6385	.068 483 9493
13	1.563 956 0604	16.113 030 2958	.062 061 5726
14	1.618 694 5225	17.676 986 3562	.056 570 7287
15	1.675 348 8308	19.295 680 8786	.051 825 0694
16	1.733 986 0398	20.971 029 7094	.047 684 8306
17	1.794 675 5512	22.705 015 7492	.044 043 1317
18	1.857 489 1955	24.499 691 3004	.040 816 8408
19	1.922 501 3174	26.357 180 4960	.037 940 3252
20	1.989 788 8635	28.279 681 8133	.035 361 0768
21	2.059 431 4737	30.269 470 6768	.033 036 5870
22	2.131 511 5753	32.328 902 1505	.030 932 0742
23	2.206 114 4804	34.460 413 7257	.029 018 8042
24	2.283 328 4872	36.666 528 2061	.027 272 8303
25	2.363 244 9843	38.949 856 6933	.025 674 0354
26	2.445 958 5587	41.313 101 6776	.024 205 3963
27	2.531 567 1083	43.759 060 2363	.022 852 4103
28	2.620 171 9571	46.290 627 3446	.021 602 6452
29	2.711 877 9756	48.910 799 3017	.020 445 3825
30	2.806 793 7047	51.622 677 2772	.019 371 3316
31	2.905 031 4844	54.429 470 9819	.018 372 3998
32	3.006 707 5863	57.334 502 4663	.017 441 5048
33	3.111 942 3518	60.341 210 0526	.016 572 4221
34	3.220 860 3342	63.453 152 4044	.015 759 6583
35	3.333 690 4459	66.674 012 7386	.014 998 3473
36	3.450 266 1115	70.007 603 1845	.014 284 1628
37	3.571 025 4254	73.457 869 2959	.013 613 2454
38	3.696 011 3152	77.028 894 7213	.012 982 1414
39	3.825 371 7113	80.724 906 0365	.012 387 7506
40	3.959 259 7212	84.550 277 7478	.011 827 2823
41	4.097 833 8114	88.509 537 4690	.011 298 2174
42	4.241 257 9948	92.607 371 2804	.010 798 2765
43	4.389 702 0246	96.848 629 2752	.010 325 3914
44	4.543 341 5955	101.238 331 2998	.009 877 6816
45	4.702 358 5513	105.781 672 8953	.009 453 4334
46	4.866 941 1006	110.484 031 4467	.009 051 0817
47	5.037 284 0392	115.350 972 5473	.008 669 1944
48	5.213 588 9805	120.388 256 5864	.008 306 4580
49	5.396 064 5948	125.601 845 5670	.007 961 6665
50	5.584 926 8557	130.997 910 1618	.007 633 7096
51	5.780 399 2956	136.582 837 0175	.007 321 5641
52	5.982 713 2710	142.363 236 3131	.007 024 2854
53	6.192 108 2354	148.345 949 5840	.006 740 9997
54	6.408 832 0237	154.538 057 8195	.006 470 8979
55	6.633 141 1445	160.946 889 8432	.006 213 2297
56	6.865 301 0846	167.580 030 9877	.005 967 2981
57	7.105 586 6225	174.445 332 0722	.005 732 4549
58	7.354 282 1543	181.550 918 6948	.005 508 0966
59	7.611 682 0297	188.905 200 8491	.005 293 6605
60	7.878 090 9008	196.516 882 8788	.005 088 6213
n	$s=(1+i)^n$	$s_{\overline{n}\rceil}=\frac{(1+i)^n-1}{i}$	$\frac{1}{s_{\overline{n}\rceil}}=\frac{i}{(1+i)^n-1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today. Periodic payment necessary to pay off a loan of $1.*	PERIODS
.966 183 5749	.966 183 5749	1.035 000 0000	1
.933 510 7004	1.899 694 2752	.526 400 4914	2
.901 942 7057	2.801 636 9809	.356 934 1806	3
.871 442 2277	3.673 079 2086	.272 251 1395	4
.841 973 1669	4.515 052 3755	.221 481 3732	5
.813 500 6443	5.328 553 0198	.187 668 2087	6
.785 990 9607	6.114 543 9805	.163 544 4938	7
.759 411 5562	6.873 955 5367	.145 476 6465	8
.733 730 9722	7.607 686 5089	.131 446 0051	9
.708 918 8137	8.316 605 3226	.120 241 3679	10
.684 945 7137	9.001 551 0363	.111 091 9658	11
.661 783 2983	9.663 334 3346	.103 483 9493	12
.639 404 1529	10.302 738 4875	.097 061 5726	13
.617 781 7903	10.920 520 2778	.091 570 7287	14
.596 890 6186	11.517 410 8964	.086 825 0694	15
.576 705 9117	12.094 116 8081	.082 684 8306	16
.557 203 7794	12.651 320 5876	.079 043 1317	17
.538 361 1396	13.189 681 7271	.075 816 8408	18
.520 155 6904	13.709 837 4175	.072 940 3252	19
.502 565 8844	14.212 403 3020	.070 361 0768	20
.485 570 9028	14.697 974 2048	.068 036 5870	21
.469 150 6308	15.167 124 8355	.065 932 0742	22
.453 285 6336	15.620 410 4691	.064 018 8042	23
.437 957 1339	16.058 367 6030	.062 272 8303	24
.423 146 9893	16.481 514 5923	.060 674 0354	25
.408 837 6708	16.890 352 2631	.059 205 3963	26
.395 012 2423	17.285 364 5054	.057 852 4103	27
.381 654 3404	17.667 018 8458	.056 602 6452	28
.368 748 1550	18.035 767 0008	.055 445 3825	29
.356 278 4106	18.392 045 4114	.054 371 3316	30
.344 230 3484	18.736 275 7598	.053 372 3998	31
.332 589 7086	19.068 865 4684	.052 441 5048	32
.321 342 7136	19.390 208 1820	.051 572 4221	33
.310 476 0518	19.700 684 2338	.050 759 6583	34
.299 976 8617	20.000 661 0955	.049 998 3473	35
.289 832 7166	20.290 493 8121	.049 284 1628	36
.280 031 6102	20.570 525 4223	.048 613 2454	37
.270 561 9422	20.841 087 3645	.047 982 1414	38
.261 412 5046	21.102 499 8691	.047 387 7506	39
.252 572 4682	21.355 072 3373	.046 827 2823	40
.244 031 3702	21.599 103 7075	.046 298 2174	41
.235 779 1017	21.834 882 8092	.045 798 2765	42
.227 805 8953	22.062 688 7046	.045 325 3914	43
.220 102 3143	22.282 791 0189	.044 877 6816	44
.212 659 2409	22.495 450 2598	.044 453 4334	45
.205 467 8656	22.700 918 1254	.044 051 0817	46
.198 519 6769	22.899 437 8023	.043 669 1944	47
.191 806 4511	23.091 244 2535	.043 306 4580	48
.185 320 2426	23.276 564 4961	.042 961 6665	49
.179 053 3745	23.455 617 8706	.042 633 7096	50
.172 998 4295	23.628 616 3001	.042 321 5641	51
.167 148 2411	23.795 764 5412	.042 024 2854	52
.161 495 8851	23.957 260 4263	.041 740 9997	53
.156 034 6716	24.113 295 0978	.041 470 8979	54
.150 758 1368	24.264 053 2346	.041 213 2297	55
.145 660 0355	24.409 713 2702	.040 967 2981	56
.140 734 3339	24.550 447 6040	.040 732 4549	57
.135 975 2018	24.686 422 8058	.040 508 0966	58
.131 377 0066	24.817 799 8124	.040 293 6605	59
.126 934 3059	24.944 734 1182	.040 088 6213	60
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\rvert} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rvert}} = \frac{i}{1-v^n}$	n

RATE

3½%

.035

per period

ANNUALLY
If compounded *annually* nominal annual rate is

3½%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

7%

QUARTERLY
If compounded *quarterly* nominal annual rate is

14%

MONTHLY
If compounded *monthly* nominal annual rate is

42%

$i = .035$
$j_{(2)} = .07$
$j_{(4)} = .14$
$j_{(12)} = .42$

RATE

4%

.04

per period

ANNUALLY
If compounded *annually* nominal annual rate is

4%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

8%

QUARTERLY
If compounded *quarterly* nominal annual rate is

16%

MONTHLY
If compounded *monthly* nominal annual rate is

48%

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.040 000 0000	1.000 000 0000	1.000 000 0000
2	1.081 600 0000	2.040 000 0000	.490 196 0784
3	1.124 864 0000	3.121 600 0000	.320 348 5392
4	1.169 858 5600	4.246 464 0000	.235 490 0454
5	1.216 652 9024	5.416 322 5600	.184 627 1135
6	1.265 319 0185	6.632 975 4624	.150 761 9025
7	1.315 931 7792	7.898 294 4809	.126 609 6120
8	1.368 569 0504	9.214 226 2601	.108 527 8320
9	1.423 311 8124	10.582 795 3105	.094 492 9927
10	1.480 244 2849	12.006 107 1230	.083 290 9443
11	1.539 454 0563	13.486 351 4079	.074 149 0393
12	1.601 032 2186	15.025 805 4642	.066 552 1727
13	1.665 073 5073	16.626 837 6828	.060 143 7278
14	1.731 676 4476	18.291 911 1901	.054 668 9731
15	1.800 943 5055	20.023 587 6377	.049 941 1004
16	1.872 981 2457	21.824 531 1432	.045 819 9992
17	1.947 900 4956	23.697 512 3889	.042 198 5221
18	2.025 816 5154	25.645 412 8845	.038 993 3281
19	2.106 849 1760	27.671 229 3998	.036 138 6184
20	2.191 123 1430	29.778 078 5758	.033 581 7503
21	2.278 768 0688	31.969 201 7189	.031 280 1054
22	2.369 918 7915	34.247 969 7876	.029 198 8111
23	2.464 715 5432	36.617 888 5791	.027 309 0568
24	2.563 304 1649	39.082 604 1223	.025 586 8313
25	2.665 836 3315	41.645 908 2872	.024 011 9628
26	2.772 469 7847	44.311 744 6187	.022 567 3805
27	2.883 368 5761	47.084 214 4034	.021 238 5406
28	2.998 703 3192	49.967 582 9796	.020 012 9752
29	3.118 651 4519	52.966 286 2987	.018 879 9342
30	3.243 397 5100	56.084 937 7507	.017 830 0991
31	3.373 133 4104	59.328 335 2607	.016 855 3524
32	3.508 058 7468	62.701 468 6711	.015 948 5897
33	3.648 381 0967	66.209 527 4180	.015 103 5665
34	3.794 316 3406	69.857 908 5147	.014 314 7715
35	3.946 088 9942	73.652 224 8553	.013 577 3224
36	4.103 932 5540	77.598 313 8495	.012 886 8780
37	4.268 089 8561	81.702 246 4035	.012 239 5655
38	4.438 813 4504	85.970 336 2596	.011 631 9191
39	4.616 365 9884	90.409 149 7100	.011 060 8274
40	4.801 020 6279	95.025 515 6984	.010 523 4893
41	4.993 061 4531	99.826 536 3264	.010 017 3765
42	5.192 783 9112	104.819 597 7794	.009 540 2007
43	5.400 495 2676	110.012 381 6906	.009 089 8859
44	5.616 515 0783	115.412 876 9582	.008 664 5444
45	5.841 175 6815	121.029 392 0365	.008 262 4558
46	6.074 822 7087	126.870 567 7180	.007 882 0488
47	6.317 815 6171	132.945 390 4267	.007 521 8855
48	6.570 528 2418	139.263 206 0438	.007 180 6476
49	6.833 349 3714	145.833 734 2855	.006 857 1240
50	7.106 683 3463	152.667 083 6570	.006 550 2004
51	7.390 950 6801	159.773 767 0032	.006 258 8497
52	7.686 588 7073	167.164 717 6834	.005 982 1236
53	7.994 052 2556	174.851 306 3907	.005 719 1451
54	8.313 814 3459	182.845 358 6463	.005 469 1025
55	8.646 366 9197	191.159 172 9922	.005 231 2426
56	8.992 221 5965	199.805 539 9119	.005 004 8662
57	9.351 910 4603	208.797 761 5083	.004 789 3234
58	9.725 986 8787	218.149 671 9687	.004 584 0087
59	10.115 026 3539	227.875 658 8474	.004 388 3581
60	10.519 627 4081	237.990 685 2013	.004 201 8451
n	$s=(1+i)^n$	$s_{\overline{n}\rceil}=\frac{(1+i)^n-1}{i}$	$\frac{1}{s_{\overline{n}\rceil}}=\frac{i}{(1+i)^n-1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today.* *Periodic payment necessary to pay off a loan of $1.*	PERIODS
.961 538 4615	.961 538 4615	1.040 000 0000	1
.924 556 2130	1.886 094 6746	.530 196 0784	2
.888 996 3587	2.775 091 0332	.360 348 5392	3
.854 804 1910	3.629 895 2243	.275 490 0454	4
.821 927 1068	4.451 822 3310	.224 627 1135	5
.790 314 5257	5.242 136 8567	.190 761 9025	6
.759 917 8132	6.002 054 6699	.166 609 6120	7
.730 690 2050	6.732 744 8750	.148 527 8320	8
.702 586 7356	7.435 331 6105	.134 492 9927	9
.675 564 1688	8.110 895 7794	.123 290 9443	10
.649 580 9316	8.760 476 7109	.114 149 0393	11
.624 597 0496	9.385 073 7605	.106 552 1727	12
.600 574 0861	9.985 647 8466	.100 143 7278	13
.577 475 0828	10.563 122 9295	.094 668 9731	14
.555 264 5027	11.118 387 4322	.089 941 1004	15
.533 908 1757	11.652 295 6079	.085 819 9992	16
.513 373 2459	12.165 668 8537	.082 198 5221	17
.493 628 1210	12.659 296 9747	.078 993 3281	18
.474 642 4240	13.133 939 3988	.076 138 6184	19
.456 386 9462	13.590 326 3450	.073 581 7503	20
.438 833 6021	14.029 159 9471	.071 280 1054	21
.421 955 3867	14.451 115 3337	.069 198 8111	22
.405 726 3333	14.856 841 6671	.067 309 0568	23
.390 121 4743	15.246 963 1414	.065 586 8313	24
.375 116 8023	15.622 079 9437	.064 011 9628	25
.360 689 2329	15.982 769 1766	.062 567 3805	26
.346 816 5701	16.329 585 7467	.061 238 5406	27
.333 477 4713	16.663 063 2180	.060 012 9752	28
.320 651 4147	16.983 714 6327	.058 879 9342	29
.308 318 6680	17.292 033 3007	.057 830 0991	30
.296 460 2577	17.588 493 5583	.056 855 3524	31
.285 057 9401	17.873 551 4984	.055 948 5897	32
.274 094 1731	18.147 645 6715	.055 103 5665	33
.263 552 0896	18.411 197 7611	.054 314 7715	34
.253 415 4707	18.664 613 2318	.053 577 3224	35
.243 668 7219	18.908 281 9537	.052 886 8780	36
.234 296 8479	19.142 578 8016	.052 239 5655	37
.225 285 4307	19.367 864 2323	.051 631 9191	38
.216 620 6064	19.584 484 8388	.051 060 8274	39
.208 289 0447	19.792 773 8834	.050 523 4893	40
.200 277 9276	19.993 051 8110	.050 017 3765	41
.192 574 9303	20.185 626 7413	.049 540 2007	42
.185 168 2023	20.370 794 9436	.049 089 8859	43
.178 046 3483	20.548 841 2919	.048 664 5444	44
.171 198 4118	20.720 039 7038	.048 262 4558	45
.164 613 8575	20.884 653 5613	.047 882 0488	46
.158 282 5553	21.042 936 1166	.047 521 8855	47
.152 194 7647	21.195 130 8814	.047 180 6476	48
.146 341 1199	21.341 472 0013	.046 857 1240	49
.140 712 6153	21.482 184 6167	.046 550 2004	50
.135 300 5917	21.617 485 2083	.046 258 8497	51
.130 096 7228	21.747 581 9311	.045 982 1236	52
.125 093 0027	21.872 674 9337	.045 719 1451	53
.120 281 7333	21.992 956 6671	.045 469 1025	54
.115 655 5128	22.108 612 1799	.045 231 2426	55
.111 207 2239	22.219 819 4037	.045 004 8662	56
.106 930 0229	22.326 749 4267	.044 789 3234	57
.102 817 3297	22.429 566 7564	.044 584 0087	58
.098 862 8171	22.528 429 5735	.044 388 3581	59
.095 060 4010	22.623 489 9745	.044 201 8451	60
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\rvert} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rvert}} = \frac{i}{1-v^n}$	n

RATE

4%

.04

per period

ANNUALLY

If compounded *annually* nominal annual rate is

4%

SEMIANNUALLY

If compounded *semiannually* nominal annual rate is

8%

QUARTERLY

If compounded *quarterly* nominal annual rate is

16%

MONTHLY

If compounded *monthly* nominal annual rate is

48%

$i = .04$
$j_{(2)} = .08$
$j_{(4)} = .16$
$j_{(12)} = .48$

RATE

4½%

.045

per period

ANNUALLY
If compounded *annually* nominal annual rate is

4½%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

9%

QUARTERLY
If compounded *quarterly* nominal annual rate is

18%

MONTHLY
If compounded *monthly* nominal annual rate is

54%

$i = .045$
$j_{(2)} = .09$
$j_{(4)} = .18$
$j_{(12)} = .54$

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.045 000 0000	1.000 000 0000	1.000 000 0000
2	1.092 025 0000	2.045 000 0000	.488 997 5550
3	1.141 166 1250	3.137 025 0000	.318 773 3601
4	1.192 518 6006	4.278 191 1250	.233 743 6479
5	1.246 181 9377	5.470 709 7256	.182 791 6395
6	1.302 260 1248	6.716 891 6633	.148 878 3875
7	1.360 861 8305	8.019 151 7881	.124 701 4680
8	1.422 100 6128	9.380 013 6186	.106 609 6533
9	1.486 095 1404	10.802 114 2314	.092 574 4700
10	1.552 969 4217	12.288 209 3718	.081 378 8217
11	1.622 853 0457	13.841 178 7936	.072 248 1817
12	1.695 881 4328	15.464 031 8393	.064 666 1886
13	1.772 196 0972	17.159 913 2721	.058 275 3528
14	1.851 944 9216	18.932 109 3693	.052 820 3160
15	1.935 282 4431	20.784 054 2909	.048 113 8081
16	2.022 370 1530	22.719 336 7340	.044 015 3694
17	2.113 376 8099	24.741 706 8870	.040 417 5833
18	2.208 478 7664	26.855 083 6970	.037 236 8975
19	2.307 860 3108	29.063 562 4633	.034 407 3443
20	2.411 714 0248	31.371 422 7742	.031 876 1443
21	2.520 241 1560	33.783 136 7990	.029 600 5669
22	2.633 652 0080	36.303 377 9550	.027 545 6461
23	2.752 166 3483	38.937 029 9629	.025 682 4930
24	2.876 013 8340	41.689 196 3113	.023 987 0299
25	3.005 434 4565	44.565 210 1453	.022 439 0280
26	3.140 679 0071	47.570 644 6018	.021 021 3674
27	3.282 009 5624	50.711 323 6089	.019 719 4616
28	3.429 699 9927	53.993 333 1713	.018 520 8051
29	3.584 036 4924	57.423 033 1640	.017 414 6147
30	3.745 318 1345	61.007 069 6564	.016 391 5429
31	3.913 857 4506	64.752 387 7909	.015 443 4459
32	4.089 981 0359	68.666 245 2415	.014 563 1962
33	4.274 030 1825	72.756 226 2774	.013 744 5281
34	4.466 361 5407	77.030 256 4599	.012 981 9119
35	4.667 347 8100	81.496 618 0005	.012 270 4478
36	4.877 378 4615	86.163 965 8106	.011 605 7796
37	5.096 860 4922	91.041 344 2720	.010 984 0206
38	5.326 219 2144	96.138 204 7643	.010 401 6920
39	5.565 899 0790	101.464 423 9787	.009 855 6712
40	5.816 364 5376	107.030 323 0577	.009 343 1466
41	6.078 100 9418	112.846 687 5953	.008 861 5804
42	6.351 615 4842	118.924 788 5371	.008 408 6759
43	6.637 438 1810	125.276 404 0213	.007 982 3492
44	6.936 122 8991	131.913 842 2022	.007 580 7056
45	7.248 248 4296	138.849 965 1013	.007 202 0184
46	7.574 419 6089	146.098 213 5309	.006 844 7107
47	7.915 268 4913	153.672 633 1398	.006 507 3395
48	8.271 455 5734	161.587 901 6311	.006 188 5821
49	8.643 671 0742	169.859 357 2045	.005 887 2235
50	9.032 636 2725	178.503 028 2787	.005 602 1459
51	9.439 104 9048	187.535 664 5512	.005 332 3191
52	9.863 864 6255	196.974 769 4560	.005 076 7923
53	10.307 738 5337	206.838 634 0815	.004 834 6867
54	10.771 586 7677	217.146 372 6152	.004 605 1886
55	11.256 308 1722	227.917 959 3829	.004 387 5437
56	11.762 842 0400	239.174 267 5551	.004 181 0518
57	12.292 169 9318	250.937 109 5951	.003 985 0622
58	12.845 317 5787	263.229 279 5269	.003 798 9695
59	13.423 356 8698	276.074 597 1056	.003 622 2094
60	14.027 407 9289	289.497 953 9753	.003 454 2558
n	$s=(1+i)^n$	$s_{\overline{n}\rceil}=\frac{(1+i)^n-1}{i}$	$\frac{1}{s_{\overline{n}\rceil}}=\frac{i}{(1+i)^n-1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today. Periodic payment necessary to pay off a loan of $1.*	PERIODS
.956 937 7990	.956 937 7990	1.045 000 0000	1
.915 729 9512	1.872 667 7503	.533 997 5550	2
.876 296 6041	2.748 964 3543	.363 773 3601	3
.838 561 3436	3.587 525 6979	.278 743 6479	4
.802 451 0465	4.389 976 7444	.227 791 6395	5
.767 895 7383	5.157 872 4827	.193 878 3875	6
.734 828 4577	5.892 700 9404	.169 701 4680	7
.703 185 1270	6.595 886 0674	.151 609 6533	8
.672 904 4277	7.268 790 4951	.137 574 4700	9
.643 927 6820	7.912 718 1771	.126 378 8217	10
.616 198 7388	8.528 916 9159	.117 248 1817	11
.589 663 8649	9.118 580 7808	.109 666 1886	12
.564 271 6410	9.682 852 4218	.103 275 3528	13
.539 972 8622	10.222 825 2840	.097 820 3160	14
.516 720 4423	10.739 545 7263	.093 113 8081	15
.494 469 3228	11.234 015 0491	.089 015 3694	16
.473 176 3854	11.707 191 4346	.085 417 5833	17
.452 800 3688	12.159 991 8034	.082 236 8975	18
.433 301 7884	12.593 293 5918	.079 407 3443	19
.414 642 8597	13.007 936 4515	.076 876 1443	20
.396 787 4255	13.404 723 8770	.074 600 5669	21
.379 700 8857	13.784 424 7627	.072 545 6461	22
.363 350 1298	14.147 774 8925	.070 682 4930	23
.347 703 4735	14.495 478 3660	.068 987 0299	24
.332 730 5967	14.828 208 9627	.067 439 0280	25
.318 402 4849	15.146 611 4476	.066 021 3674	26
.304 691 3731	15.451 302 8206	.064 719 4616	27
.291 570 6919	15.742 873 5126	.063 520 8051	28
.279 015 0162	16.021 888 5288	.062 414 6147	29
.267 000 0155	16.288 888 5443	.061 391 5429	30
.255 502 4072	16.544 390 9515	.060 443 4459	31
.244 499 9112	16.788 890 8627	.059 563 1962	32
.233 971 2069	17.022 862 0695	.058 744 5281	33
.223 895 8917	17.246 757 9613	.057 981 9119	34
.214 254 4419	17.461 012 4031	.057 270 4478	35
.205 028 1740	17.666 040 5772	.056 605 7796	36
.196 199 2096	17.862 239 7868	.055 984 0206	37
.187 750 4398	18.049 990 2266	.055 401 6920	38
.179 665 4926	18.229 655 7192	.054 855 6712	39
.171 928 7011	18.401 584 4203	.054 343 1466	40
.164 525 0728	18.566 109 4931	.053 861 5804	41
.157 440 2611	18.723 549 7542	.053 408 6759	42
.150 660 5369	18.874 210 2911	.052 982 3492	43
.144 172 7626	19.018 383 0536	.052 580 7056	44
.137 964 3661	19.156 347 4198	.052 202 0184	45
.132 023 3169	19.288 370 7366	.051 844 7107	46
.126 338 1023	19.414 708 8389	.051 507 3395	47
.120 897 7055	19.535 606 5444	.051 188 5821	48
.115 691 5842	19.651 298 1286	.050 887 2235	49
.110 709 6500	19.762 007 7785	.050 602 1459	50
.105 942 2488	19.867 950 0273	.050 332 3191	51
.101 380 1424	19.969 330 1697	.050 076 7923	52
.097 014 4903	20.066 344 6600	.049 834 6867	53
.092 836 8328	20.159 181 4928	.049 605 1886	54
.088 839 0745	20.248 020 5673	.049 387 5437	55
.085 013 4684	20.333 034 0357	.049 181 0518	56
.081 352 6013	20.414 386 6370	.048 985 0622	57
.077 849 3793	20.492 236 0163	.048 798 9695	58
.074 497 0137	20.566 733 0299	.048 622 2094	59
.071 289 0083	20.638 022 0382	.048 454 2558	60
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\rceil} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rceil}} = \frac{i}{1-v^n}$	n

RATE

4½%

.045

per period

ANNUALLY
If compounded *annually* nominal annual rate is

4½%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

9%

QUARTERLY
If compounded *quarterly* nominal annual rate is

18%

MONTHLY
If compounded *monthly* nominal annual rate is

54%

$i = .045$
$j_{(2)} = .09$
$j_{(4)} = .18$
$j_{(12)} = .54$

RATE

5%

.05

per period

ANNUALLY If compounded *annually* nominal annual rate is **5%**

SEMIANNUALLY If compounded *semiannually* nominal annual rate is **10%**

QUARTERLY If compounded *quarterly* nominal annual rate is **20%**

MONTHLY If compounded *monthly* nominal annual rate is **60%**

$i = .05$
$j_{(2)} = .1$
$j_{(4)} = .2$
$j_{(12)} = .6$

PERIODS	AMOUNT OF 1 *How \$1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How \$1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to \$1 at future date.*
1	1.050 000 0000	1.000 000 0000	1.000 000 0000
2	1.102 500 0000	2.050 000 0000	.487 804 8780
3	1.157 625 0000	3.152 500 0000	.317 208 5646
4	1.215 506 2500	4.310 125 0000	.232 011 8326
5	1.276 281 5625	5.525 631 2500	.180 974 7981
6	1.340 095 6406	6.801 912 8125	.147 017 4681
7	1.407 100 4227	8.142 008 4531	.122 819 8184
8	1.477 455 4438	9.549 108 8758	.104 721 8136
9	1.551 328 2160	11.026 564 3196	.090 690 0800
10	1.628 894 6268	12.577 892 5355	.079 504 5750
11	1.710 339 3581	14.206 787 1623	.070 388 8915
12	1.795 856 3260	15.917 126 5204	.062 825 4100
13	1.885 649 1423	17.712 982 8465	.056 455 7652
14	1.979 931 5994	19.598 631 9888	.051 023 9695
15	2.078 928 1794	21.578 563 5882	.046 342 2876
16	2.182 874 5884	23.657 491 7676	.042 269 9080
17	2.292 018 3178	25.840 366 3560	.038 699 1417
18	2.406 619 2337	28.132 384 6738	.035 546 2223
19	2.526 950 1954	30.539 003 9075	.032 745 0104
20	2.653 297 7051	33.065 954 1029	.030 242 5872
21	2.785 962 5904	35.719 251 8080	.027 996 1071
22	2.925 260 7199	38.505 214 3984	.025 970 5086
23	3.071 523 7559	41.430 475 1184	.024 136 8219
24	3.225 099 9437	44.501 998 8743	.022 470 9008
25	3.386 354 9409	47.727 098 8180	.020 952 4573
26	3.555 672 6879	51.113 453 7589	.019 564 3207
27	3.733 456 3223	54.669 126 4468	.018 291 8599
28	3.920 129 1385	58.402 582 7692	.017 122 5304
29	4.116 135 5954	62.322 711 9076	.016 045 5149
30	4.321 942 3752	66.438 847 5030	.015 051 4351
31	4.538 039 4939	70.760 789 8782	.014 132 1204
32	4.764 941 4686	75.298 829 3721	.013 280 4189
33	5.003 188 5420	80.063 770 8407	.012 490 0437
34	5.253 347 9691	85.066 959 3827	.011 755 4454
35	5.516 015 3676	90.320 307 3518	.011 071 7072
36	5.791 816 1360	95.836 322 7194	.010 434 4571
37	6.081 406 9428	101.628 138 8554	.009 839 7945
38	6.385 477 2899	107.709 545 7982	.009 284 2282
39	6.704 751 1544	114.095 023 0881	.008 764 6242
40	7.039 988 7121	120.799 774 2425	.008 278 1612
41	7.391 988 1477	127.839 762 9546	.007 822 2924
42	7.761 587 5551	135.231 751 1023	.007 394 7131
43	8.149 666 9329	142.993 338 6575	.006 993 3328
44	8.557 150 2795	151.143 005 5903	.006 616 2506
45	8.985 007 7935	159.700 155 8699	.006 261 7347
46	9.434 258 1832	168.685 163 6633	.005 928 2036
47	9.905 971 0923	178.119 421 8465	.005 614 2109
48	10.401 269 6469	188.025 392 9388	.005 318 4306
49	10.921 333 1293	198.426 662 5858	.005 039 6453
50	11.467 399 7858	209.347 995 7151	.004 776 7355
51	12.040 769 7750	220.815 395 5008	.004 528 6697
52	12.642 808 2638	232.856 165 2759	.004 294 4966
53	13.274 948 6770	245.498 973 5397	.004 073 3368
54	13.938 696 1108	258.773 922 2166	.003 864 3770
55	14.635 630 9164	272.712 618 3275	.003 666 8637
56	15.367 412 4622	287.348 249 2439	.003 480 0978
57	16.135 783 0853	302.715 661 7060	.003 303 4300
58	16.942 572 2396	318.851 444 7913	.003 136 2568
59	17.789 700 8515	335.794 017 0309	.002 978 0161
60	18.679 185 8941	353.583 717 8825	.002 828 1845
n	$s=(1+i)^n$	$s_{\overline{n}\rvert} = \frac{(1+i)^n-1}{i}$	$\frac{1}{s_{\overline{n}\rvert}} = \frac{i}{(1+i)^n-1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today.* *Periodic payment necessary to pay off a loan of $1.*	PERIODS
.952 380 9524	.952 380 9524	1.050 000 0000	1
.907 029 4785	1.859 410 4308	.537 804 8780	2
.863 837 5985	2.723 248 0294	.367 208 5646	3
.822 702 4748	3.545 950 5042	.282 011 8326	4
.783 526 1665	4.329 476 6706	.230 974 7981	5
.746 215 3966	5.075 692 0673	.197 017 4681	6
.710 681 3301	5.786 373 3974	.172 819 8184	7
.676 839 3620	6.463 212 7594	.154 721 8136	8
.644 608 9162	7.107 821 6756	.140 690 0800	9
.613 913 2535	7.721 734 9292	.129 504 5750	10
.584 679 2891	8.306 414 2183	.120 388 8915	11
.556 837 4182	8.863 251 6364	.112 825 4100	12
.530 321 3506	9.393 572 9871	.106 455 7652	13
.505 067 9530	9.898 640 9401	.101 023 9695	14
.481 017 0981	10.379 658 0382	.096 342 2876	15
.458 111 5220	10.837 769 5602	.092 269 9080	16
.436 296 6876	11.274 066 2478	.088 699 1417	17
.415 520 6549	11.689 586 9027	.085 546 2223	18
.395 733 9570	12.085 320 8597	.082 745 0104	19
.376 889 4829	12.462 210 3425	.080 242 5872	20
.358 942 3646	12.821 152 7072	.077 996 1071	21
.341 849 8711	13.163 002 5783	.075 970 5086	22
.325 571 3058	13.488 573 8841	.074 136 8219	23
.310 067 9103	13.798 641 7943	.072 470 9008	24
.295 302 7717	14.093 944 5660	.070 952 4573	25
.281 240 7350	14.375 185 3010	.069 564 3207	26
.267 848 3190	14.643 033 6200	.068 291 8599	27
.255 093 6371	14.898 127 2571	.067 122 5304	28
.242 946 3211	15.141 073 5782	.066 045 5149	29
.231 377 4487	15.372 451 0269	.065 051 4351	30
.220 359 4749	15.592 810 5018	.064 132 1204	31
.209 866 1666	15.802 676 6684	.063 280 4189	32
.199 872 5396	16.002 549 2080	.062 490 0437	33
.190 354 7996	16.192 904 0076	.061 755 4454	34
.181 290 2854	16.374 194 2929	.061 071 7072	35
.172 657 4146	16.546 851 7076	.060 434 4571	36
.164 435 6330	16.711 287 3405	.059 839 7945	37
.156 605 3647	16.867 892 7053	.059 284 2282	38
.149 147 9664	17.017 040 6717	.058 764 6242	39
.142 045 6823	17.159 086 3540	.058 278 1612	40
.135 281 6022	17.294 367 9562	.057 822 2924	41
.128 839 6211	17.423 207 5773	.057 394 7131	42
.122 704 4011	17.545 911 9784	.056 993 3328	43
.116 861 3344	17.662 773 3128	.056 616 2506	44
.111 296 5089	17.774 069 8217	.056 261 7347	45
.105 996 6752	17.880 066 4968	.055 928 2036	46
.100 949 2144	17.981 015 7113	.055 614 2109	47
.096 142 1090	18.077 157 8203	.055 318 4306	48
.091 563 9133	18.168 721 7336	.055 039 6453	49
.087 203 7270	18.255 925 4606	.054 776 7355	50
.083 051 1685	18.338 976 6291	.054 528 6697	51
.079 096 3510	18.418 072 9801	.054 294 4966	52
.075 329 8581	18.493 402 8382	.054 073 3368	53
.071 742 7220	18.565 145 5602	.053 864 3770	54
.068 326 4019	18.633 471 9621	.053 666 8637	55
.065 072 7637	18.698 544 7258	.053 480 0978	56
.061 974 0607	18.760 518 7865	.053 303 4300	57
.059 022 9149	18.819 541 7014	.053 136 2568	58
.056 212 2999	18.875 754 0013	.052 978 0161	59
.053 535 5237	18.929 289 5251	.052 828 1845	60
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\vert} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\vert}} = \frac{i}{1-v^n}$	n

RATE

5%

.05

per period

ANNUALLY

If compounded *annually* nominal annual rate is

5%

SEMIANNUALLY

If compounded *semiannually* nominal annual rate is

QI

MC

If

$i = .05$
$j_{(2)} = .1$
$j_{(4)} = .2$
$j_{(12)} = .6$

RATE

5%

.05

per period

ANNUALLY
If compounded *annually* nominal annual rate is

5%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

10%

QUARTERLY
If compounded *quarterly* nominal annual rate is

20%

MONTHLY
If compounded *monthly* nominal annual rate is

60%

$i = .05$
$j_{(2)} = .1$
$j_{(4)} = .2$
$j_{(12)} = .6$

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.050 000 0000	1.000 000 0000	1.000 000 0000
2	1.102 500 0000	2.050 000 0000	.487 804 8780
3	1.157 625 0000	3.152 500 0000	.317 208 5646
4	1.215 506 2500	4.310 125 0000	.232 011 8326
5	1.276 281 5625	5.525 631 2500	.180 974 7981
6	1.340 095 6406	6.801 912 8125	.147 017 4681
7	1.407 100 4227	8.142 008 4531	.122 819 8184
8	1.477 455 4438	9.549 108 8758	.104 721 8136
9	1.551 328 2160	11.026 564 3196	.090 690 0800
10	1.628 894 6268	12.577 892 5355	.079 504 5750
11	1.710 339 3581	14.206 787 1623	.070 388 8915
12	1.795 856 3260	15.917 126 5204	.062 825 4100
13	1.885 649 1423	17.712 982 8465	.056 455 7652
14	1.979 931 5994	19.598 631 9888	.051 023 9695
15	2.078 928 1794	21.578 563 5882	.046 342 2876
16	2.182 874 5884	23.657 491 7676	.042 269 9080
17	2.292 018 3178	25.840 366 3560	.038 699 1417
18	2.406 619 2337	28.132 384 6738	.035 546 2223
19	2.526 950 1954	30.539 003 9075	.032 745 0104
20	2.653 297 7051	33.065 954 1029	.030 242 5872
21	2.785 962 5904	35.719 251 8080	.027 996 1071
22	2.925 260 7199	38.505 214 3984	.025 970 5086
23	3.071 523 7559	41.430 475 1184	.024 136 8219
24	3.225 099 9437	44.501 998 8743	.022 470 9008
25	3.386 354 9409	47.727 098 8180	.020 952 4573
26	3.555 672 6879	51.113 453 7589	.019 564 3207
27	3.733 456 3223	54.669 126 4468	.018 291 8599
28	3.920 129 1385	58.402 582 7692	.017 122 5304
29	4.116 135 5954	62.322 711 9076	.016 045 5149
30	4.321 942 3752	66.438 847 5030	.015 051 4351
31	4.538 039 4939	70.760 789 8782	.014 132 1204
32	4.764 941 4686	75.298 829 3721	.013 280 4189
33	5.003 188 5420	80.063 770 8407	.012 490 0437
34	5.253 347 9691	85.066 959 3827	.011 755 4454
35	5.516 015 3676	90.320 307 3518	.011 071 7072
36	5.791 816 1360	95.836 322 7194	.010 434 4571
37	6.081 406 9428	101.628 138 8554	.009 839 7945
38	6.385 477 2899	107.709 545 7982	.009 284 2282
39	6.704 751 1544	114.095 023 0881	.008 764 6242
40	7.039 988 7121	120.799 774 2425	.008 278 1612
41	7.391 988 1477	127.839 762 9546	.007 822 2924
42	7.761 587 5551	135.231 751 1023	.007 394 7131
43	8.149 666 9329	142.993 338 6575	.006 993 3328
44	8.557 150 2795	151.143 005 5903	.006 616 2506
45	8.985 007 7935	159.700 155 8699	.006 261 7347
46	9.434 258 1832	168.685 163 6633	.005 928 2036
47	9.905 971 0923	178.119 421 8465	.005 614 2109
48	10.401 269 6469	188.025 392 9388	.005 318 4306
49	10.921 333 1293	198.426 662 5858	.005 039 6453
50	11.467 399 7858	209.347 995 7151	.004 776 7355
51	12.040 769 7750	220.815 395 5008	.004 528 6697
52	12.642 808 2638	232.856 165 2759	.004 294 4966
53	13.274 948 6770	245.498 973 5397	.004 073 3368
54	13.938 696 1108	258.773 922 2166	.003 864 3770
55	14.635 630 9164	272.712 618 3275	.003 666 8637
56	15.367 412 4622	287.348 249 2439	.003 480 0978
57	16.135 783 0853	302.715 661 7060	.003 303 4300
58	16.942 572 2396	318.851 444 7913	.003 136 2568
59	17.789 700 8515	335.794 017 0309	.002 978 0161
60	18.679 185 8941	353.583 717 8825	.002 828 1845
n	$s=(1+i)^n$	$s_{\overline{n}\vert}=\frac{(1+i)^n-1}{i}$	$\frac{1}{s_{\overline{n}\vert}}=\frac{i}{(1+i)^n-1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today.* *Periodic payment necessary to pay off a loan of $1.*	PERIODS
.952 380 9524	.952 380 9524	1.050 000 0000	1
.907 029 4785	1.859 410 4308	.537 804 8780	2
.863 837 5985	2.723 248 0294	.367 208 5646	3
.822 702 4748	3.545 950 5042	.282 011 8326	4
.783 526 1665	4.329 476 6706	.230 974 7981	5
.746 215 3966	5.075 692 0673	.197 017 4681	6
.710 681 3301	5.786 373 3974	.172 819 8184	7
.676 839 3620	6.463 212 7594	.154 721 8136	8
.644 608 9162	7.107 821 6756	.140 690 0800	9
.613 913 2535	7.721 734 9292	.129 504 5750	10
.584 679 2891	8.306 414 2183	.120 388 8915	11
.556 837 4182	8.863 251 6364	.112 825 4100	12
.530 321 3506	9.393 572 9871	.106 455 7652	13
.505 067 9530	9.898 640 9401	.101 023 9695	14
.481 017 0981	10.379 658 0382	.096 342 2876	15
.458 111 5220	10.837 769 5602	.092 269 9080	16
.436 296 6876	11.274 066 2478	.088 699 1417	17
.415 520 6549	11.689 586 9027	.085 546 2223	18
.395 733 9570	12.085 320 8597	.082 745 0104	19
.376 889 4829	12.462 210 3425	.080 242 5872	20
.358 942 3646	12.821 152 7072	.077 996 1071	21
.341 849 8711	13.163 002 5783	.075 970 5086	22
.325 571 3058	13.488 573 8841	.074 136 8219	23
.310 067 9103	13.798 641 7943	.072 470 9008	24
.295 302 7717	14.093 944 5660	.070 952 4573	25
.281 240 7350	14.375 185 3010	.069 564 3207	26
.267 848 3190	14.643 033 6200	.068 291 8599	27
.255 093 6371	14.898 127 2571	.067 122 5304	28
.242 946 3211	15.141 073 5782	.066 045 5149	29
.231 377 4487	15.372 451 0269	.065 051 4351	30
.220 359 4749	15.592 810 5018	.064 132 1204	31
.209 866 1666	15.802 676 6684	.063 280 4189	32
.199 872 5396	16.002 549 2080	.062 490 0437	33
.190 354 7996	16.192 904 0076	.061 755 4454	34
.181 290 2854	16.374 194 2929	.061 071 7072	35
.172 657 4146	16.546 851 7076	.060 434 4571	36
.164 435 6330	16.711 287 3405	.059 839 7945	37
.156 605 3647	16.867 892 7053	.059 284 2282	38
.149 147 9664	17.017 040 6717	.058 764 6242	39
.142 045 6823	17.159 086 3540	.058 278 1612	40
.135 281 6022	17.294 367 9562	.057 822 2924	41
.128 839 6211	17.423 207 5773	.057 394 7131	42
.122 704 4011	17.545 911 9784	.056 993 3328	43
.116 861 3344	17.662 773 3128	.056 616 2506	44
.111 296 5089	17.774 069 8217	.056 261 7347	45
.105 996 6752	17.880 066 4968	.055 928 2036	46
.100 949 2144	17.981 015 7113	.055 614 2109	47
.096 142 1090	18.077 157 8203	.055 318 4306	48
.091 563 9133	18.168 721 7336	.055 039 6453	49
.087 203 7270	18.255 925 4606	.054 776 7355	50
.083 051 1685	18.338 976 6291	.054 528 6697	51
.079 096 3510	18.418 072 9801	.054 294 4966	52
.075 329 8581	18.493 402 8382	.054 073 3368	53
.071 742 7220	18.565 145 5602	.053 864 3770	54
.068 326 4019	18.633 471 9621	.053 666 8637	55
.065 072 7637	18.698 544 7258	.053 480 0978	56
.061 974 0607	18.760 518 7865	.053 303 4300	57
.059 022 9149	18.819 541 7014	.053 136 2568	58
.056 212 2999	18.875 754 0013	.052 978 0161	59
.053 535 5237	18.929 289 5251	.052 828 1845	60
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\rvert} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rvert}} = \frac{i}{1-v^n}$	n

RATE

5%

.05

per period

ANNUALLY

If compounded *annually* nominal annual rate is

5%

SEMIANNUALLY

If compounded *semiannually* nominal annual rate is

10%

QUARTERLY

If compounded *quarterly* nominal annual rate is

20%

MONTHLY

If compounded *monthly* nominal annual rate is

60%

$i = .05$
$j_{(2)} = .1$
$j_{(4)} = .2$
$j_{(12)} = .6$

RATE

5½%

.055

per period

ANNUALLY
If compounded *annually* nominal annual rate is

5½%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

11%

QUARTERLY
If compounded *quarterly* nominal annual rate is

22%

MONTHLY
If compounded *monthly* nominal annual rate is

66%

$i = .055$
$j_{(2)} = .11$
$j_{(4)} = .22$
$j_{(12)} = .66$

PERIODS	AMOUNT OF 1 *How $1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How $1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to $1 at future date*
1	1.055 000 0000	1.000 000 0000	1.000 000 0000
2	1.113 025 0000	2.055 000 0000	.486 618 0049
3	1.174 241 3750	3.168 025 0000	.315 654 0747
4	1.238 824 6506	4.342 266 3750	.230 294 4853
5	1.306 960 0064	5.581 091 0256	.179 176 4362
6	1.378 842 8068	6.888 051 0320	.145 178 9476
7	1.454 679 1611	8.266 893 8388	.120 964 4178
8	1.534 686 5150	9.721 572 9999	.102 864 0118
9	1.619 094 2733	11.256 259 5149	.088 839 4585
10	1.708 144 4584	12.875 353 7882	.077 667 7687
11	1.802 092 4036	14.583 498 2466	.068 570 6532
12	1.901 207 4858	16.385 590 6502	.061 029 2312
13	2.005 773 8975	18.286 798 1359	.054 684 2587
14	2.116 091 4618	20.292 572 0334	.049 279 1154
15	2.232 476 4922	22.408 663 4952	.044 625 5976
16	2.355 262 6993	24.641 139 9875	.040 582 5380
17	2.484 802 1478	26.996 402 6868	.037 041 9723
18	2.621 466 2659	29.481 204 8346	.033 919 9163
19	2.765 646 9105	32.102 671 1005	.031 150 0559
20	2.917 757 4906	34.868 318 0110	.028 679 3300
21	3.078 234 1526	37.786 075 5016	.026 464 7754
22	3.247 537 0310	40.864 309 6542	.024 471 2319
23	3.426 151 5677	44.111 846 6852	.022 669 6472
24	3.614 589 9039	47.537 998 2528	.021 035 8037
25	3.813 392 3486	51.152 588 1567	.019 549 3529
26	4.023 128 9278	54.965 980 5054	.018 193 0713
27	4.244 401 0188	58.989 109 4332	.016 952 2817
28	4.477 843 0749	63.233 510 4520	.015 814 3996
29	4.724 124 4440	67.711 353 5268	.014 768 5720
30	4.983 951 2884	72.435 477 9708	.013 805 3897
31	5.258 068 6093	77.419 429 2592	.012 916 6543
32	5.547 262 3828	82.677 497 8685	.012 095 1895
33	5.852 361 8138	88.224 760 2512	.011 334 6865
34	6.174 241 7136	94.077 122 0651	.010 629 5769
35	6.513 825 0078	100.251 363 7786	.009 974 9266
36	6.872 085 3833	106.765 188 7865	.009 366 3488
37	7.250 050 0793	113.637 274 1697	.008 799 9295
38	7.648 802 8337	120.887 324 2490	.008 272 1659
39	8.069 486 9896	128.536 127 0827	.007 779 9139
40	8.513 308 7740	136.605 614 0723	.007 320 3434
41	8.981 540 7565	145.118 922 8463	.006 890 9001
42	9.475 525 4982	154.100 463 6028	.006 489 2731
43	9.996 679 4006	163.575 989 1010	.006 113 3667
44	10.546 496 7676	173.572 668 5015	.005 761 2757
45	11.126 554 0898	184.119 165 2691	.005 431 2651
46	11.738 514 5647	195.245 719 3589	.005 121 7512
47	12.384 132 8658	206.984 233 9237	.004 831 2858
48	13.065 260 1734	219.368 366 7895	.004 558 5424
49	13.783 849 4830	232.433 626 9629	.004 302 3035
50	14.541 961 2045	246.217 476 4458	.004 061 4501
51	15.341 769 0708	260.759 437 6504	.003 834 9523
52	16.185 566 3697	276.101 206 7211	.003 621 8603
53	17.075 772 5200	292.286 773 0908	.003 421 2975
54	18.014 940 0086	309.362 545 6108	.003 232 4534
55	19.005 761 7091	327.377 485 6194	.003 054 5778
56	20.051 078 6031	346.383 247 3284	.002 886 9756
57	21.153 887 9262	366.434 325 9315	.002 729 0020
58	22.317 351 7622	387.588 213 8577	.002 580 0578
59	23.544 806 1091	409.905 565 6199	.002 439 5863
60	24.839 770 4451	433.450 371 7290	.002 307 0692
n	$s = (1+i)^n$	$s_{\overline{n}\rvert} = \frac{(1+i)^n - 1}{i}$	$\frac{1}{s_{\overline{n}\rvert}} = \frac{i}{(1+i)^n - 1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today.* *Periodic payment necessary to pay off a loan of $1.*	PERIODS
.947 867 2986	.947 867 2986	1.055 000 0000	1
.898 452 4157	1.846 319 7143	.541 618 0049	2
.851 613 6642	2.697 933 3785	.370 654 0747	3
.807 216 7433	3.505 150 1218	.285 294 4853	4
.765 134 3538	4.270 284 4756	.234 176 4362	5
.725 245 8330	4.995 530 3086	.200 178 9476	6
.687 436 8086	5.682 967 1172	.175 964 4178	7
.651 598 8707	6.334 565 9879	.157 864 0118	8
.617 629 2613	6.952 195 2492	.143 839 4585	9
.585 430 5794	7.537 625 8286	.132 667 7687	10
.554 910 5018	8.092 536 3304	.123 570 6532	11
.525 981 5183	8.618 517 8487	.116 029 2312	12
.498 560 6809	9.117 078 5296	.109 684 2587	13
.472 569 3658	9.589 647 8954	.104 279 1154	14
.447 933 0481	10.037 580 9435	.099 625 5976	15
.424 581 0883	10.462 162 0317	.095 582 5380	16
.402 446 5292	10.864 608 5609	.092 041 9723	17
.381 465 9044	11.246 074 4653	.088 919 9163	18
.361 579 0563	11.607 653 5216	.086 150 0559	19
.342 728 9633	11.950 382 4849	.083 679 3300	20
.324 861 5766	12.275 244 0615	.081 464 7754	21
.307 925 6650	12.583 169 7266	.079 471 2319	22
.291 872 6683	12.875 042 3949	.077 669 6472	23
.276 656 5576	13.151 698 9525	.076 035 8037	24
.262 233 7039	13.413 932 6564	.074 549 3529	25
.248 562 7525	13.662 495 4089	.073 193 0713	26
.235 604 5047	13.898 099 9136	.071 952 2817	27
.223 321 8055	14.121 421 7191	.070 814 3996	28
.211 679 4364	14.333 101 1555	.069 768 5720	29
.200 644 0156	14.533 745 1711	.068 805 3897	30
.190 183 9010	14.723 929 0722	.067 916 6543	31
.180 269 1005	14.904 198 1727	.067 095 1895	32
.170 871 1853	15.075 069 3580	.066 334 6865	33
.161 963 2088	15.237 032 5668	.065 629 5769	34
.153 519 6292	15.390 552 1960	.064 974 9266	35
.145 516 2362	15.536 068 4322	.064 366 3488	36
.137 930 0817	15.673 998 5140	.063 799 9295	37
.130 739 4140	15.804 737 9279	.063 272 1659	38
.123 923 6151	15.928 661 5431	.062 779 9139	39
.117 463 1423	16.046 124 6854	.062 320 3434	40
.111 339 4714	16.157 464 1568	.061 890 9001	41
.105 535 0440	16.262 999 2007	.061 489 2731	42
.100 033 2170	16.363 032 4177	.061 113 3667	43
.094 818 2152	16.457 850 6329	.060 761 2757	44
.089 875 0855	16.547 725 7184	.060 431 2651	45
.085 189 6545	16.632 915 3729	.060 121 7512	46
.080 748 4877	16.713 663 8606	.059 831 2858	47
.076 538 8509	16.790 202 7114	.059 558 5424	48
.072 548 6738	16.862 751 3853	.059 302 3035	49
.068 766 5155	16.931 517 9007	.059 061 4501	50
.065 181 5312	16.996 699 4320	.058 834 9523	51
.061 783 4419	17.058 482 8739	.058 621 8603	52
.058 562 5042	17.117 045 3781	.058 421 2975	53
.055 509 4827	17.172 554 8608	.058 232 4534	54
.052 615 6234	17.225 170 4841	.058 054 5778	55
.049 872 6288	17.275 043 1129	.057 886 9756	56
.047 272 6339	17.322 315 7468	.057 729 0020	57
.044 808 1838	17.367 123 9307	.057 580 0578	58
.042 472 2121	17.409 596 1428	.057 439 5863	59
.040 258 0210	17.449 854 1638	.057 307 0692	60
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\rvert} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rvert}} = \frac{i}{1-v^n}$	n

RATE

5½%

.055

per period

ANNUALLY

If compounded *annually* nominal annual rate is

5½%

SEMIANNUALLY

If compounded *semiannually* nominal annual rate is

11%

QUARTERLY

If compounded *quarterly* nominal annual rate is

22%

MONTHLY

If compounded *monthly* nominal annual rate is

66%

$i = .055$
$j_{(2)} = .11$
$j_{(4)} = .22$
$j_{(12)} = .66$

RATE

6%

.06

per period

ANNUALLY
If compounded *annually* nominal annual rate is
6%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is
12%

QUARTERLY
If compounded *quarterly* nominal annual rate is
24%

MONTHLY
If compounded *monthly* nominal annual rate is
72%

$i = .06$
$j_2 = .12$
$j_4 = .24$
$j_{12} = .72$

PERIODS	AMOUNT OF 1 *How \$1 left at compound interest will grow*	AMOUNT OF 1 PER PERIOD *How \$1 deposited periodically will grow*	SINKING FUND *Periodic deposit that will grow to \$1 at future date*
1	1.060 000 0000	1.000 000 0000	1.000 000 0000
2	1.123 600 0000	2.060 000 0000	.485 436 8932
3	1.191 016 0000	3.183 600 0000	.314 109 8128
4	1.262 476 9600	4.374 616 0000	.228 591 4924
5	1.338 225 5776	5.637 092 9600	.177 396 4004
6	1.418 519 1123	6.975 318 5376	.143 362 6285
7	1.503 630 2590	8.393 837 6499	.119 135 0181
8	1.593 848 0745	9.897 467 9088	.101 035 9426
9	1.689 478 9590	11.491 315 9834	.087 022 2350
10	1.790 847 6965	13.180 794 9424	.075 867 9582
11	1.898 298 5583	14.971 642 6389	.066 792 9381
12	2.012 196 4718	16.869 941 1973	.059 277 0294
13	2.132 928 2601	18.882 137 6691	.052 960 1053
14	2.260 903 9558	21.015 065 9292	.047 584 9090
15	2.396 558 1931	23.275 969 8850	.042 962 7640
16	2.540 351 6847	25.672 528 0781	.038 952 1436
17	2.692 772 7858	28.212 879 7628	.035 444 8042
18	2.854 339 1529	30.905 652 5485	.032 356 5406
19	3.025 599 5021	33.759 991 7015	.029 620 8604
20	3.207 135 4722	36.785 591 2035	.027 184 5570
21	3.399 563 6005	39.992 726 6758	.025 004 5467
22	3.603 537 4166	43.392 290 2763	.023 045 5685
23	3.819 749 6616	46.995 827 6929	.021 278 4847
24	4.048 934 6413	50.815 577 3545	.019 679 0050
25	4.291 870 7197	54.864 511 9957	.018 226 7182
26	4.549 982 9629	59.156 382 7155	.016 904 3467
27	4.822 345 9407	63.705 765 6784	.015 697 1663
28	5.111 686 6971	68.528 111 6191	.014 592 5515
29	5.418 387 8990	73.639 798 3162	.013 579 6135
30	5.743 491 1729	79.058 186 2152	.012 648 9115
31	6.088 100 6433	84.801 677 3881	.011 792 2196
32	6.453 386 6819	90.889 778 0314	.011 002 3374
33	6.840 589 8828	97.343 164 7133	.010 272 9350
34	7.251 025 2758	104.183 754 5961	.009 598 4254
35	7.686 086 7923	111.434 779 8719	.008 973 8590
36	8.147 251 9999	119.120 866 6642	.008 394 8348
37	8.636 087 1198	127.268 118 6640	.007 857 4274
38	9.154 252 3470	135.904 205 7839	.007 358 1240
39	9.703 507 4879	145.058 458 1309	.006 893 7724
40	10.285 717 9371	154.761 965 6188	.006 461 5359
41	10.902 861 0134	165.047 683 5559	.006 058 8551
42	11.557 032 6742	175.950 544 5692	.005 683 4152
43	12.250 454 6346	187.507 577 2434	.005 333 1178
44	12.985 481 9127	199.758 031 8780	.005 006 0565
45	13.764 610 8274	212.743 513 7907	.004 700 4958
46	14.590 487 4771	226.508 124 6181	.004 414 8527
47	15.465 916 7257	241.098 612 0952	.004 147 6805
48	16.393 871 7293	256.564 528 8209	.003 897 6549
49	17.377 504 0330	272.958 400 5502	.003 663 5619
50	18.420 154 2750	290.335 904 5832	.003 444 2864
51	19.525 363 5315	308.756 058 8582	.003 238 8028
52	20.696 885 3434	328.281 422 3897	.003 046 1669
53	21.938 698 4640	348.978 307 7331	.002 865 5076
54	23.255 020 3718	370.917 006 1970	.002 696 0209
55	24.650 321 5941	394.172 026 5689	.002 536 9634
56	26.129 340 8898	418.822 348 1630	.002 387 6472
57	27.697 101 3432	444.951 689 0528	.002 247 4350
58	29.358 927 4238	472.648 790 3959	.002 115 7359
59	31.120 463 0692	502.007 717 8197	.001 992 0012
60	32.987 690 8533	533.128 180 8889	.001 875 7215
n	$s = (1+i)^n$	$s_{\overline{n}\rvert} = \frac{(1+i)^n - 1}{i}$	$\frac{1}{s_{\overline{n}\rvert}} = \frac{i}{(1+i)^n - 1}$

PRESENT WORTH OF 1 *What $1 due in the future is worth today.*	PRESENT WORTH OF 1 PER PERIOD *What $1 payable periodically is worth today.*	PARTIAL PAYMENT *Annuity worth $1 today.* *Periodic payment necessary to pay off a loan of $1.*	PERIODS
.943 396 2264	.943 396 2264	1.060 000 0000	1
.889 996 4400	1.833 392 6664	.545 436 8932	2
.839 619 2830	2.673 011 9495	.374 109 8128	3
.792 093 6632	3.465 105 6127	.288 591 4924	4
.747 258 1729	4.212 363 7856	.237 396 4004	5
.704 960 5404	4.917 324 3260	.203 362 6285	6
.665 057 1136	5.582 381 4396	.179 135 0181	7
.627 412 3713	6.209 793 8110	.161 035 9426	8
.591 898 4635	6.801 692 2745	.147 022 2350	9
.558 394 7769	7.360 087 0514	.135 867 9582	10
.526 787 5254	7.886 874 5768	.126 792 9381	11
.496 969 3636	8.383 843 9404	.119 277 0294	12
.468 839 0222	8.852 682 9626	.112 960 1053	13
.442 300 9644	9.294 983 9270	.107 584 9090	14
.417 265 0607	9.712 248 9877	.102 962 7640	15
.393 646 2837	10.105 895 2715	.098 952 1436	16
.371 364 4186	10.477 259 6901	.095 444 8042	17
.350 343 7911	10.827 603 4812	.092 356 5406	18
.330 513 0105	11.158 116 4917	.089 620 8604	19
.311 804 7269	11.469 921 2186	.087 184 5570	20
.294 155 4027	11.764 076 6213	.085 004 5467	21
.277 505 0969	12.041 581 7182	.083 045 5685	22
.261 797 2612	12.303 378 9794	.081 278 4847	23
.246 978 5483	12.550 357 5278	.079 679 0050	24
.232 998 6305	12.783 356 1583	.078 226 7182	25
.219 810 0288	13.003 166 1870	.076 904 3467	26
.207 367 9517	13.210 534 1387	.075 697 1663	27
.195 630 1431	13.406 164 2818	.074 592 5515	28
.184 556 7388	13.590 721 0206	.073 579 6135	29
.174 110 1309	13.764 831 1515	.072 648 9115	30
.164 254 8405	13.929 085 9920	.071 792 2196	31
.154 957 3967	14.084 043 3887	.071 002 3374	32
.146 186 2233	14.230 229 6119	.070 272 9350	33
.137 911 5314	14.368 141 1433	.069 598 4254	34
.130 105 2183	14.498 246 3616	.068 973 8590	35
.122 740 7720	14.620 987 1336	.068 394 8348	36
.115 793 1811	14.736 780 3147	.067 857 4274	37
.109 238 8501	14.846 019 1648	.067 358 1240	38
.103 055 5190	14.949 074 6838	.066 893 7724	39
.097 222 1877	15.046 296 8715	.066 461 5359	40
.091 719 0450	15.138 015 9165	.066 058 8551	41
.086 527 4010	15.224 543 3175	.065 683 4152	42
.081 629 6235	15.306 172 9410	.065 333 1178	43
.077 009 0788	15.383 182 0198	.065 006 0565	44
.072 650 0743	15.455 832 0942	.064 700 4958	45
.068 537 8060	15.524 369 9002	.064 414 8527	46
.064 658 3075	15.589 028 2077	.064 147 6805	47
.060 998 4033	15.650 026 6110	.063 897 6549	48
.057 545 6635	15.707 572 2746	.063 663 5619	49
.054 288 3618	15.761 860 6364	.063 444 2864	50
.051 215 4357	15.813 076 0721	.063 238 8028	51
.048 316 4488	15.861 392 5208	.063 046 1669	52
.045 581 5554	15.906 974 0762	.062 865 5076	53
.043 001 4674	15.949 975 5436	.062 696 0209	54
.040 567 4221	15.990 542 9657	.062 536 9634	55
.038 271 1529	16.028 814 1186	.062 387 6472	56
.036 104 8612	16.064 918 9798	.062 247 4350	57
.034 061 1898	16.098 980 1696	.062 115 7359	58
.032 133 1979	16.131 113 3676	.061 992 0012	59
.030 314 3377	16.161 427 7052	.061 875 7215	60
$v^n = \frac{1}{(1+i)^n}$	$a_{\overline{n}\rceil} = \frac{1-v^n}{i}$	$\frac{1}{a_{\overline{n}\rceil}} = \frac{i}{1-v^n}$	n

RATE

6%

.06

per period

ANNUALLY
If compounded *annually* nominal annual rate is

6%

SEMIANNUALLY
If compounded *semiannually* nominal annual rate is

12%

QUARTERLY
If compounded *quarterly* nominal annual rate is

24%

MONTHLY
If compounded *monthly* nominal annual rate is

72%

$i = .06$
$j_{(2)} = .12$
$j_{(4)} = .24$
$j_{(12)} = .72$

Commissioners 1941 Standard Ordinary Mortality Table

Age(x)	l_x	d_x	q_x	$\mathring{e}_x$
0	1 023 102	23 102	0 02258	62 33
1	1 000 000	5 770	0 00577	62 76
2	994 230	4 116	0 00414	62 12
3	990 114	3 347	0 00338	61 37
4	986 767	2 950	0 00299	60 58
5	983 817	2 715	0 00276	59 76
6	981 102	2 561	0 00261	58 92
7	978 541	2 417	0 00247	58 08
8	976 124	2 255	0 00231	57 22
9	973 869	2 065	0 00212	56 35
10	971 804	1 914	0 00197	55 47
11	969 890	1 852	0 00191	54 58
12	968 038	1 859	0 00192	53 68
13	966 179	1 913	0 00198	52 78
14	964 266	1 996	0 00207	51 89
15	962 270	2 069	0 00215	50 99
16	960 201	2 103	0 00219	50 10
17	958 098	2 156	0 00225	49 21
18	955 942	2 199	0 00230	48 32
19	953 743	2 260	0 00237	47 43
20	951 483	2 312	0 00243	46 54
21	949 171	2 382	0 00251	45 66
22	946 789	2 452	0 00259	44 77
23	944 337	2 531	0 00268	43 88
24	941 806	2 609	0 00277	43 00
25	939 197	2 705	0 00288	42 12
26	936 492	2 800	0 00299	41 24
27	933 692	2 904	0 00311	40 36
28	930 788	3 025	0 00325	39 49
29	927 763	3 154	0 00340	38 61
30	924 609	3 292	0 00356	37 74
31	921 317	3 437	0 00373	36 88
32	917 880	3 598	0 00392	36 01
33	914 282	3 767	0 00412	35 15
34	910 515	3 961	0 00435	34 29
35	906 554	4 161	0 00459	33 44
36	902 393	4 386	0 00486	32 59
37	898 007	4 625	0 00515	31 75
38	893 382	4 878	0 00546	30 91
39	888 504	5 162	0 00581	30 08
40	883 342	5 459	0 00618	29 25
41	877 883	5 785	0 00659	28 43
42	872 098	6 131	0 00703	27 62
43	865 967	6 503	0 00751	26 81
44	859 464	6 910	0 00804	26 01
45	852 554	7 340	0 00861	25 21
46	845 214	7 801	0 00923	24 43
47	837 413	8 299	0 00991	23 65
48	829 114	8 822	0 01064	22 88
49	820 292	9 392	0 01145	22 12

Commissioners 1941 Standard Ordinary Mortality Table

Age(x)	l_x	d_x	q_x	$\overset{o}{e}_x$
50	810 900	9 990	0.01232	21.37
51	800 910	10 628	0.01327	20.64
52	790 282	11 301	0.01430	19.91
53	778 981	12 020	0.01543	19.19
54	766 961	12 770	0.01665	18.48
55	754 191	13 560	0.01798	17.78
56	740 631	14 390	0.01943	17.10
57	726 241	15 251	0.02100	16.43
58	710 990	16 147	0.02271	15.77
59	694 843	17 072	0.02457	15.13
60	677 771	18 022	0.02659	14.50
61	659 749	18 988	0.02878	13.88
62	640 761	19 979	0.03118	13.27
63	620 782	20 958	0.03376	12.69
64	599 824	21 942	0.03658	12.11
65	577 882	22 907	0.03964	11.55
66	554 975	23 842	0.04296	11.01
67	531 133	24 730	0.04656	10.48
68	506 403	25 553	0.05046	9.97
69	480 850	26 302	0.05470	9.47
70	454 548	26 955	0.05930	8.99
71	427 593	27 481	0.06427	8.52
72	400 112	27 872	0.06966	8.08
73	372 240	28 104	0.07550	7.64
74	344 136	28 154	0.08181	7.23
75	315 982	28 009	0.08864	6.82
76	287 973	27 651	0.09602	6.44
77	260 322	27 071	0.10399	6.07
78	233 251	26 262	0.11259	5.72
79	206 989	25 224	0.12186	5.38
80	181 765	23 966	0.13185	5.06
81	157 799	22 502	0.14260	4.75
82	135 297	20 857	0.15416	4.46
83	114 440	19 062	0.16657	4.18
84	95 378	17 157	0.17988	3.91
85	78 221	15 185	0.19413	3.66
86	63 036	13 198	0.20937	3.42
87	49 838	11 245	0.22563	3.19
88	38 593	9 378	0.24300	2.98
89	29 215	7 638	0.26144	2.77
90	21 577	6 063	0.28099	2.58
91	15 514	4 681	0.30173	2.39
92	10 833	3 506	0.32364	2.21
93	7 327	2 540	0.34666	2.03
94	4 787	1 776	0.37100	1.84
95	3 011	1 193	0.39621	1.63
96	1 818	813	0.44719	1.37
97	1 005	551	0.54826	1.08
98	454	329	0.72467	0.78
99	125	125	1.00000	0.50

x	D_x	N_x	C_x	M_x	$a_x = \frac{N_x}{D_x}$	$A_x = \frac{M_x}{D_x}$
0	1 023 102 00	31 374 230	22 538 536 6	257 876 88	30 665 8	0 252 054
1	975 609 76	30 351 128	5 491 969 1	235 338 35	31 109 9	0 241 222
2	946 322 43	29 375 518	3 822 115 2	229 846 38	31 041 8	0 242 884
3	919 419 28	28 429 196	3 032 216 8	226 024 26	30 920 8	0 245 834
4	893 962 20	27 509 776	2 607 370 2	222 992 05	30 772 9	0 249 442
5	869 550 88	26 615 814	2 341 136 0	220 384 68	30 608 7	0 253 447
6	846 001 18	25 746 263	2 154 480 3	218 043 54	30 432 9	0 257 734
7	823 212 53	24 900 262	1 983 744 5	215 889 06	30 247 7	0 262 252
8	801 150 42	24 077 050	1 805 642 5	213 905 32	30 053 1	0 266 998
9	779 804 53	23 275 899	1 613 174 7	212 099 67	29 848 4	0 271 991
10	759 171 73	22 496 095	1 458 745 1	210 486 50	29 632 4	0 277 258
11	739 196 60	21 736 923	1 377 065 5	209 027 75	29 406 1	0 282 777
12	719 790 36	20 997 726	1 348 556 5	207 650 69	29 172 0	0 288 488
13	700 885 94	20 277 936	1 353 882 1	206 302 13	28 931 9	0 294 345
14	682 437 28	19 577 050	1 378 169 3	204 948 25	28 687 0	0 300 318
15	664 414 29	18 894 613	1 393 730 0	203 570 08	28 438 0	0 306 390
16	646 815 33	18 230 198	1 382 081 2	202 176 35	28 184 5	0 312 572
17	629 657 27	17 583 383	1 382 353 7	200 794 27	27 925 3	0 318 895
18	612 917 42	16 953 726	1 375 535 5	199 411 91	27 660 7	0 325 349
19	596 592 68	16 340 808	1 379 212 3	198 036 38	27 390 2	0 331 946
20	580 662 42	15 744 216	1 376 533 1	196 657 17	27 114 2	0 338 677
21	565 123 40	15 163 553	1 383 619 6	195 280 63	26 832 3	0 345 554
22	549 956 28	14 598 430	1 389 541 6	193 897 01	25 544 7	0 352 568
23	535 153 17	14 048 474	1 399 327 5	192 507 47	26 251 3	0 359 724
24	520 701 32	13 513 320	1 407 270 0	191 108 14	25 952 2	0 367 021
25	506 594 02	12 992 619	1 423 464 9	189 700 88	25 647 0	0 374 463
26	492 814 61	12 486 025	1 437 519 2	188 277 41	25 336 2	0 382 045
27	479 357 22	11 993 210	1 454 549 1	186 839 89	25 019 4	0 389 772
28	466 211 03	11 513 853	1 478 200 3	185 385 34	24 696 7	0 397 643
29	453 361 83	11 047 642	1 503 646 4	183 907 14	24 368 3	0 405 652
30	440 800 58	10 594 280	1 531 158 0	182 403 50	24 034 2	0 413 800
31	428 518 18	10 153 480	1 559 609 4	180 872 34	23 694 4	0 422 088
32	416 506 91	9 724 962	1 592 845 3	179 312 73	23 348 9	0 430 516
33	404 755 37	9 308 455	1 626 987 4	177 719 88	22 997 7	0 439 080
34	393 256 29	8 903 699	1 669 050 8	176 092 90	22 641 0	0 447 781
35	381 995 63	8 510 443	1 710 561 0	174 423 84	22 278 9	0 456 612
36	370 968 10	8 128 447	1 759 080 1	172 713 28	21 911 4	0 465 574
37	360 161 02	7 757 479	1 809 692 8	170 954 20	21 538 9	0 474 660
38	349 566 90	7 397 318	1 862 134 5	169 144 51	21 161 4	0 483 869
39	339 178 75	7 047 751	1 922 486 9	167 282 38	20 778 9	0 493 198
40	328 983 61	6 708 573	1 983 511 0	165 359 89	20 391 8	0 502 639
41	318 976 11	6 379 589	2 050 694 7	163 376 38	20 000 2	0 512 190
42	309 145 51	6 060 613	2 120 338 1	161 325 68	19 604 4	0 521 844
43	299 485 04	5 751 467	2 194 136 7	159 205.35	19 204 5	0 531 597
44	289 986 39	5 451 982	2 274 595 1	157 011 21	18 800 8	0 541 443
45	280 638 95	5 161 996	2 357 209 9	154 736 61	18 393 7	0 551 373
46	271 436 89	4 881 357	2 444 154 2	152 379 40	17 983 4	0 561 381
47	262 372 33	4 609 920	2 536 765 0	149 935 25	17 570 1	0 571 460
48	253 436 24	4 347 548	2 630 859 4	147 398 48	17 154 4	0 581 600
49	244 624 00	4 094 112	2 732 529 2	144 767 62	16 736 3	0 591 796

x	D_x	N_x	C_x	M_x	$\ddot{a}_x = \frac{N_x}{D_x}$	$A_x = \frac{M_x}{D_x}$
50	235 925.04	3 849 488	2 835.622 1	142 035.10	16.316 6	0.602 035
51	227 335.15	3 613 563	2 943.137 4	139 199.47	15.895 3	0.612 310
52	218 847.25	3 386 227	3 053.177 2	136 256.34	15.473 0	0.622 609
53	210 456.33	3 167 380	3 168.222 9	133 203.16	15.050 1	0.632 925
54	202 155.03	2 956 924	3 283.812 1	130 034.94	14.627 0	0.643 244
55	193 940.61	2 754 769	3 401.913 1	126 751.12	14.204 2	0.653 556
56	185 808.43	2 560 828	3 522.090 1	123 349.21	13.782 1	0.663 852
57	177 754.43	2 375 020	3 641.783 5	119 827.12	13.361 2	0.674 116
58	169 777.17	2 197 265	3 761.696 8	116 185.34	12.942 1	0.684 340
59	161 874 57	2 027 488	3 880.185 4	112 423.64	12.525 1	0.694 511
60	154 046.23	1 865 614	3 996.199 9	108 543.46	12.110 7	0.704 616
61	146 292.80	1 711 567	4 107.708 0	104 547.26	11.699 6	0.714 644
62	138 616.97	1 565 275	4 216.676 0	100 439.55	11.292 1	0.724 583
63	131 019.40	1 426 658	4 315.413 8	96 222.87	10.888 9	0.734 417
64	123 508.39	1 295 638	4 407.831 2	91 907.46	10.490 3	0.744 139
65	116 088.15	1 172 130	4 489.449 7	87 499.63	10.096 9	0.753 734
66	108 767.29	1 056 042	4 558.728 2	83 010.18	9.709 2	0.763 191
67	101 555.70	947 274.4	4 613.189 3	78 451.45	9.327 6	0.772 497
68	94 465.545	845 718.7	4 650.452 1	73 838.26	8.952 7	0.781 642
69	87 511.050	751 253.1	4 670.014 3	69 187.81	8.584 7	0.790 618
70	80 706.625	663 742.1	4 669.226 0	64 517.79	8.224 1	0.799 411
71	74 068.942	583 035.4	4 644.235 4	59 848.57	7.871 5	0.808 012
72	67 618.148	505 966.5	4 595.428 1	55 204.33	7.527 1	0.816 413
73	61 373.498	441 348.3	4 520.662 7	50 608.90	7.191 2	0.824 605
74	55 355.921	379 974.8	4 418.249 2	46 088.24	6.864 2	0.832 580
75	49 587.526	324 618.9	4 288.286 9	41 669.99	6.546 4	0.840 332
76	44 089.787	275 031.4	4 130.220 2	37 381.70	6.238 0	0.847 854
77	38 884.206	230 941.6	3 944.961 8	33 251.48	5.939 2	0.855 141
78	33 990.850	192 057.4	3 733.725 8	29 306.52	5.650 3	0.862 189
79	29 428.077	158 066.6	3 498.684 1	25 572.80	5.371 3	0.868 993
80	25 211.636	128 638.5	3 243.115 8	22 074.11	5.102 3	0.875 553
81	21 353.602	103 426.8	2 970.736 8	18 831.00	4.843 5	0.881 865
82	17 862.047	82 073.24	2 686.402 0	15 860.26	4.594 8	0.887 931
83	14 739.984	64 211.19	2 395.321 2	13 173.86	4.356 3	0.893 750
84	11 985.151	49 471.21	2 103.356 1	10 778.54	4.127 7	0.899 324
85	9 589.474 6	37 486.06	1 816.194 6	8 675.180	3.909 1	0.904 656
86	7 539.390 5	27 896.58	1 540.039 4	6 858.986	3.700 1	0.909 753
87	5 815.463 2	20 357.19	1 280.145 4	5 318.946	3.500 5	0.914 621
88	4 393.477 3	14 541.73	1 041.564 6	4 038.801	3.309 8	0.919 272
89	3 244.754 6	10 148.25	827.621 5	2 997.236	3.127 6	0.923 717
90	2 337.992 9	6 903.496	640.937 1	2 169.615	2.952 7	0.927 982
91	1 640.030 9	4 565.503	482.773 6	1 528.677	2.783 8	0.932 103
92	1 117.257 1	2 925.472	352.770 7	1 045.904	2.618 4	0.036 136
93	737.236 9	1 808.215	249.339 1	693.133 5	2.452 7	0.940 178
94	469.915 8	1 070.979	170.088 8	443.794 4	2.279 1	0.944 413
95	288.365 6	601.062 8	111.467 8	273.705 6	2.084 4	0.949 162
96	169.864 4	312.697 2	74.109 8	162.237 8	1.840 9	0.955 101
97	91.611 4	142.832 6	49.001 2	88.128 0	1.559 1	0.961 973
98	40.375 5	51.220 9	28.541 1	39.126 1	1.268 6	0.969 058
99	10.845 4	10.845 4	10.580 0	10.580 9	1.000 0	0.975 610

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